REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON FIXED POINT THEOREMS FOR SINGLE AND MULTIVALUED MAPPINGS AND APPLICATIONS TO DIFFERENTIAL PROBLEMS

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Ph.D. THESIS
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Nour El Houda BOUZARA

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LIST OF SYMBOLS

$\mathbb{N}$	The set of all natural numbers.
$\mathbb{Z}$	The set of all integer numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{F}$	The set of scalars $\mathbb{R}$ or $\mathbb{C}$.
$Conv(A)$	The convex hull of A.
$B(X)$	The space of all bounded mappings from X into X.
c_0	The space of 0 convergent sequences.
c	The space of convergent sequences.
ℓ_p	The space of sequences $(x_n)_n$ that satisfy $(\sum_n \lvert x_n \rvert^p)^{\frac{1}{p}} < \infty$.
ℓ_∞	The space of bounded sequences.
$BC\ (\mathbb{R}^+)$	The space of all continuous bounded functions defined on $\mathbb{R}^+$.
$C(J, X)$	The space of all continuous functions from J into X.
$C^n(a, b)$	The space of n-times continuously differentiable real functions on (a, b).
$L^p[a, b]$	The space of all p-Bochner integrable functions.
$PC(\mathbb{R}_+, X)$	The space of piecewise continuous functions from $\mathbb{R}_+$ into X.
$\mathcal{P}(X)$	The family of all nonempty subsets of X.
$\mathcal{P}_p(X)$	The family of all nonempty subsets of X that satisfy the property p.
$\mathcal{P}_b(X)$	The family of all nonempty bounded subsets of X .
$\mathcal{P}_{cl}(X)$	The family of all nonempty closed subsets of X .
$\mathcal{P}_{cv}(X)$	The family of all nonempty convex subsets of X .
$\mathcal{P}_{cp}(X)$	The family of all nonempty compact subsets of X .
$S_F(y)$	The set of $f \in L^1(I, X)\ \ for\ a.e. t \in I$ such that $f\ (t)\ \in\ F\ (t, y\ (t))$ where F is a multivalued mapping.

LIST OF ABBREVIATIONS

a. e.	almost everywhere
DE	Differential Equations
Iff	if and only if
l.s.c	lower semi-continuous
MNC	Measure of Non-Compactness
ODE	Ordinay Differential Equations
u.s.c	upper semi-continuous
WMNC	Measure of Weak Non-Compactness

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ON FIXED POINT THEOREMS FOR SINGLE AND MULTIVALUED MAPPINGS AND APPLICATIONS TO DIFFERENTIAL PROBLEMS

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Fixed point theory is an intersting field of mathematics that have applications in many sciences like economy, computer sciences, medecine and game theory. It provides tools applied in other domains with no immediate connection with mathematics at first sight.

Due to their importance, a chapter of this work is destinated to discuss most important theorems with their generalisations and applications. Furthermore, extensions of Darbo fixed point theorem were given; some of them are done by weaking the set contractive condition while others by weakening the space working on. As application, the existence of mild solutions for an impulsive differential equation with nonlocal conditions was investigated. Moreover, the existence of common fixed points for commutative set contraction mappings was garanteed. The obtained results were used to solve an integral equation. Finally, in virtue of the measure of noncompactness, the existence of fixed points for multivalued mappings under different type of conditions was established and as example, using these results, the existence of mild solutions for a nonlocal evolution differential inclusion was inquired.

Key words: Fixed point , measure of noncompactness, condensing mappings, impulsive differential equation, evolution differential inclusion.

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TEK VE ÇOK DEĞERLİ DÖNÜŞÜMLER İÇİN SABİT NOKTA TEOREMLERİ VE DİFERANSİYEL PROBLEMLERE UYGULAMALARI

Nour El Houda BOUZARA

Matematik Anabilim Dalı

Doktora Tezi

Tez Danışmanı: Prof. Dr. Vatan KARAKAYA

Sabit nokta teorisi matematiğin önemli bir alanıdır. Bazı alanlarda ilk bakışta sabit nokta teorisinin uygulaması görülmesine rağmen ekonomi, bilgisayar bilimleri, tıp, oyun teorisi gibi alanlarda uygulamaları olan önemli bir konudur.

Sabit nokta teoremlerinin öneminden dolayı bu çalışmanın bir bölümünde temel varlık teoremleri ve uygulamalarına yer verilmiştir. Ayrıca, diğer bir bölümde, Darbo teoremi küme büzülme koşulu altında zayıflatarak genelleştirildi. Çalışmanın devamında, bu teorem Frechet uzayları için ispatlandı ve daha sonra yerel olmayan impulsif diferansiyel denklem çözmek için uygulandı. Ayrıca, komutatif küme büzülme dönüşümleri için ortak sabit noktaları elde edilmiş ve ilgili sonuçlar kullanılarak integral denklemlerin çözümü yapılmıştır. Son olarak, çok değerli dönüşümler için farklı koşullar altında kompakt olmama ölçüsü kullanılarak sabit noktalarının varlığı garanti edilmiş ve evolution diferansiyel içerme için çözümler elde edilmiştir.

Anahtar Kelimeler: Sabit nokta, kompakt olmama ölçüsü, impulsif diferansiyel denklem, evolusiyon diferansiyel içerme.

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Fixed point theory is a thrilling branch of mathematics. It is a combination of analysis, topology and geometry. It is a mathematical domain that had known a flourishing development in the last five decades ([1-20]).

It provides very popular tools which take part in solving existence problems in many branches of mathematical analysis ([21-29]).

One of the main tools of the theory is Darbo [30] fixed point theorem which uses the concept of measure of noncompactness ([31]) and provides a generalization of two classical theorems of the theory: Schauder [4] fixed point theorem and Banach [3] contraction principle. For this reason, many authors are interested in extending Darbo fixed point theorem ([32-36] and many others).

There are also works dealing with its applications to solving different types of integral and differential problems ([37-43]).

Also, Darbo's fixed point theorem was extended to multivalued mappings ([44-45]) and there are a large number of works that combine these concepts to study the existence of solutions for different type of integral and differential inclusions ([46-50]).

1.2 Objective of the Thesis

On the contrary of the finite case, the boundedness condition is not enough to have control over the behavior of sets in the infinite dimension spaces. It is natural to ask how to predict the behavior of sets in the infinite dimension case.

The answer is in the compactness condition. Compact sets can be treated as finite sets. They can act according to only finitely many possible behaviors. This property makes the compactness of a great essence in different mathematical fields, in particular, the field of fixed point theory. It is essential to guarantee the existence of fixed points. However, this condition is strong and its satisfaction may be hard to achieve. In fact, only a few numbers of compact spaces are known, like Hilbert cubes and Cantor sets. Again it is natural to wonder how this condition could be weakened. The answer to this question is given by Kuratowski [31] where he introduced the concept of measure of noncompactness. Then, Darbo employed this concept to prove a fixed point theorem that revealed to be of great importance in fixed point theory and had the attention of a large number of mathematicians.

1.3 Hypothesis

In this work due to its importance, a bouquet of fixed point theorems was gathered and presented. Also, they were illustrated by treating their applications. In further, the usefulness of the measure of noncompactness for extending the fixed point results to the noncompact case was investigated. Generalizations of Darbo's fixed point theorem were shown; some of them were done by weakening the set contractive condition, others were done by weakening the space working on. As an application, the existence of a nonlocal impulsive differential equation was inquired. Moreover, the existence of common fixed point for commutative set contraction mappings was guaranteed. These results were used to study the solvability of some integral equations. Finally, the existence of fixed point for multivalued mappings under different types of conditions was treated. As an application, the solvability of a nonlocal evolution inclusion was studied.

CHAPTER 2

HISTORY AND PRELIMINARIES

2.1 Brief Glance at Fixed Point Theory

The fixed point theory deals with finding conditions on a mapping N and a space X sufficient to guarantee the existence of a point p such that $Np = p$. It is of high importance in the mathematic fields and for a better understanding of the prehistory of the fixed point theorems, passing through differential equations is of essence. On the 19th century, the stability of the solar system problem returned into the focus of the mathematical community. Its solution required new methods. As noted by Henri Poincaré, there is no hope to find an exact solution: "Nothing is more proper to give us an idea of the hardness of the three-body problem. He studied a question about the trajectories on a surface animated by a constant flow (which is analogous to that of the surface movement in Brouwer's cup of coffee). In 1880, he presented a theorem (without proof) which simply states that there is no smooth vector field on a sphere having no sources or sinks.

Later in 1912, motivated by Poincaré's analysis the Dutch Mathematician, and father of the Intuitionist School of Mathematics, Luitzen Egbertus Jan Brouwer [1] proved a theorem originated from his observation on a cup of coffee. The story goes that as he was stirring sugar into a cup of coffee, he noticed there always seemed to be a point on the surface which wasn't moving. He concluded that at any given instant, there was always a point which wasn't in motion, even though this point could change in different instances. He also claimed: "I can formulate this splendid result differently; I take a horizontal sheet and another identical one which I crumple, flatten and place on the other. Then, a point of the crumpled sheet is in the same place as on the other sheet". This example is better than the coffee cup one, as it shows that uniqueness of the fixed point may fail. Although it was yet apparent, this theorem revealed to be an equivalent of Poincaré theorem by Miranda several decades later (1941).

Meanwhile, the Polish mathematician Stefan Banach established a remarkably fixed point theorem which has appeared in his Ph.D. thesis and was published in 1922. The Banach contraction based on shrinking map ensures the existence and uniqueness of a fixed point and when abstract metric spaces were introduced by Hausdorff, it provided the general framework for the principle of contraction mappings in a complete metric space.

In 1930, Juliusz Schauder [4] proved an extension of the Brouwer fixed point theorem to spaces of infinite dimension under a condition of compactness.

At the same year, Kuratowski [31] introduced a new concept that called MNC which Darbo used in 1955 to extend the Schauder fixed point theorem by generalizing the compactness condition.

Darbo fixed point theorem received the attention of many researchers since it has combined between two classical theorems Banach contraction principle [3] and Schauder fixed point theorem [4]. It revealed to be a very useful tool to solve differential equations that model different real world phenomena like in economics physics, computer sciences and etc. A great deal of authors has been working on extending this theorem. Since the more the conditions are weakened, the more differential problems could be solved ([32-36] and the references therein).

In 1955, Alfred Tarski [19] proved a result that guaranteed the existence of fixed points for the discrete case. The key idea was to replace the continuity by the monotonicity. This theorem became the foundational block in the discrete fixed point theory.

Later in 1969, using the concept of Hausdorff metric, Sam Bernard Nadler [20] established multivalued version of the Banach contraction principle. This result initialized investigations on the existence of fixed points for multivalued mappings.

As a side note, the generalizations mentioned above are not the only ones but during the 20th century, there were vast amount of literature dealing with generalizations of these remarkable theorems and even a branch of mathematics called fixed-point theory was developed.

It is also worth mentioning that the importance of fixed point theorems cannot be underestimated and by some examples we will show that these theorems have applications everywhere from mathematics, physics and logic to economics and medicine.

2.2 Basic Concepts

This chapter surveys a number of mathematical techniques that are needed throughout this work. Some of the topics are treated in more details in later chapters.

Definition 2.1 A topological space is an ordered pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X satisfying the following axioms,

1) The empty set and X itself belong to $\mathcal{T}$;
2) Any (finite or infinite) union of members of $\mathcal{T}$ still belongs to $\mathcal{T}$;
3) The intersection of any finite number of members of $\mathcal{T}$ still belongs to $\mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets and the collection $\mathcal{T}$ is a topology on X.

Definition 2.2 Let X be a non-empty set, the function $d: X \times X \to \mathbb{R}_+$ is said to be a metric iff for all $x, y, z \in X$, the following conditions are satisfied:

1) $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a metric space.

Definition 2.3 V is said to be a vector space if for any $x, y, z \in V$ and scalars $a, b \in \mathbb{F}$ the following properties are satisfied:

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$;
3. $x + 0 = 0 + x = x$;
4. $x + (-x) = (-x) + x = 0$;
5. $a \times (x + y) = (a \times x) + (a \times y)$;
6. $(a + b) \times x = (a \times x) + (b \times x)$;
7. $a \times (b \times x) = (a \times b) \times x$;
8. $1 \times x = x \times 1 = x$;

where 0 is the identity with respect to + and 1 is the identity with respect to $\times$.

Definition 2.4 Let X be a non-empty set, the function $\|.\|: X \to \mathbb{R}_+$ is said to be a norm iff for all $x, y, z \in X$, the following conditions are satisfied:

1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$;

2) $\|\lambda x\| = |\lambda|\|x\|$, for any scalar $\lambda \in \mathbb{F}$;

3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|.\|)$ is called a normed space.

Remark 2.5 A functional that satisfies only the second and third assumptions of the above definition is called seminorm.

Definition 2.6 Let X be a non-empty set, the function $\langle .,. \rangle: X \times X \to \mathbb{R}_+$ is said to be an inner product iff for al $x, y, z \in X$ the following conditions are satisfied:

1) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$;

2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;

3) $\langle \lambda x, x \rangle = |\lambda| \langle x, x \rangle$, for any scalar $\lambda \in \mathbb{F}$;

4) $\langle x + y, z \rangle \leq \langle x, z \rangle + \langle y, z \rangle$.

The pair $(X, \langle .,. \rangle)$ is called an inner product space.

Remark 2.7

- Every inner product space gives rise to a normed space with the norm $\|x\| = \sqrt{< x, x >}$.
- Every normed space gives rise to a metric space where its metric is defined by $d(x, y) = \|x - y\|$.
- Every metric space gives rise to a topological space where its elements are the open balls.

Definition 2.8 A sequence $(x_n)_{n \in \mathbb{N}}$ in a normed space X is a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for any $m > n_0$, we have

$\|x_n - x_m\| < \varepsilon$.

Definition 2.9 A sequence $(x_n)_{n \in \mathbb{N}}$ in a normed space X converges to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for any $n > n_0$, we have $\|x_n - x\| < \varepsilon$.

Remark 2.10 Every convergent sequence is Cauchy but the inverse is not true.

Definition 2.11 A space X is said to be complete if and only if every Cauchy sequence in X converges. Moreover, a normed complete space is called Banach space and inner product complete space is called Hilbert space.

Definition 2.12 A Fréchet space is a locally convex complete and metrizable space.

Remark 2.13 The topology of a Fréchet space is defined by a countable family of seminorms.

Definition 2.14 The topological dual space of a vector space V is defined as the set of all the linear functional $f: V \to \mathbb{F}$ and it is denoted by V'.

Definition 2.15 The weak topology on a topological space X is the initial topology with respect to its dual X'.

Definition 2.16 A subset A of a space X is said to be:

1) Closed, if a set having an open set as its complement.

2) Convex, if for any $x, y \in A$ we have $\lambda x + (1 - \lambda)y \in A$ where $\lambda \in (0,1)$.

3) Compact, if every sequence in A has a subsequence that converges to a limit that is also in A.

4) Relatively compact (precompact), if it is closure $\bar{A}$ is compact.

Remark 2.17 Subsets of a topological vector space X are called weakly closed (respectively, weakly compact, etc.) if they are closed (respectively, compact, etc.) with respect to the weak topology.

Theorem 2.18 (Eberlein—Šmulian) a set $A \subset X$ is weakly compact iff every sequence of elements of A has a weakly convergent subsequence, in other word, weakly compact set is a compact set in a space with the weak topology.

Definition 2.19 The convex hull of a set A is the intersection of all convex sets containing A.

Definition 2.20 Let X be a Banach space. A Schauder basis is a sequence $(x_n)_n$ of elements of X such that for any $x \in X$ there exists a unique sequence $(a_n)_n$ of scalars in $\mathbb{F}$ so that $x = \sum_{n=0}^{\infty} a_n x_n$.

Definition 2.21 Let (X, d) be a metric space, then for all bounded $A, B, C \in \mathcal{P}_b(X)$ the functional $H_d(A, b)$ is called Hausdorff distance if it satisfies the following properties:

1) $H_d(A, B) \geq 0$ and $H_d(A, B) = 0$ iff $A = B$;

2) $H_d(A, B) = H_d(B, A)$;

3) $H_d(A, C) \leq H_d(A, B) + H_d(B, C)$.

The pair $(\mathcal{P}_b(X), H_d)$ is called a Hausdorff metric space.

Remark 2.22 We can also define the Hausdorff metric by the following expression

$H_d(A, B) = max\{\rho(A, B), \rho(B, A) \ \ for\ all\ A, B \in \mathcal{P}_b(X)\}$,

where $\rho(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A}\{\inf_{y\in B} d(x, y)\}$.

Theorem 2.23 Let (X, d) be a complete metric space, then $(\mathcal{P}_b(X), H_d)$ is also a complete metric space where H_d is the Hausdorff metric relative to d .

The above definitions and results could be found in any function analysis book (for example [47] or [52].

Definition 2.24 A lattice is a partially ordered set $(S, \preccurlyeq)$ such that any two elements x, y have a least upper bound (supremum) denoted $x \wedge y$ and called "joint" and a greatest lower bound (infimum) denoted $x \vee y$ and called "meet".

Definition 2.25 A lattice is complete if every nonempty subset of S has a supremum and an infimum in S. For more details on Banach lattices see [53].

Definition 2.26 Let X be a Banach space. A semigroup is a family $\{T(t)\}_{t\geq 0}$ of continuous linear operators $T(t): X \to X$ such that,

1) $T(0) = I$, where I is the identity operator;
2) $T(s) \circ T(t) = T(t + s)$, for all $t, s \geq 0$.

Definition 2.27 The infinitesimal generator of a semigroup $\{T(t)\}_{t\geq 0}$ is the operator $T(t): D(A) \to X$ such that,

$$D(A) = \left\{x \in X,\ \ \lim_{h\to 0^+} \frac{T(h)x - x}{h} \text{ exists in } X\right\}$$

And $A(x) = \lim_{h\to 0^+} \frac{T(h)x - x}{h}$, for all $x \in X$.

Definition 2.28 A family $\{U(t, s)\}_{(t,s)\in\Delta}$ of bounded linear operators $U(t, s): X \longrightarrow X$ where $(t, s) \in \Delta := \{(t, s):\ 0 \leq s \leq t < +\infty\}$ is called an evolution system if the following properties are satisfied:

1) $U(t, t) = I$ where I is the identity operator in X and $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t < +\infty$;
2) $U(t, s) \in B(X)$ where for every $(t, s) \in \Delta$ and for each $y \in Y$, the mapping

$(t, s) \to U(t, s)y$ is continuous;

3) For $s \leq t < a$, the function $(s, t] \to X$, $t \to U(t, s)$ is differentiable with $\frac{\partial U(t,s)}{\partial t} = A(t)U(t, s)$.

Remark 2.29 An evolution system $U(t, s)$ is said to be compact if $U(t, s)$ is compact for any $t - s > 0$. $U(t, s)$ is said to be equicontinuous if $\{U(t, s)x: x \in \Omega\}$ is equicontinuous at $0 \leq s \leq t < b$, for any bounded subset $\Omega \subset X$. Obviously, if $U(t, s)$ is a compact evolution system, it must be equicontinuous. The inverse is not necessarily true.

More details on semigroup theory and evolution systems and their properties could be found on the books of [54] and [55].

2.3 Single Valued Mappings

Definition 2.30 A mapping N is said to be, a function, functional and operator (respectively) if,

1) it maps a subset of the scalars set $\mathbb{F}$ into a subset of $\mathbb{F}$.
2) it maps a subset of a space X to a subset of $\mathbb{F}$.
3) it maps a subset of a space X into a subset of a space Y.

Definition 2.31 Let X, Y be two normed spaces. A mapping $N: X \to Y$ is said to be continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ whenever $x, y \in X$ satisfy $\|x - y\|_X < \delta$ implies $\|N(x) - N(y)\|_Y < \varepsilon$.

Definition 2.32 Let X, Y be two normed spaces. A mapping $N: X \to Y$ is said to be bounded if there exists a real positive constant M such that $\|Nx\|_Y \leq M\|x\|_X$.

Definition 2.33 A family $\mathcal{N}$ of mappings defined on a metric space X is said to be equicontinuous if for every $\varepsilon > 0$ and for every $x \in X$, there exists $\delta > 0$ such that for all any $N \in \mathcal{N}$ we have $d\big(N(x), N(y)\big) < \varepsilon$ whenever $d(x, y) < \delta$.

Definition 2.34 A continuous function f is said to be p −Bochner integral if it satisfies

$$\int |f(x)|^p dx < \infty.$$

Definition 2.35 Let (X, d) be a metric space. A mapping $N: X \to X$ is called:

1) Lipschitzian (or k - Lipschitzian) if there exists $k > 0$ such that for all $x, y \in X$,

$d(Nx, Ny) \leq kd(x, y)$.

2) Contraction if N is k-Lipschitzian with $0 \leq k < 1$ for all $x, y \in X$.

3) Nonexpansive if N is 1-Lipschitzian for all $x, y \in X$.

4) Contractive if for all $x, y \in X$ with $x \neq y$, we have $d(x, y) < d(x, y)$.

Definition 2.36 A mapping $N: J \times C(J, X) \to X$ is said to be a Carathéodory map if it satisfies:

- $x \to N(t, x)$ is continuous for almost all $t \in J$.
- $t \to N(t, x)$ is measurable for each $x \in C(J, X)$.

Definition 2.37 A mapping *N* is compact if it maps each bounded to a compact set.

Lemma 2.38 Let $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then,
$\lim_{n\to\infty} \psi^n(t) = 0 \Leftrightarrow \psi(t) < t \quad \forall t > 0$.

Definition 2.39 Let (*X*, *d*) be a metric space and $N, H : X \to X$. The mappings *N* and *H* are commuting if $NHx = HNx$ for all $x \in X$.

For more details on single valued mappings and the existence of their fixed points see [21], [26] and [72].

Definition 2.40 A differential equation is an equation that expresses a relationship between a function and its derivatives this amount to saying that differential equations describe things that change.

Differential equations may have different type, which can be resumed as following:

n is the order of the ODE		*Linear/Nonlinear*		*Explicite*
	↖	↑	↗	
Partial	←	$y = f(t, x^{(n)}, \ldots, x^{(1)}, x)$	→	*Ordinary*
	↙	↓	↘	
Constant/variables		*Homogenous/Nonhomogenous*		*Autonomous*

Differential equations are very important since sometimes it is easy to know the change of a thing rather than the thing itself. They have a remarkable ability to predict the real world life.

Applications of differential equations are now used in modeling motions and changes in all areas of science:

1. In meteorology: for predicting weather and storm: where DE represented changes in temperature, pressure, wind velocity …etc.
2. In ecology: for modeling population growth.
3. In economics: to find optimum investment strategies, express the changes of the Gross Domestic Product over and any economic quantity that changes with time.
4. In physics: to express physical laws, describe the motion of waves, heat, pendulums or chaotic systems.
5. In chemistry: for modeling a Chemical Reaction.
6. In medicine: for modeling immune system, tumors growth or the spread of disease (for example, the Malthusian Law of population tumor growth or Logistic law of cancer tumor growth, where the tumor is considered as a population of cancer cells during a time period).

For more details see [56] which contains a great deal of applications and give reviews of the entire field.

2.4 Multivalued Mappings

Multimap or set-valued or multivalued are mappings that map a set to a set. They are very important since they have a direct application to the real world as an example the set of signals emitted by bats and the echo received from nearby objects can be described by a multivalued functions. Another example can be seen when a ray of light strikes a plane mirror, the light ray reflects off the mirror, this phenomenon can also be described by a multivalued function.

In this section, we gather the fundamental concepts and results associated with set-valued maps and various continuity properties (upper and lower semi-continuity, Lipschitzian continuity) of set-valued maps.

Let X and Y be two normed spaces. A set-valued map N from X to Y is a map that associates with any $x \in X$ and a subset $N(x) \in Y$. More formally, let the following definition:

Definition 2.41 Let *X* and *Y* be two metric spaces. A multivalued (or set-valued) map *N* from X to Y is a map that associates with any $x \in X$ a subset $N(x) \in Y$. We denote $\mathcal{P}(Y)$ (also noted 2^Y) the collection of all non-empty subsets of *Y* and we write

$N: X \longrightarrow \mathcal{P}(Y)$ or $N: X \longrightarrow 2^Y$ or $N: X \multimap Y$.

We associate with *N* the set $N(A) = \cup_{x \in A} N(x)$ called the image (range) of A under *N* where *A* is a subset of *X*.

Definition 2.42 A multivalued map *N* is said to have a fixed point if there exists a point $p \in X$ such that $p \in Np$.

A set-valued map *N* is completely characterized by its graph. The graph is denoted G_N and is defined by

$G_N = \{(x, y) \in X \times Y \ where \ y \in N(x)\}$. The set

$D_N = \{x \in X, N(x) \neq 0\}$,

is called the domain of *N*. The image of *N* is defined as

$ImN = \{y, x \in X : y \in N(x)\}$.

The inverse $N^{-1}: Y \multimap X$ is defined by

$N^{-1}(y) = \{x \in X, \ (x, y) \in G_N\}$.

By $N /_A$ we denote the restriction of *N* to a set $A \subseteq X$.

Let $N: X \to \mathcal{P}(Y)$ be a multivalued map and $f: X \to Y$ be a single-valued map. We say that *f* is a selection of *N* (written $f \subset N$) if $f(x) \subset N(x)$ for every $x \in X$.

Let us present more general examples which give motivation for consideration of multivalued maps.

Example 2.43 Let $f: X \longrightarrow Y$ be a continuous single-valued map. Then, its inverse $f^{-1}: Y \longrightarrow X$ is usually a set-valued mapping.

Example 2.44 Let $f: X \times Y \longrightarrow Z$ and $g: X \longrightarrow Z$ be two continuous maps such that for every $x \in X$ there exists $y \in Y$ such that $f(x, y) = g(x)$. These type of functions are called implicit functions and we can consider them as multivalued maps defined by

$N(x) = \{y \in Y, f(x,y) = g(x)\}$.

Example 2.45 Dynamical systems determined by ordinary differential equations without unique property are multivalued maps.

Example 2.46 Let the following problem which appears widely in control theory,

$$\begin{cases} x'(t) = f\big(t, x(t), u(t)\big) \\ x(0) = x_0 \end{cases}, \tag{2.1}$$

where $u(t)$ are the controls parameters and $f: [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

To solve this problem we need to define a new map as follows,

$F(t,x) = \bigcup_{u \in U} f(t,x,u)$.

Clearly, F is a multivalued map such that $F: [0,a] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$.

Then, the solutions of problem (2.1) are the solutions of the following differential inclusion

$$\begin{cases} x'(t) \in F\big(t, x(t)\big) \\ x(0) = x_0 \end{cases}.$$

Remark 2.47 There are many other examples provided by game theory, mathematical economies, convex analysis and nonlinear analysis. For more details on multivalued mappings and their applications see [47] or [58].

If X and Y are topological spaces (or more particular metric or normed spaces) then a natural question about defining a proper notion of continuity of a multivalued mapping $N: X \to \mathcal{P}(Y)$ arises. Formally, one can apply "word by word" the definition of continuity of single valued mappings regarding N as "continuous" if the pre-image of every open subset of Y is an open subset of X. However, there are different notions of the image under multivalued mappings.

In order to study the continuity of multivalued mappings we need to introduce the following two images sets:

$N_+^{-1}(A) = \{x \in X: N(x) \subset A\}$ and $N_-^{-1}(A) = \{x \in X: N(x) \cap A \neq \emptyset\}$.

Remark 2.48 In the single valued case these notions coincide.

Definition 2.49 A multimap $N: X \to \mathcal{P}(Y)$ is upper semi-continuous (u.s.c) at a point $x \in X$ if for every open set $W \subseteq Y$ that contains $N(x)$, there exists a neighborhood $V(x)$ contained in W i.e. $N(V(x)) \subset W$.

Remark 2.50 A multimap is upper semi-continuous if it is u.s.c at every point $x \in X$.

Lemma 2.51 The following assumptions are equivalents:

1) The multimap N is u. s. c.

2) The set $N_+^{-1}(O)$ is open for every open set $O \subset Y$;

3) The set $N_-^{-1}(C)$ is closed for every closed set $C \subset Y$;

4) $N_-^{-1}(\bar{A}) \supseteq \overline{N_-^{-1}(A)}$ for every $A \subset Y$.

Definition 2.52 A multimap $N: X \to \mathcal{P}(Y)$ is lower semi-continuous (l.s.c) at a point $x \in X$ if for every open set $W \subseteq Y$ such that $N(x) \cap W \neq \emptyset$ for all $x' \in V(x)$ of x with the property that $V(x') \cap W \neq \emptyset$ for all $x' \in X$.

Remark 2.53 A multimap is l.s.c if it is lower semi-continuous at every point $x \in X$.

Lemma 2.54 The following assumptions are equivalents.

1) The multimap N is l. s. c.

2) The set $N_+^{-1}(C)$ is closed for every closed set $C \subset Y$.

3) The set $N_-^{-1}(O)$ is open for every open set $O \subset Y$.

4) $N_+^{-1}(\bar{A}) \supseteq \overline{N_+^{-1}(A)}$ for every $A \subset Y$.

5) $N(\bar{A}) \subseteq \overline{N(A)}$ for every set $A \subset X$.

Definition 2.55 If a multimap N is both u.s.c and l.s.c, then it is said to be continuous.

Definition 2.56 Let the set-valued mappings $N: \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$. A single-valued map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfying $f(x) \in N(x)$ for all $x \in \mathbb{R}^n$ is called a selection.

Remark 2.57 If all sets $N(x)$ for $x \in X$ are singletons then N can be seen as a usual single valued mapping, the restriction of a multivalued mappings is called selection.

Theorem 2.58 (Kuratowski, Ryll-Nardzewski) Let $(\mathcal{A}, \sigma)$ be a measurable space and (X, d) be a complete separable metric space and $N: \mathcal{A} \to \mathcal{P}_{\mathrm{cl}}(\mathrm{X})$ be weak measurable. Then, N has a measurable selection.

Theorem 2.59 Let X be a Banach space and N be an upper Carathéodory multivalued mapping. Let $\Phi: L_1([0,b],X) \to C([0,b],X)$ be a linear continuous mapping. Then, the composition

$\Phi \circ \mathrm{S_N}: \mathrm{C}([0,b],X) \to \mathcal{P}_{cl,c}([0,b],X)$, is a closed graph mapping in $C([0,b],X) \times C([0,b],X)$. For details see [48].

It is clear from Definition 2.56 that a selection mapping always exists. But finding a continuous selection of a multivalued mapping is more interesting.

Theorem 2.60 Let $N: X \to \mathcal{P}(Y)$ be a multivalued mapping. Suppose that for every $x \in X$ and $y \in N(x)$, there exist a neighborhood $V = V(x)$ and a continuous selection f of the restriction $N /_V$, with $f(x) = y$. Then, N is l.s.c.

Definition 2.61 A multivalued map $N: A \subseteq \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a-selectionable if there exists a decreasing sequence of compact valued maps $N_k: A \subseteq \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ with closed graphs such that for any $k = 1, 2, \ldots$ N_k has a continuous selection and $N(x) = \bigcap_{k=1}^{\infty} N_k \subset Y$ for any $x \in A$.

Definition 2.62 A multimap $N: X \to \mathcal{P}(Y)$ is said to be compact, locally compact and quasicompact (respectively) if,

1) for $x \in X$ we have $N(x) = \bigcap_{k=1}^{\infty} N_k(x)$ is relatively compact in Y;

2) every point $x \in X$ has a neighborhood $V(x)$ such that the restriction of N to $V(x)$ is compact;

3) its restriction to any compact subset $A \subseteq X$ is compact.

Remark 2.63 It is clear that $1 \Rightarrow 2 \Rightarrow 3$ but the inverse is not always true.

Definition 2.64 A mapping $F: J \times C(J,X) \longrightarrow \mathcal{P}_{cp,cv}(X)$ is said to be an upper Carathéodory multivalued map if it satisfies,

- $x \longrightarrow F(t,x)$ is an upper semicontinuous (with respect to the metric H_d) for almost all $t \in J$.
- $t \longrightarrow F(t,x)$ is measurable for each $x \in C(J,X)$.

Definition 2.65 A multi-valued operator $N: X \longrightarrow \mathcal{P}(Y)$ is called Lipchitz if for any $x, y \in X$ we have,

$H_d(Nx, Ny) \leq k d(x,y)$ where $k > 0$.

If $0 \leq k < 1$, then N is called a contraction mapping.

Moreover, if for every $u \in X$ there is an open neighborhood V_u of u in X and $k_u > 0$ such that,

$H_d(Nx, Ny) \leq k_u d(x, y)$ for each $x, y \in V_u$.

Then, N is called locally Lipchitz.

Lemma 2.66

- Let (Y, δ) be a complete metric space and $N: X \to \mathcal{P}(Y)$ be a Lipchitz (or locally Lipchitz) with closed graph, then N is u.s.c.
- If N is u.s.c. and D is closed, then N has a closed graph i.e., $x_n \to x$ and $y_n \to y$ such that $y_n \in N(x_n)$ implies $y \in N(x)$.
- If $N(D)$ is compact and D is closed, then N is u.s.c. iff N has a closed graph.

Definition 2.67 A differential inclusion is given by $x'(t) \in N\big(t, x(t)\big)$ where N is a multivalued mapping.

Differential inclusions serve as models for many dynamical systems. Obviously, any process described by an ordinary differential equation $x'(t) = f(t, x(t))$ can be described by a differential inclusion of the form $x'(t) \in \{f(t, x(t))\}$. They are also used to study ordinary differential equations with an inaccurately known right-hand side. In this case, one can solve this type of equations by using the following differential inclusions

$x'(t) \in f\big(t, x(t)\big) + \varepsilon B_n$ where B_n is the ball centered in 0.

They also model the performance of various mechanical and electrical devices as well as the behavior of automatic control systems. For more details on differential inclusions see [46-50].

2.5 Measure of Noncompactness

The degree of noncompactness of a set is measured by means of functions called MNC. In this section, we study the main and most common used MNC's. The first MNC was defined and studied by Kuratowski [31] while he was trying to generalize the Cantor's intersection. Its importance came after that the mathematician Darbo [30] used it to generalize his fixed point theorem which extended two classical theorems: Schauder fixed point principle and Banach's contraction mapping principle for so-called condensing operators. The Hausdorff MNC was introduced by Goldenstein et al. [60] in 1957 and later studied by Goldenstein and Markus [61].

Apart from the mentioned MNC's, an axiomatic approach is also given to this concept. Such an approach helps to define useful and handy MNC's in several Banach spaces. MNC's are very useful tools in Banach spaces. They are frequently used in fixed point theory, differential equations, functional equations, integral and integrodifferential equations, optimization, etc.

This section surveys the most important basic properties of MNC's on bounded sets of complete metric spaces and Banach spaces. In the next chapters, we will demonstrate how the theory of MNC's can be applied in fixed point theory and the theory of differential and integral equations.

Definition 2.68 Let (X, d) be a metric space and A be a bounded subset of X. The Kuratowski MNC is given by the expression

$\alpha(A) = \inf\{\varepsilon > 0: A\ can\ be\ covered\ by\ a\ finite\ number\ of\ sets\ of\ diamter \leq\ \varepsilon\}$.

This function has been introduced by Kuratowski [31] in 1930. Since it is not always easy to calculate the diameter of an arbitrary set, another MNC more convenient and useful was introduced and instead of sets balls were used.

Definition 2.69 The Hausdorff MNC is given by

$\beta(A) = inf\{r > 0: A\ is\ covered\ by\ a\ finite\ number\ of\ balls\ with\ diameter\ \leq \mathrm{r}\}$.

Remarks 2.70 The diameter of a set A is the number $sup\{d(x, y): x, y \in A\}$ denoted by $diam(A)$. It is clear that,

$0 \leq \alpha(A) \leq diam(A) < +\infty$.

If $diam(A) = 0$, then $\alpha(A) = 0$. But we know that $diam(A) = 0$ iff A is an empty set or consists of exactly one point and these both cases correspond to $\alpha(A) = 0$.

From the above definitions, one can see that loosely speaking, a MNC is the smallest distance between the set A and the compact sets containing A. In other words, the smaller $\mu(A)$, the closer is A to be precompact.

The relation between Kuratowski and Hausdorff MNC's is given by the following inequalities

$\beta(A) \leq \alpha(A) \leq 2\beta(A)$.

In some Banach spaces, having good geometrical structure, the above inequalities may be even strengthened. For example in Hilbert space, the following inequalities are given

$\sqrt{2}\beta(A) \leq \alpha(A) \leq 2\beta(A)$.

From these definitions we can obtain following facts:

1) $\mu(A) = 0 \Leftrightarrow A$ is a precompact set;

2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

3) $\mu(A) = \mu(\bar{A}), \ \forall A \in \mathcal{P}_b(X)$;

4) $\mu(A \cup B) = max\big(\mu(A), \mu(B)\big), \forall A, B \in \mathcal{P}_b(X)$;

5) $\mu(A \cap B) = inf\big(\mu(A), \mu(B)\big), \forall A, B \in \mathcal{P}_b(X)$.

Remark 2.71 The family of sets A that satisfies $\mu(A) = 0$ is called the kernel of the MNC and noted by $\ker \mu$.

The MNC was used by Kuratowski to prove the following generalization of the well known Cantor's intersection theorem.

Theorem 2.72 Let (X, d) be a complete metric space and (A_n) be a decreasing sequence of nonempty, closed subsets of $\mathcal{P}_b(X)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.

Observe that the intersection set A_∞ defined in the above theorem is a member of the kernel $\ker \mu$. In fact, since $\mu(A_\infty) \leq \mu(A_n)$ for any $n \in \mathbb{N}$ and we have $\mu(A_\infty) = 0$.

Remark 2.73 There are some other MNC's which were investigated for example by Istratescu and Dane. Those are less regular since they lack some of the properties aforementioned.

However, the best way of dealing with MNC's is the axiomatic approach. Obviously, there is a possibility of using several not necessarily equivalent, systems of axioms. Also, measures with real values or the measures taking their values in some abstract ordered sets can be considered. In this way, one can get wider classes of measures. But it is better when it satisfies two requirements: having natural realizations and providing useful tools or applications.

In this work, the following axiomatic definition of the MNC is accepted.

Definition 2.74 Let *X* be a normed space and $\mathcal{P}_b(X)$ the family of bounded subset of *X*. A map $\mu: \mathcal{P}_b(X) \to [0, \infty[$ is called MNC defined on *X* if it satisfies the following:

1) $\mu(A) = 0 \Leftrightarrow A$ is a precompact set;

2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

3) $\mu(A) = \mu(\bar{A}), \ \forall A \in \mathcal{P}_b(X)$;

4) $\mu(A \cup B) = max\big(\mu(A), \mu(B)\big), \forall A, B \in \mathcal{P}_b(X)$;

5) $\mu(A \cap B) = inf\big(\mu(A), \mu(B)\big), \forall A, B \in \mathcal{P}_b(X)$;

6) $\mu(A) = \mu(convA), \ \forall A \in \mathcal{P}_b(X)$;

7) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B), \ for\ \lambda \in (0,1)$;

8) Let (A_n) be a decreasing sequence of nonempty, closed subsets of $\mathcal{P}_b(X)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.

Theorem 2.75 Each MNC is locally Lipschitzian (hence continuous) with respect to the Hausdorff distance, and we have

$|\mu(A) - \mu(B)| \leq \mu\big(B(0,1)\big)H_d(X, Y).$

Lemma 2.76 If $f \subset L^1([0,T], X)$ is an equicontinuous function, then

$\mu\left(\int_0^t f(s,A)ds\right) \leq 2\int_0^t \mu\big(f(s,A)\big)ds, \ \ t \in [0,T].$

Remark 2.77 It is possible to construct new MNC's where compactness criteria are defined.

For example, let the following criterion.

Lemma 2.78 Let X be a Banach space with a Schauder basis $(a_n)_n$. Then, a bounded subset A of X is relatively compact iff for any $x \in A$ $\lim_{n\to\infty}\|R_n\| = 0$ where R_n is the rest part.

Using this criterion, one can obtain the following MNC's.

Example 2.79 The MNC for sequence spaces c_0, c and ℓ_p for $1 \leq p \leq \infty$ (respectively) are given by the following expressions:

1. $\mu(A) = \lim_{n\to\infty} \sup_{x\in A}(\max_{k\geq n}|x_k|)$.
2. $\mu(A) = \lim_{n\to\infty} \sup_{x\in A}(\max_{k\geq n}|x_k - x|)$.
3. $\mu(A) = \lim_{n\to\infty} \sup_{x\in A}(\sum_{k\geq n}^{\infty}|x_k|^p)^{\frac{1}{p}}$.

Remark 2.80 Lemma 2.78 cannot be applied to space ℓ_∞ since it has no Schauder basis.

Example 2.81 Let $C[a,b]$ the space of continuous functions defined on $[a,b]$. Then,

$$\mu(A) = \frac{1}{2}\lim_{\delta\to 0} \sup\nolimits_{x\in A}\left(\max_{0\le r\le\delta}|x_r - x|\right),$$

where x_r denotes the *r* translate of the function *x* i.e.

$$x_r = \begin{cases} x(r+t), & a \le t \le b-r \\ x(b), & b-r \le t \le b \end{cases}.$$

Example 2.82 Let for $1 \le p \le \infty$, $L_p[a,b]$ the space of equivalence classes x of measurable functions $f: [a,b] \to \mathbb{R}$ which are *p*-Bohner integrable, endowed with the norm,

$$\|x\|_p = \left(\int_a^b |x(t)|^p dt\right)^{\frac{1}{p}}.$$

Then, one of the MNC's of this space is given by

$$\mu(A) = \frac{1}{2}\lim_{\delta\to 0} \sup\nolimits_{x\in A}\left(\max_{0\le h\le\delta}\|x_h - x\|_p\right),$$

where,

$$x_h(t) = \frac{1}{2h}\int_{t-h}^{t+h} x(s)ds.$$

Moreover, other MNC's can be constructed using the following compactness criterion.

Lemma 2.83 (Ascoli-Arzela) Let *A* be a bounded subset of a Banach space *X*, then *A* is relatively compact iff all functions belonging to *A* are equicontinuous.

Example 2.84 Denote the diameter of the set *A* by

$diam(A(t)) = sup\{|x(t) - x(t)|: x, y \in A\}$, where $A(t) = \{x(t): x \in A\}$.

And let $x \in A$ and $\varepsilon > 0$,

$\omega^T(x,\varepsilon) = sup\{|x(t) - x(s)|: t, s \in [0,T], |t-s| \le \varepsilon\}$ for $T > 0$,

is the modulus of continuity of *x* on $[0,T]$.

Then, the MNC on $BC(\mathbb{R}^+)$ for a positive fixed *t* and $A \in \mathcal{P}_b(BC(\mathbb{R}^+))$ is defined as follows:

$\mu(A) = \omega_0(A) + \lim\sup_{t\to\infty} diam\big(A(t)\big),$

where,

$\omega_0(A) = \lim_{T\to\infty}\{\lim_{\varepsilon\to 0}\sup\{\omega^T(x,\varepsilon): x \in A\}\}.$

Remark 2.85 There is also another compactness criterion called Kolmogorov criterion which is used for function spaces L^p.

It is also possible to define MNC's in some spaces of differentiable functions.

Example 2.86 Let $C^n(a,b)$ be the space of n-times continuously differentiable real functions on (a,b) with an arbitrary norm generating the uniform convergence of the derivatives up to the order n. Let A be a subset of $C^n(a,b)$, then

$$\|A^{(n)}\| = \sum_{i=0}^{n}\|A^{(i)}\| = \sum_{i=0}^{n}\max\{\|A^{(i)}(t)\|: t \in (a,b)\},$$

with $A^{(i)} = [x^{(i)}: x \in A]$ for $i = 0,\dots,n$ where $x(0) = x$.

Then, the MNC of the space $C^n(a,b)$ is given by,

$\mu_{C^n}(A) = \mu\big(A^{(n)}\big).$

The following example shows that we can also define a MNC in product spaces.

Example 2.87 Let μ_1, μ_2, …, μ_n be MNC's in Banach spaces E_1, E_2, …, E_n (resp.). Then the function,

$\tilde{\mu}(A) = N\big(\mu_1(A_1), \mu_2(A_2), \dots, \mu_n(A_n)\big),$

defines a MNC in $E_1 \times E_2 \times \ldots \times E_n$ where A_i is the natural projection of A on E_i for $i = 1,2,\dots,n$ and N be a convex function defined by $N: [0,\infty[^n \to [0,\infty[$ such that

$N(x_1, x_2, \dots, x_n) = 0 \Leftrightarrow x_i = 0, for\ i = 1,2,\dots,n.$

Remark 2.88 Researchers attempt to define new MNC's but it is not obvious to define them for spaces that do not have compactness criteria.

Example 2.89 A MNC in a Frechet space is a functional $\mu: \mathcal{P}_b(F) \longrightarrow \mathbb{R}^+$ such that

$$\mu(A) = \inf\{\varepsilon > 0: A\ is\ the\ finite\ union\ of\ A_i:\ \ sup_{x,y\in A_i}\ \{p(x-y)\} \le \varepsilon, \forall i\},$$

where $\mathcal{P}_b(F)$ is the family of all bounded subsets of F which its topology is defined by the family $\mathcal{P}$ of all continuous semi-norms p on F.

A local base of closed 0-neighborhoods of F is formed by the sets $\{x \in F: \max_{1\leq i\leq n} p_i(x) \leq \varepsilon\}$,

where ε is a positive constant and $p_i \in \mathcal{P}$.

Lemma 2.90 A MNC μ in a Frechet space F satisfies the following properties:

1) $\mu(A) = 0 \iff A$ is a precompact set;

2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

3) $\mu(A) = \mu(\bar{A}),\ \forall A \in \mathcal{P}_b(F)$;

4) $\mu(A) = \mu(convA),\ \forall A \in \mathcal{P}_b(F)$;

5) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B),\ \ for\ \lambda \in (0,1)$;

6) Let (A_n) be a decreasing sequence of nonempty, closed subsets of $\mathcal{P}_b(F)$ such that $\lim_{n\to\infty} \mu(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.

For more details on the theory of MNC and its examples see [37-42].

2.5.1 Measure of Weak Non-compactness

WMNC's are expressions that quantify different characterizations of weak compactness in Banach spaces. It was introduced by De Blasi in the paper [62] and was applied successfully to nonlinear functional analysis to operator theory and to the theory of differential and integral equations.

Definition 2.91 A functional $\omega: \mathcal{P}_b(X) \longrightarrow \mathbb{R}^+$ such that

$\omega = \inf\{\varepsilon > 0: there\ exists\ W \in \mathcal{P}_b(X)\ with\ A \subseteq W + B_\varepsilon\}$

is called WMNC.

Lemma 2.92 A WMNC ω satisfies the following properties:

1) $\omega(A) = 0 \iff A$ is weakly precompact set;

2) $A \subset B \Rightarrow \omega(A) \leq \omega(B)$;

3) $\omega(A) = \omega(\overline{A^w}),\ \forall A \in \mathcal{P}_b(X)$, where $\overline{A^w}$ denotes the weak closure of A;

4) $\omega(A) = \omega(convA),\ \forall A \in \mathcal{P}_b(X)$;

5) $\omega(\lambda A+(1-\lambda)B) \leq \lambda\omega(A)+(1-\lambda)\omega(B),\ \ for\ \lambda \in (0,1)$;

6) Let (A_n) be a decreasing sequence of nonempty, weakly closed subsets of $\mathcal{P}_b(X)$ such that $\lim_{n\to\infty} \omega(A_n) = 0$. Then, the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and relatively weakly compact.

Remark 2.93 The concept of WMNC is more general than MNC. However, it is rather difficult to express De Blasi WMNC by a convenient formula. Up to now the only formula of this type was obtained by Appell and De Pascale [63] in the Lebesgue space $L^1(\Omega, X)$,

$\omega(A) = \lim\sup_{\delta\to 0}\{\sup_{\varphi\in A}[\int_D \|\varphi(t)\|_X :\ LM(D) \leq \varepsilon]\}$,

for every bounded subset A of $L^1(\Omega, X)$ where X is finite dimensional Banach space and $LM(D)$ is the Lebesgue measure of D.

2.5.2 Condensing Mappings

This subsection is principal in our treatment of the theory of condensing single or multivalued maps.

Definition 2.94 Let A be a closed subset of a Banach space X and N be a single / multivalued self-mapping defined on A. N is said to be a k-set contraction mapping if for every A, there exists a positive constant k such that

$\mu(NA) \leq k\mu(A)$.

If $0 \leq k \leq 1$, then N is said to be a k-set contraction.

In further, N is said to be a condensing map if we have

$\mu(NA) < \mu(A)$.

Example 2.95 We consider the following examples of condensing mappings:

1) Let N and H be two set-contraction mappings,

2) Or, N be a compact map and H be a set-contraction map.

Then, in both cases the family of mappings

$F(\lambda, x) = (1-\lambda)N(x) + \lambda H(x),$

is a family of single / multivalued k-set contraction mappings.

CHAPTER 3

A SURVEY ON MOST IMPORTANT THEOREMS OF FIXED POINT THEORY

In this chapter, an evolution of the fixed point theory is presented to form a brief survey of the theory from the beginning up to now. It is divided into two sections. The first one is about the theory dealing with single valued mappings. The second section, deals with the theory for multivalued mappings.

3.1 Fixed Point Theorems For Single Valued Mappings

Fixed point theory was extended in various directions, classical instruments were generalized, new notions and results have been given and constantly improved. In addition, they are important tools applied in other domains with no immediate connection with mathematics at first glance. Many stability and equilibrium problems can be modeled using fixed points. Such examples can be found in economics, game theory, compiler theory and many others.

In this section, the most important theorems and related generalizations and applications for single valued mappings are stated in chronological order.

3.1.1 Poincaré

"If one looks at the different problems of the integral calculus which arise naturally when one wishes to go deep into the different parts of physics, it is impossible not to be struck by the analogies existing".

H. Poincaré

Poincaré stated his theorem only for two dimensions spaces but later in 1926, the German mathematician Heinz Hopf generalized the theorem to higher dimensions. This theorem is also referred as the Bolzano-Poincare-Miranda theorem because in this result

Poincaré generalized the theorem of intermediate value of Bolzano and Miranda was who proved this result is equivalent to the Brouwer's theorem.

Theorem 3.1 Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and suppose that $|x_i| \leq a_i$, for some prescribed set of reals $a_i > 0$, $i = \overline{1,n}$ such that on each face $x_i = a_i$, we have $f_i(x) > 0$ and for $x_i = -a_i$, we have $f_i(x) < 0$. Then there exists x such that $f(x) = 0$.

The main application of this theorem is the hairy ball theorem.

Theorem 3.2 A continuous vector field on a spherical surface has at least one zero.

Application 3.3 One can daily see the application the hairy ball theorem when trying to comb their hair (it is more clearly seen for short hair) and find a persistent whorl at the top of their heads or another example "You can't comb the hair on a coconut", the hairy ball theorem states that it is impossible to comb a spherical ball covered in hair so that there are no whorls.

Figure 3.1 Illustration of hairy ball theorem

This event is mathematically explained as following:

Let N be a surface of $\mathbb{R}^3$, we know that for every point p of N there is a plane that is tangent to N at p. If we consider the mathematical equivalent of the head the sphere and the hair as tangent vectors, then for any two points p and q, let the tangent vectors V_p and V_q of N at p and q respectively. Imagine that p and q are very close to each other, the tangent vectors V_p and V_q also approach each other in length and the angle between them approaches zero. Denote V the collection of the tangent vectors on N, then V forms a continuous vector field on N. If at some point p on N we have $V_p = 0$, then p is a zero of V and in this case if we consider the $\mathbb{R}^3$ to be the head and V the hair then p is the whorl.

Application 3.4 Another interesting application states that "somewhere on the surface of the earth, there is a point with zero horizontal wind velocity".

Mathematics explanation: In this case we define the wind at every point over the earth as a continuous vector field, so the hairy ball theorem states that there must be at least one point on a planet at all times with no wind at all which is can be clearly seen in the eye of cyclones or anticyclones.

It is worthwhile to mention that saying there is no wind at all is physically unrealistic (there is always wind) since the air above the earth has multiple layers but for each layer, there must be a point with zero horizontal wind speed.

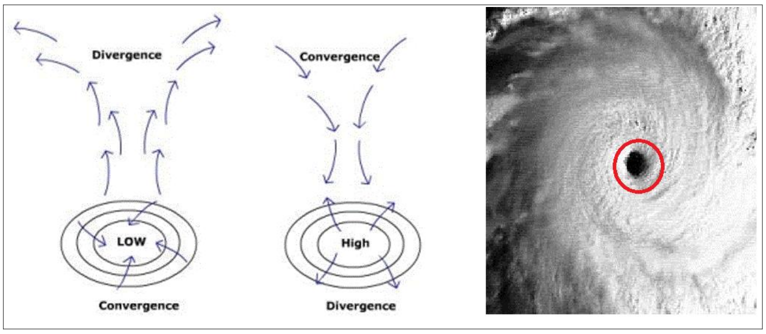

Figure 3.2 Illustration of hairy ball theorem on wind velocity

3.1.2 Brouwer

"One cannot inquire into the foundations and nature of mathematics without delving into the question of the operations by which the mathematical activity of the mind is conducted. If one failed to take that into account, then one would be left studying only the language in which mathematics is represented rather than the essence of mathematics". L. E. J. Brouwer

Theorem 3.5 Every continuous mapping from an n-dimensional disk to itself contains a fixed point. In other words, there is always a point that remains invariant under a continuous map from an n-dimensional disk to itself.

The generalizations of Brouwer's theorem have appeared in relation to the theory of topological vector spaces in mathematical analysis: the compactness, convexity, single-valuedness, continuity, self-mapiness, etc ([64] and [65]).

Example 3.6 As we mentioned before, Brouwer's theorem says that if you take a cup of coffee, and slosh it around, then after the sloshing there must be some point in the coffee which is in the exact spot that it was before you did the sloshing (though it might have moved around in between). Moreover, if you tried to slosh that point out of its original position, you can't help but slosh another point back into its original position!

Figure 3.3 Illustration of Brouwer’s theorem on cup of coffee

Application 3.7 Another application if you take two sheets of paper, one lying directly above the other. If you crumple the top sheet and place it on top of the other sheet, then Brouwer's theorem says that there must be at least one point on the top sheet that is directly above the corresponding point on the bottom sheet.

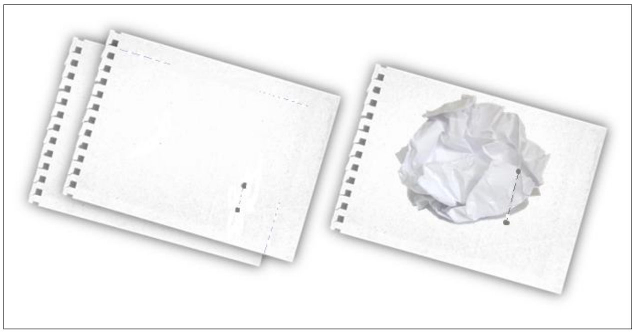

Figure 3.4 Illustration of Brouwer’s theorem with sheets of paper

This example is better than the coffee cup one as it shows that uniqueness of the fixed point may fail.

Example 3.8 Another example, if you take a map for example of Turkey. By shrinking the map and placing it directly on the original map, then Brouwer’s theorem says that there must be at least one point on the shrinking map that beneath a point of the original map.

Figure 3.5 Illustration of Brouwer’s theorem using maps

3.1.2.1 Application on Differential Equations

Brouwer’s theorem can guarantee the existence of a solution (but not its uniqueness) of many nonlinear equations but only when the equation is posed in some finite dimensional vector space and even establishes a priori estimate of the size of the solution by precising the ball that containing all the solutions.

The most important example of this situation is the stationary Navier-Stokes equation, with Dirichlet condition $u = 0$ on the boundary of the domain. Firstly, one needs to construct an operator Φ such that their fixed points consist the solutions of our equation. Secondly, since the space is infinite dimensional, one establishes the existence of an approximate solution in a subspace of dimension *n* (Galerkin procedure) and by using Brouwer's theorem one can ensure the existence of a point u such that $\Phi\mathrm{u} = \mathrm{u}$ and the following theorem is given.

Theorem 3.10 Let $f: \bar{B}_n \times \mathbb{R} \to \mathbb{R}^n$ be a mapping with continuous derivative at any point $(x, t) \in \bar{B}_n \times \mathbb{R}$ and *T*-periodic in the second variable. Consider the differential equation

$$x'(t) = f(x(t), t), \qquad (3.1)$$

where *x* takes values in $\bar{B}_n$. If *f* satisfies the boundary condition $f(x, t).\, x < 0$ for every $x \in S_n(0,1)$ and any $t \in \mathbb{R}$, then the equation (3.1) has a periodic solution of period *T*.

3.1.2.2 Application in Economy and Game Theory

In 1932, John von Neumann, using the Brouwer's fixed point theorem as his main analytical tool, was the first who discussed a theory of economic processes that established the existence of the best techniques of production to achieve maximum outputs of all goods at the lowest possible prices with the outputs growing at the highest possible rates. Known in economic circles as general equilibrium theory, the theory states that there always exists a set of prices at which supply equals demand for all goods, a result whose only known proof comes from showing that these prices are the fixed points of a particular transformation. This is a consequence of the fact that one can regard the prices as elements in a vector space, each of whose entries is a nonnegative real number. The set of all such vectors having n elements constitutes a topological space and under reasonable conditions the price-setting mechanism in the economy is a continuous transformation of that space to itself, that is, it moves prices from one point in the space to another.

It is used to prove the existence of general equilibrium in market economies, for which Kenneth Arrow and Gérard Debreu received the Nobel Prize in Economics in 1972, 1983 respectively.

It was also used in the original proof of the Nash Equilibria. His idea was if you imagine that each player is trying to improve his results based on the current actions of his opponents, then there is some combination of strategies so that each player can't improve. It won John Forbes Nash the Nobel Prize in Economics in 1994.

3.1.2.3 Application in Computer Sciences

Imagine a huge library containing billions of documents with different formats, no centralized organization and no librarians. Even worse, anyone can add a document at any time without informing anyone else.

Under these circumstances how can one hope to quickly find a specific document in this library?

Actually, something similar happens whenever one visits the World Wide Web and posed in this way, the problem seems without solutions. But this is exactly what happens continuously on the Internet. Using one of the search engines (such as Google, Yahoo, etc) one finds the sought after document in some seconds. Back in the 1990's when the first search engines were designed, they used text based systems to decide which web pages were the most relevant to the query. Today's search engines work differently. Loosely speaking, to find a page on the web, a system $Nx = x$ where x stands for a Pagerank and N is some web matrix that needs to be solved (for more details see [21]).

3.1.3 Banach Contraction Theorem

Although the basic idea of the Banach contraction theorem was known to others earlier, the principle first appeared in explicit form in Banach's 1922 thesis. But on the contrary of the other fixed point theorems which give no information about the uniqueness and location of fixed points the Banach contraction not only ensures the uniqueness but also present a constructive method to approach the fixed point.

Theorem 3.11 Let (X, d) be a complete metric space and $N: X \to X$ be a map such that $d(Nx, Ny) \leq kd(x, y)$,

for some $0 \leq k < 1$ and all $x, y \in X$. Then, N has a unique fixed point in X.

Moreover, for any $x_0 \in X$ the sequence of iterates $x_{n+1} = N(x_n) = N^n(x_0)$ converges to the fixed point of N.

3.1.3.1 Generalizations

The first generalization in this direction which received a significant importance is the following result of Rakotch [66].

Theorem 3.12 Let (X, d) be a complete metric space and suppose that $N: X \to X$ satisfies:

$$d(Nx, Ny) \leq \eta\big(d(x, y)\big)d(x, y), \ for\ all\ x, y \in X.$$

where η is a decreasing function on $\mathbb{R}^n$ to $[0,1[$. Then, N has a unique fixed point.

Remark 3.13 Geraghty [67] proved the same theorem but replaced the condition on η by the simple condition that $\eta(t_n) \to 1 \Rightarrow t_n \to 0$.

Then, Boyd and Wong presented a more general result in [68] as follows:

Theorem 3.14 Let (X, d) be a complete metric space and suppose that $N: X \longrightarrow X$ satisfies:

$d(Nx, Ny) \leq \Psi\big(d(x, y)\big)$ for all $x, y \in X$,

where $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ is upper semi-continuous from the right, that is, for any sequence $t_n \to t \geq 0 \Rightarrow \limsup_{n\to\infty} \Psi(t_n) \leq \Psi(t)$ for $t > 0$.

Then, N has a unique fixed point.

Theorem 3.15 Let (X, d) be a complete metric space and suppose that $N: X \longrightarrow X$ satisfies:

$d(Nx, Ny) \leq \Psi\big(d(x, y)\big)$ for all $x, y \in X$,

where $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and satisfies $\lim_{n\to\infty} \Psi(t_n) = 0$ for all $t > 0$.

Then, N has a unique fixed point.

Another important extension of the Banach contraction theorem is due to Meir and Keeler where they introduced a new class of mappings.

Theorem 3.16 Let (X, d) be a complete metric space and suppose that $N: X \longrightarrow X$ satisfies the condition: for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Nx, Ny) < \varepsilon$.

Then, N has a unique fixed point.

The following theorem is also a generalization of Banach contraction principle obtained by Kirk [69] using the Cantor's intersection theorem.

Theorem 3.17 Let (X, d) be a complete metric space, and suppose that $N: X \longrightarrow X$ satisfies:

$d(N^n x, N^n y) \leq \Psi_n\big(d(x, y)\big)$ for all $x, y \in X$,

where $\Psi_n: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous such that $\Psi_n(t) < t$ for all $t > 0$ and satisfy $\Psi_n \to \Psi$ uniformly.

Then, N has a unique fixed point.

Another wonderful generalization of the Banach contraction theorem is due to Caristi [70] given as follows.

Theorem 3.18 Let (X, d) be a complete metric space, and suppose that $N: X \to X$ satisfies:

$d(x, Nx) \leq \phi(x) - \phi(Nx)$,

where $\phi: X \to \mathbb{R}_+$ is a lower semi-continuous function.

Then, *N* has a unique fixed point.

There are many generalizations beyond those mentioned here most of them are gathered in [71].

3.1.3.2 Applications

Banach used his theorem to establish the existence of a solution to an integral equation. One of the more well-known uses of the method of successive approximations is Picard's original proof of the existence and uniqueness of solutions to ordinary differential equations. Despite the modern presentations which almost always pass through Banach's fixed point theorem, Picard "brute-forced" the construction. (Lindelof's improvement on Picard's original theorem was published in 1894, 25 years before Banach even formulated his fixed-point theorem). The Picard-Lindelöf theorem is about the existence and uniqueness of solutions to certain ordinary differential equations. The sought solution of the differential equation is expressed as a fixed point of a suitable integral operator which transforms continuous functions into continuous functions. The Banach fixed-point theorem is then used to show that this integral operator has a unique fixed point.

The iteration $x_{n+1} = Nx_n$ is used to solve equations of the form $g(x) = 0$ (numerically) and we have at least the following possibilities

1) Simplest version : $Nx = x - g(x)$;
2) Linear relaxation : $Nx = x - \omega g(x)$;
3) Nonlinear relaxation : $Nx = x - \omega F(f(x))$;
4) Newton's method $Nx = x - \frac{g(x)}{g'(x)}$;
5) Splitting method $Nx = h^{-1}(g(x) + f(x))$ where $g(x) = h(x) + f(x)$;

Here ω denotes a real, nonzero parameter, while *F*, g and *h* are suitable functions. For example, the iteration corresponding to the simplest version is

$x_{n+1} = x_n - g(x_n)$.

These iterations are explained in details in Zeidler's book [72].

Remark 3.19 There are a large amount of literature that deal with ameliorating and extending those iterations (Picard [18] Mann [73], Krasnosel'skii (1955), Ishikawa [74], Noor [75], Karakaya et al. [76]).

Example 3.20 Consider the metric space $X = C[a, b]$ of continuous real-valued functions defined on the compact interval $[a, b]$ endowed with the metric $d(x, y) = sup|x - y|$. Define $N: X \to X$ such that,

$$Nx(t) = \int_a^t x(s)ds.$$

Then,

$$d(Nx, Ny) = sup|Nx - Ny|$$

$$= sup\left|\int_a^t x(s)ds - \int_a^t y(s)ds\right|$$

$$\leq \int_a^t sup|x(s) - y(s)|ds \leq \int_a^t sup|x(s) - y(s)|ds$$

$$\leq (b - a)d(x, y).$$

In further, we compute

$$N^2x(t) = \int_a^t (t - s)x(s)ds.$$

Then,

$$N^n x(t) = \frac{1}{(n-1)!}\int_a^t (t - s)^{n-1}x(s)ds.$$

Hence,

$$d(N^n x, N^n y) = sup|N^n x - N^n y|$$

$$= \frac{1}{(n-1)!} sup\left|\int_a^t (t - s)^{n-1}[x(s) - y(s)]ds\right|$$

$$\leq \frac{(b-a)^n}{n!} sup|x(s) - y(s)|$$

$$\leq \frac{(b-a)^n}{n!} d(x, y).$$

Finally, we conclude that for $\frac{(b-a)^n}{n!} < 1$ N has a unique fixed point p such that

$$p = \lim_{n\to\infty} N^n x = \lim_{n\to\infty} \frac{1}{(n-1)!}\int_a^t (t - s)^{n-1}x(s)ds = 0.$$

3.1.4 Schauder

Example 3.21 Let B be the unit ball in the infinite dimensional Hilbert space ℓ_2. That is, $B = \{x \in \ell_2 : \|x\|_2 \leq 1\}$ and $N: B \longrightarrow B$ is given by

$$Nx = x.\sqrt{1 - \|x\|_2}.$$

It is clear that N is a self mapping, since for any $x \in B$,

$$\|Nx\|_2 = \left\|x.\sqrt{1 - \|x\|_2}\right\|_2 = \sqrt{1 - \|x\|_2}\|x\|_2 \leq 1.$$

In further, N is continuous. Indeed,

$$\|Nx\|_2 \leq \sqrt{1 - \|x\|_2}\|x\|_2.$$

Therefore, $\|Nx\|_2 \leq M\|x\|_2$ where $M = \sqrt{1 - \|x\|_2}$.

However, N has no fixed point because if it there is an x such that $Nx = x$, then $x.\sqrt{1 - \|x\|_2^2} = x$.

Hence, N is a continuous self mapping on the closed bounded ball B but N has no fixed point.

From this example, it is clearly that Brouwer's theorem is not true in infinite dimensional spaces. The first fixed point theorem in infinite dimensional Banach space was due to Schauder [4] in 1930. There exist two version of the theorem as given below.

Theorem 3.22 Let X be a Banach space and A be a nonempty compact and convex subset of X. Then, any continuous mapping $N: A \to A$ has at least one fixed point.

Theorem 3.23 Let X be a Banach space and A be a nonempty closed bounded and convex subset of X. Suppose $N: A \to A$ is a continuous and compact mapping, then N has at least one fixed point.

3.1.4.1 Generalizations

In the following, we cite the most important generalizations of Schauder theorem.

In 1934, Tychonoff proved the following theorem.

Theorem 3.24 (Schauder—Tychonoff): Let A be a compact convex subset of a locally convex space X. Then, every continuous mapping $N: A \to A$ has a fixed point.

In 2006, Latrach et all [36] have extended the Schauder fixed point theorem to the following theorem.

Theorem 3.25 Let *A* be a weakly relatively compact set under the supposition that for every weakly convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(N)$ in *X*, the sequence $N(x_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in *X*. Then, *N* has at least one fixed point.

3.1.4.2 Applications

The Schauder theorem is a very popular tool in proving the existence of integral and differential equations since it is valid in infinite dimensional spaces so one does not need to find an approximate solution in a subspace of *n*-dimension.

The following theorem is a very important application of the Schauder fixed point theorem.

Theorem 3.26 (Leray-Schauder alternative) Let Ω be a convex subset of a normed linear space X and let U be an open subset of A with $p \in U$. Then, every compact, continuous map $N: \overline{U} \to \Omega$ satisfies one of the following properties:

- *N* has a fixed point,
- or there is an $x \in U$ with $x = (1-\lambda)p + \lambda Nx$ for any $0 < \lambda < 1$.

To solve differential equations usually we study the inverse problem "integral equations" because the integral operators with sufficient regular kernels provide the most important examples of nonlinear compact operators on infinite dimensional Banach spaces.

3.1.5 Darbo

Example 3.27 Let consider the Hilbert space $\ell_2(\mathbb{Z})$ with the standard basis consisting of the sequence $\{e_n : n \in \mathbb{Z}\}$ where $e_n = (0, \dots, 0,1,0, \dots)$ with the one in position *n* and denote by *B* the closed unit ball in this space. For $x \in \ell_2(\mathbb{Z})$ one can write

$$x = (\dots, x^{-1}, x^0, x^2, \dots) = \sum_{n\in\mathbb{Z}} x^n e_{n.}$$

We now define the right shift operator $N: \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ by $Nx = \sum x^n e_{n+1}$.

The relation,

$$x - Nx = \sum_{n\in\mathbb{Z}} (x^n - x^{n-1})e_n = ce_0$$

requires that $x^n = x^0$ for all $n > 0$ and that $x^n = x^{-1}$ for all $n < 0$, for a point of $\ell_2(\mathbb{Z})$ this is possible only if $x^n = x^{-1} = 0$, hence, $x - Nx \neq 0$.

The above example shows that the compactness condition is crucial in fixed point theory. However, it is very powerful hypothesis and we don't know compact spaces beside the cantor set and Hilbert cube. Therefore, by assuming the compactness of the set or the mappings one restrict the number of mappings or sets for which the theory holds. Many researchers were and still seeking a way to weaken the compactness conditions. One way to circumvent this difficulty is to use the MNC. As defined above, the MNC is a functional that measures the distance between a set and its closest compact set, it is big if the set is far from compactness and small if it is close to be compact. It has been very popular and researchers are trying to generalize the aforementioned theorems in virtue of this concept.

Darbo was the first who used the concept of MNC and get over the problem of lack of compactness in fixed point theorems, where he used condensing instead of compact mappings. Roughly speaking, condensing operators are the sum of a compact and contraction mappings (but one should be aware that not all condensing mappings are of this form see [42]). That is remaining to say that the Darbo's fixed point theorem makes a combination of two classical theorems in the fixed point theory: Schauder's fixed point theorem and Banach Contraction Principal.

Theorem 3.28 Let A be a nonempty closed, bounded and convex subset of X. If $N: A \to A$ is a continuous mapping such that

$$\mu(NA) \leq k\mu(A),\ k \in [0,1[,$$

then N has a fixed point in A.

In the following, some generalizations of Darbo's fixed point theorem.

Theorem 3.29 Let A be a nonempty closed bounded and convex subset of X. If $N: A \to A$ is a continuous mapping such that

$$\mu(NA) \leq \Phi\big(\mu(A)\big),\ k \in [0,1[,$$

where $\Phi: [0,\infty[\to [0,\infty[$ is a nondecreasing function such that $\lim_{n\to\infty} \Phi^n(t) = 0$, $\forall\, t > 0$.

Then, N has at least one fixed point.

Theorem 3.30 Let A be a nonempty, bounded, closed, and convex subset of a Banach space X and μ be an arbitrary MNC on X. If $N: A \to A$ is a continuous mapping and for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(A) < \varepsilon + \delta \Rightarrow \mu(NA) < \varepsilon.$$

Then, N has at least one fixed point and the set of all fixed points of N in A is compact.

Theorem 3.31 Let A be a nonempty closed, bounded and convex subset of X. If $N: A \to A$ is a continuous mapping such that

$$\int_0^{\mu(NA)} \varphi(s)ds \leq \Psi\left(\int_0^{\mu(A)} \varphi(s)ds\right),$$

where φ is an integrable and summable function and $\Phi: [0, \infty[\to [0, \infty[$ is a nondecreasing function such that $\lim_{n\to\infty} \Phi^n(t) = 0$.

Then, N has at least one fixed point.

Theorem 3.32 Let X be a Banach space and A be a nonempty, closed, bounded and convex subset of a Banach space X and let $N: A \to A$ be a continuous operator which satisfies the following inequality

$$\Phi\left(\int_0^{\mu(NA)} \varphi(s)ds\right) \leq \Psi\left(\int_0^{\mu(A)} \varphi(s)ds\right),$$

where μ is a MNC and

1) $\Psi: [0, \infty[\to [0, \infty[$ is a nondecreasing and concave function such that $\lim_{n\to\infty} \Psi^n(t) = 0$ for every $t > 0$.

2) $\Phi: [0, \infty[\to [0, \infty[$ is a nondecreasing subadditive function such that $\Phi(t) > t$ and $\lim_{n\to\infty} \Phi^n(x_n) = 0 \Leftrightarrow \lim_{n\to\infty} x_n = 0$.

3) $\varphi: [0, \infty[\to [0, \infty[$ is an integral mapping which is summable on each compact subset of $[0, \infty[$ and for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(s)ds > 0$.

Then, N has at least one fixed point in X.

Darbo's theorem can also be generalized to product spaces as shown in the following results.

Theorem 3.33 Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $\varphi: [0, \infty[\to [0, \infty[$ be a nondecreasing and upper semi-continuous function such that $\varphi(t) < t$ for all $t > 0$. Then for any MNC μ, the continuous operator $G: \Omega \times \Omega \times \Omega \to \Omega$ satisfying

$$\mu\big(G(A_1 \times A_2 \times A_3)\big) \leq \varphi\left(\frac{\mu(A_1) + \mu(A_2) + \mu(A_3)}{3}\right), \qquad A_1, A_2, A_3 \in \Omega,$$

has at least one tripled fixed point.

Theorem 3.33 can be used in the study of the existence of solutions for systems of integral equations define on the Banach space $BC(\mathbb{R}^+)$. For more details see [43].

3.1.6 Tarski

The following example shows that the continuity is a crucial condition to guarantee the existence of fixed points.

Example 3.34 Let $X = [0,1]$ and $Nx = \begin{cases} 1, & x \in [0,1[\\ 0, & x = 1 \end{cases}$. It is easy to see that N has no fixed points.

N does not possess any fixed points since in the discontinuity points N jumps down. However, one can come over this handicap by suggesting N to be weakly increasing. Following this intuition Bronisław Knaster and Alfred Tarski proved there fixed point theorem which generated the discrete fixed point theory.

It was Tarski who stated the result in its most general form and so the theorem is often known as Tarski's fixed point theorem.

Theorem 3.35 Let $(L, \preccurlyeq)$ be any complete lattice. Suppose $N: L \to L$ is an order preserving mapping, then N has a fixed point. In fact, the set of fixed points of N is a complete lattice with respect to $\preccurlyeq$.

Remark 3.36 N has a greatest fixed point $\bar{u}$ and a least fixed point $\underline{u}$. Moreover, for all $x \in L$, $x \preccurlyeq N(x)$ implies $x \preccurlyeq \bar{u}$, whereas $Nx \preccurlyeq x$ implies $x \preccurlyeq \underline{u}$.

Later in the same year, Anne Davis [77] proved the converse of Tarski theorem is also true.

Theorem 3.37 Every lattice with the fixed point property is complete.

The classical Tarski’s fixed point theorem has many applications, in particular, in analysis theory, set theory, game theory, economics and many others. This emphasizes the importance of an order-theoretic fixed point theory.

3.1.6.1 Application in Analysis

Theorem 3.38 Let $x, y \in \mathbb{R}$ with $x \leq y$. Since the closed interval $[a, b]$ is a complete lattice with respect to $\leq$, every monotone increasing function $f: [a, b] \rightarrow [a, b]$ must have a greatest fixed point and a least fixed point. Note here *f* need not to be continuous.

3.1.6.2 Application in Set Theory

One of the main applications is the fundamental Schroder-Cantor-Bernstein theorem.

Theorem 3.39 Let N and H be two sets. If $f: N \longrightarrow H$ and $g: H \longrightarrow N$ are injections, then there is a bijection $h: N \longrightarrow H$.

3.1.6.3 Application to Solve Differential Equations

The following differential equations which are given by

$$x'(t) = F\big(t, x(t)\big),$$

with F being discontinuous are rather unpleasant from the mathematical point of view. However, discontinuous differential equations are faced in many applications. A wide number of problems from mechanics and electrical engineering are modeled to differential equations with discontinuous right-hand sides because many physical laws are expressed by discontinuous functions, like a dry friction force or jump-like transition characteristic of some electronic devices.

Another motivation to consider discontinuous right-hand sides is of a mathematical reason. That is, when the right-hand side is continuous but complicated, it is more comfortable to approximate it by a simple discontinuous function, for example, by a piecewise linear function. This method is frequently used in technical problems. For more details and concrete examples see [50].

To solve this class of differential equation one needs to formulate them to fixed point problems then applies Tarksi's theorem to investigate the existence of fixed points which represent also the solutions of differential problems.

3.1.6.4 Application in Computer Science

There are many other uses to the notion of fixed-points in computer science. Especially when writing a recursive program or model loops, one needs some stop criteria. Such programs contain a body (pack of steps) which is applying repeatedly until no change is possible. Mathematically, one need a mathematical notion of what it means for a function to apply itself and the answer is given by the fixed points. Therefore, proving the existence of some fixed-points helps to show that certain functions or constructs are well-defined in the framework.

3.1.6.5 Application in Game Theory

A useful application of Tarski's fixed point theorem is that every supermodular game (mostly games with strategic complementaries) has a smallest and a largest pure strategy Nash equilibrium. Informally, this means that when one player takes a higher action, the others want to do the same, so increasing a player's strategy raises with increases in the other players' strategies. This is mathematically captured by lattices.

To study supermodular games one need lattice theory and monotonicity results. The methods used are non-topological and they exploit order properties.

To compute Nash equilibrium of a supermodular game, a generic approach is to convert it into the computation of a fixed point of an order preserving mapping. So the Tarski's theorem is the best tool to study the existence problem. We can use it when the best-response correspondences of players have a monotone increasing selection. This monotonicity property is guaranteed for supermodular games.

Supermodular games are interesting for several reasons. Firstly, they encompass many applied models. Secondly, they have the remarkable property that many solution concepts yield the same predictions. Finally, they tend to be analytically appealing - they have nice comparative statics properties and behave well under various learning rules.

Much of the theory is due to Topkis (1979), Vives (1990) and Milgrom and Roberts (1990). The literature is huge. By a slight modification of the theorem, one can actually show that the set of pure strategy Nash equilibria forms a complete lattice in itself.

3.2 Existence Theorems for Multivalued Mappings and Applications

3.2.1 Kakutani

The Kakutani's fixed point theorem [78] was developed by Shizuo Kakutani in 1941 as a generalization of Brouwer's fixed point theorem for set-valued mappings

Theorem 3.40 Let A be a nonempty compact convex subset of $\mathbb{R}^n$, and let $N: A \to \mathcal{P}_{cl,cv}(A)$ be an upper semicontinuous map. Then *N* has a fixed point.

3.2.1.1 Applications in Game Theory

Mathematician John Nash used the Kakutani's fixed point theorem to prove a major result in game theory [79]. Stated informally, the theorem implies the existence of a Nash equilibrium in every finite game with mixed strategies for any number of players. This work later earned him a Nobel Prize in Economics.

3.2.1.2 General Equilibrium

In general equilibrium theory in economics, Kakutani's theorem has been used to prove the existence of set of prices which simultaneously equate supply with demand in all markets of an economy.

3.2.1.3 Fair Cake Cutting

Kakutani's fixed-point theorem was used to prove the existence of cake allocations that are both envy-free and Pareto efficient. This result is known as Weller's theorem.

3.2.2 Bohnenblust-Karlin

In 1950, Kakutani's fixed point theorem which holds only in $\mathbb{R}^n$ was generalized by Bohnenblust and Karlin [80] to more general spaces. Namely, to Banach spaces, they obtained a fixed point theorem similar to the Schauder's theorem.

Theorem 3.41 Let *A* be a nonempty compact convex subset of a Banach space *X*. Let $N: A \to \mathcal{P}_{cl,cv}(A)$ be an upper semicontinuous mapping. Then, N has a fixed point.

In 1972, Himmelberg generalized the second version of Schauder's fixed point theorem to obtain the following theorem.

Theorem 3.42 If A is a nonempty convex subset of a locally convex space X and

$N: A \to \mathcal{P}_{cl,cv,cp}(A)$ is an u.s.c, then *N* has a fixed point.

As an application of the above theorem, let the following classical multi-valued version of the nonlinear alternative of Leray and Schauder.

Theorem 3.43 Let X be a normed space and let $N: A \to \mathcal{P}_{cl,cv}(A)$ be an u.s.c compact multivalued map. Then, either one of the following assumptions holds:

1) N has a fixed point,
2) or the set $W = \{x \in X:\ x \in \lambda N(x)\ \ for\ \lambda \in (0,1)\}$ is unbounded.

As a side note, differential inclusions are not only models for many dynamical processes but they also provide a powerful tool for various branches of mathematical analysis. In further, they occur naturally in a wide variety of variational problems. Its techniques are applied to prove existence theorems in optimal control theory. They are used to derive sufficient conditions for optimality, play an essential role in the theory of control under conditions of uncertainty and in differential games theory.

A way to study those problems is by formulating them as fixed point problems for multi-valued operators. Then, appealed to some ingredients from multi-valued analysis and topological fixed point theory to get the existence of solutions.

3.2.3 Fan-Glicksberg

In 1952, the Bohnenblust-Karlin theorem was extended by Fan and Glicksberg (independently) to locally convex spaces.

Theorem 3.44 If A is a nonempty compact convex subset of a locally convex space *X* and $N: A \to \mathcal{P}_{cl,cv}(A)$ is an u.s.c multivalued mapping, then *N* has a fixed point.

The following extension is due to Arino, Gautier and Penot [81].

Theorem 3.45 Let X be a metrizable locally convex linear topological space and let *A* be a weakly compact, convex subset of *X*, suppose $N: A \to \mathcal{P}_{cl,cv,b}(A)$ has a weakly sequentially closed graph. Then, *N* has a fixed point.

The following theorem is an application of the above theorem.

Theorem 3.46 Let A and C be closed, bounded, convex subsets of a Banach space *X* with $A \subseteq C$. In addition, let U be a weakly open subset of *A* with $0 \in U$. Assume $\overline{U^w}$ is a weakly compact subset of A and $N: \overline{U^w} \to \mathcal{P}_{cl,wcp}(C)$ has weakly sequentially closed graph. Finally, suppose $N: \overline{U^w} \to \mathcal{P}_{cl,wcp}(C)$ is a weakly compact map. Then either,

1) N has a fixed point,

2) or there is a point $x \in \partial_A U$ (the weak boundary of $U \subseteq A$) and $\lambda \in (0,1)$ with $x \in \lambda N x$.

3.2.4 Nadler

In 1969, Nadler [20] using Hausdorff metric proved a fixed point theorem for the set-valued contractions, which is of fundamental importance in nonlinear analysis. Inspired by this fixed point result a fixed point theory of set-valued contraction was further developed in different directions and by many authors.

Theorem 3.47 Let (X, d) be a complete metric space and $N: X \to \mathcal{P}_{cl,b}(X)$ be a multi-valued contraction mapping. Then, N has a fixed point.

Remark 3.48 Nadler's fixed point theorem received a great attention in applicable mathematics and was extended and generalized to various settings and by many authors. The first results were established by J. Markin [82] in a Hilbert space setting. Then, Reich in [83] proved the theorem for multi-valued contractive mappings. Later, Mizoguchi and Takahashi in [84] generalized it under some Caristi type conditions. Ciric in [85] proved it for multivalued quasi-contraction maps. Another interesting generalization is due to T.C. Lim [86] where he obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space. For details, see [87].

Theorem 3.49 Let U be an open subset of a Banach space X containing the element 0 and $N: U \to X$ be a multivalued contraction with nonempty, closed values such that $N(U)$ is bounded. Then, one of the following two assertions holds:

1) N has a fixed point in U,

2) or there exists $x \in \partial U$ and $x \in \lambda N(x)$ for $\lambda \in (0,1)$.

3.2.5 Tarski-Knaster Multivalued Version

In 1997 an analog of Knaster-Tarski Theorem for multivalued mappings was proved.

Theorem 3.50 Let $(A, \leq)$ be a partial order such that every chain in A has a supremum and let $N: A \to \mathcal{P}(A)$ be a multivalued mapping which is monotonic in the sense that whenever $x \leq y$ and $a \in N(x)$, then there exists some $b \in N(y)$ with $a \leq b$. Furthermore, let there be some $x_0 \in A$ such that there exists some $x_1 \in N(x_0)$ with $x_0 \leq x_1$. Then, N has a fixed point.

This version of Knaster-Tarski theorem has many applications in games, economics and pragmatic theory for more details see [88].

3.2.6 Darbo Multivalued Version

The following theorem presents the generalized Darbo's theorem for multivalued mappings, see [89].

Theorem 3.51 Let *A* be a closed convex and bounded subset of a Banach space X and let $N: A \rightarrow \mathcal{P}_{cl,cv}(A)$ be an u.s.c multivalued mapping such that

$\mu(NC) \leq \Phi(\mu(C))$ for any bounded $C \subseteq A$,

where $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous nondecreasing function that satisfies $\Phi(t) < t$

Then, *N* has a fixed point and the set of fixed points is compact.

The following result can be considered as a generalization of the above theorem.

Theorem 3.52 Let *X* be a Banach space and μ be a regular and set additive WMNC on *X*. Let *A* be a nonempty closed convex subset of *X*. Assume that $N: A \rightarrow \mathcal{P}_{cl,cv,b}(A)$ has weakly sequentially closed graph with $N(A)$ bounded. Let *N* be a μ -condensing, i.e.

$\mu(NC) < \mu(C)$ for any bounded $C \subseteq A$.

Then, *N* has a fixed point.

CHAPTER 4

ON DARBO'S FIXED POINT THEOREM: GENERALIZATIONS AND APPLICATIONS

As mentioned before, Darbo's fixed point theorem makes a combination of two classical theorems in the fixed point theory: Schauder's fixed point theorem and Banach contraction Principal. However, it does guarantee only the existence and not the unicity of fixed points or a way to reach them.

Our aim in this chapter is to introduce an iteration that converges to the set of fixed points for set contraction mappings. Also, establish results of stability for these type of mappings. Then, prove extensions of Darbo's theorem, one time by generalizing the contractive condition and another time by extending the space working on. Finally, the solvability of a nonlocal impulsive differential equation is investigated.

Theorem 4.1 Let B be a nonempty closed, bounded and convex subset of X. If $N: B \to B$ is a continuous mapping such that for any bounded $A \subseteq B$,

$\mu(NA) \leq k\mu(A)$ for $k \in [0,1[$,

then N has a fixed point in B. Moreover, let the closed bounded convex sequence $(A_n)_{n\in\mathbb{N}}$, then for any A_1, the sequence of subsets $A_{n+1} = conv(NA_n)$ converges to the set of fixed points of N.

Proof The existence part is given by Darbo [30] fixed point theorem.

In the next, we prove the convergence of the iteration $A_{n+1} = conv(NA_n)$.

Let $\mathcal{F} \in \mathcal{P}_b(X)$, then

$$\|\mu(A_{n+1}) - \mu(\mathcal{F})\| = \|\mu(NA_n) - \mu(N\mathcal{F})\|$$

$$\leq \|k\mu(A_n) - k\mu(\mathcal{F})\|$$

$$\leq \|k\mu(A_n) - k\mu(\mathcal{F})\|$$

$$\leq k\|\mu(A_n) - \mu(\mathcal{F})\|. \tag{4.1}$$

In further,

$$\|\mu(A_n) - \mu(\mathcal{F})\| \leq k\|\mu(A_{n-1}) - \mu(\mathcal{F})\|. \tag{4.2}$$

Substituting (4.2) in (4.1), we get

$$\|\mu(A_{n+1}) - \mu(\mathcal{F})\| \leq k^2\|\mu(A_{n-1}) - \mu(\mathcal{F})\|.$$

Repeating this process *n*-1 times, we get

$$\|\mu(A_{n+1}) - \mu(\mathcal{F})\| \leq k^n\|\mu(A_1) - \mu(\mathcal{F})\|.$$

Since $k \in [0,1[$, $\lim_{n\to\infty} k^n = 0$, then

$$\lim_{n\to\infty} \|\mu(A_{n+1}) - \mu(\mathcal{F})\| = 0.$$

Thus, $\lim_{n\to\infty} \mu(A_{n+1}) = \mu(\mathcal{F})$ and using the fact that μ is continuous with respect to Hausdorff metric, then $\lim_{n\to\infty} A_{n+1} = \mathcal{F}$.

Remark 4.2 These results don't remind true for metric spaces unless if the metric verifies the following inequality,

$$d(kx, ky) \leq kd(x, y).$$

Corollary 4.3 Let X be a Banach space and a mapping $N: X \to X$ such that

$$diam(NA) \leq k\ diam(A).$$

Then, *N* has a unique fixed point in *X*.

Proof Let the iteration $A_{n+1} = conv(NA_n)$ and from the properties of diameter we get

$$diam(A_{n+1}) = diam\big(conv(NA_n)\big)$$

$$= diam(NA_n)$$

$$\leq k\ diam(A_n).$$

Repeating these process *n*-times we get,

$$diam(A_{n+1}) \leq k^n\ diam(A_1).$$

Since $k \in [0,1[$, then $\lim_{n\to\infty} k^n = 0$. Thus, $\lim_{n\to\infty} diam(A_{n+1}) = 0$.

using the fact that $\lim_{n\to\infty} A_{n+1} = \mathcal{F}$, we get $diam(\mathcal{F}) = 0$.

We conclude that either the set of fixed points of $\mathcal{F}$ is empty or contains only one point. However, it couldn't be empty since N is k-set contraction, hence $\mathcal{F}$ contains only one point.

Remark 4.4 The Corollary 4.3 includes the Banach contraction principle as a special case. Indeed, suppose that N is a contraction mapping, then

$$\|Nx - Ny\| \leq k\|x - y\|, \text{for any } x, y \in X. \tag{4.3}$$

In further, we know that the diameter is the simplest MNC and by definition for any bounded set A,

$$diam(A) = \sup_{x,y\in A}\|x - y\|.$$

By taking supremum in inequality (4.3), we get

$$\sup_{Nx,Ny\in NA}\|Nx - Ny\| \leq k \sup_{x,y\in A}\|x - y\|.$$

Then, $diam(NA) \leq k\ diam(A)$.

The concept of stability was introduced by Harder [90], Harder and Hicks [91-92]. A fixed point iteration procedure is numerically stable if by affecting small modifications in initial data involved in the computation process we get a small influence on the computed value of the fixed point. There are also other definitions of stability considered by several authors. For example, Imoru and Olatinwo [93], Osilike [94], Osilike and Udomene [96], Rhoades [96-97] and many others.

Inspired by the definition given by Harder and Hicks, we introduce the following definition:

Definition 4.5 Let N be a k-set contraction mapping and (A_n) is a sequence of nonempty, closed, bounded and convex subset of X such that $A_{n+1} = conv(NA_n)$ and converges to $\mathcal{F}$ (the set of fixed points of N). Let $\varepsilon_n = \left\|\mu(B_{n+1}) - \mu\big(f(N, A_{n+1})\big)\right\|$ where (B_n) is a sequence of nonempty, closed, bounded and convex subsets of X such that $B_{n+1} = conv(NB_n)$. The mapping N is said to be N-set stable if the following assumption holds,

$$\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} A_{n+1} = \lim_{n\to\infty} B_{n+1} = \mathcal{F}.$$

Theorem 4.6 Let N be a k-set contraction and let the iteration $A_{n+1} = conv(NA_n)$ where (A_n) is a sequence of nonempty, closed, bounded and convex subsets of X. Then, the mapping N is set stable.

Proof Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and let show that

$$\lim_{n\to\infty} A_{n+1} = \lim_{n\to\infty} B_{n+1} = \mathcal{F}.$$

We have,

$$\|\mu(B_{n+1}) - \mu(\mathcal{F})\| = \|\mu(B_{n+1}) - \mu(A_{n+1}) + \mu(A_{n+1}) - \mu(\mathcal{F})\|$$
$$\leq \|\mu(B_{n+1}) - \mu(A_{n+1})\| + \|\mu(A_{n+1}) - \mu(\mathcal{F})\|.$$

Since,

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|\mu(B_{n+1}) - \mu(A_{n+1})\| = 0$$

and

$$\lim_{n\to\infty} \|\mu(A_{n+1}) - \mu(\mathcal{F})\| = 0.$$

Then, $\lim_{n\to\infty} \|\mu(B_{n+1}) - \mu(\mathcal{F})\| = 0$.

On the other hand, suppose that $\lim_{n\to\infty} B_{n+1} = \mathcal{F}$ and let show that,

$$\lim_{n\to\infty} \varepsilon_n = 0.$$

Let,

$$\|\mu(B_{n+1}) - \mu(A_{n+1})\| = \|\mu(B_{n+1}) - \mu(\mathcal{F}) + \mu(\mathcal{F}) - \mu(A_{n+1})\|$$
$$\leq \|\mu(B_{n+1}) - \mu(\mathcal{F})\| + \|\mu(\mathcal{F}) - \mu(A_{n+1})\|.$$

By taking the limit, we get

$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \|\mu(B_{n+1}) - \mu(A_{n+1})\| = 0$. This ends the proof.

Darbo's theorem revealed to be a very important tool to solve differential and integral equation, it has been in the center of researches. In the following we mention some generalizations and their applications.

4.1 Extensions of Darbo's Theorem by Generalizing Contractive Conditions

It is known that a contraction mapping N on a Banach space $(X, \|.\|)$ with contraction constant k is also a contraction on X with a contraction constant k^n. Then, it is natural to ask if this result remains true for set contraction mappings?

This section is devoted to investigate the answer to this question. A part of the answer is given in the following theorem.

Theorem 4.7 Let A be a nonempty closed, bounded and convex subspace of a Banach space $(X, \|.\|)$ and $N: A \to A$ be a k-set contraction mapping on A. Then for an integer $n > 0$, N^n is a k^n-set contraction on A.

Proof Let A be a nonempty closed, bounded and convex subset of X, then

$$\begin{aligned} \mu(N^n A) &= \mu\big(N(N^{n-1}A)\big) \\ &\leq k\ \mu(N^{n-1}A) \\ &\leq k^2\ \mu(N^{n-1}A) \\ &\ \ \vdots \\ &\leq k^n\ \mu(A) \end{aligned}$$

Since $0 \leq k < 1$, hence $0 \leq k^n < 1$ and so N^n is a k-set contraction mapping.

Remark 4.8 The inverse is not true that is if N^n is k-set contraction mapping then N could be not a k-set contraction mapping.

Example 4.9 Let X be a Banach space and $N: X \to X$ be a mapping defined by $Nx = 1 - \frac{x}{2}$ for any $x \in X$.

Let $B(0, r)$ be an open ball with center 0 and radius $r \leq 1$. Then,

$$NB(0, r) = B\left(0,1 - \frac{r}{2}\right).$$

It is easy to see that $B\left(0,1 - \frac{r}{2}\right) \not\subseteq B(0, r)$, hence N isn't a k-set contraction mapping.

However, $N^2 x = N\left(1 - \frac{x}{2}\right) = \frac{x}{2}$, then

$$N^2 B(0, r) = B\left(0, \frac{r}{2}\right) \subseteq B(0, r).$$

Thus for any MNC μ, we have $\mu\left(N^2 B(0,r)\right) \leq \frac{1}{2}\mu\left(B(0,r)\right)$. Hence, N^2 is a k -set contraction mapping with $k = \frac{1}{2}$.

Theorem 4.10 Let A be a nonempty closed, bounded and convex subspace of a Banach space $(X, \|.\|)$ and $N: A \to A$ be a mapping such that for any $n \geq 1$ we have $N^n\left(conv(A)\right) \subseteq conv(N^n A)$

and

$\mu(N^n A) \leq k_n \mu(A)$ where $k_n \to 0$ while $n \to +\infty$. (4.4)

Then, N has at least one fixed point.

Proof Let the iteration $A_n = conv(NA_{n-1})$ where $(A_n)_n$ is a sequence of nonempty closed, bounded and convex subsets of X.

It is clear that $(A_n)_n$ is decreasing and by using the properties of the MNC, we get

$$\mu(A_n) = \mu(conv\ NA_{n-1})$$

$$\leq \mu(conv\ NA_{n-1}) = \mu(NA_{n-1}) \leq \mu\left(N\left(conv(NA_{n-2})\right)\right)$$

$$\leq \mu\left(N^2(A_{n-2})\right).$$

Repeating this process many times, we get

$$\mu(A_n) \leq \mu(N^n A_0).$$

Using inequality (4.4), we obtain $\mu(A_n) \leq \mu(N^n A_0) \leq k_n \mu(A_0)$.

By taking the limit, we get $\lim_{n\to\infty} \mu(A_n) = 0$ which imply that A_∞ is compact and hence N has at least one fixed point in A_∞.

Corollary 4.11 Let X be a Banach space and N be a mapping such that for each $n \geq 1$ there exists a constant k_n such that

$$sup_{N^n x, N^n y \in N^n A} \|N^n x - N^n y\| \leq k_n \sup_{x,y\in A} \|x - y\|, \text{ for all } A \in \mathcal{P}_b(X),$$

where $k_n \to 0$, $n \to +\infty$.

Then, N has a unique fixed point.

Proof Easy to see, since by definition the $diam(A) = \sup_{x,y\in A} \|x - y\|$.

The above corollary includes the theorem given by Caccioppoli as a special case.

Corollary 4.12 Let N be a mapping such that for each $n \geq 1$, there exists a constant k_n such that,

$\|N^n x - N^n y\| \leq k_n \|x - y\|$ for all $x, y \in X$,

where $\sum_{n=1}^{\infty} k_n < \infty$. Then, N has a unique fixed point.

Proof As we know the simplest MNC is the diameter

$diam(A) = \sup_{x,y \in A} d(x, y)$.

Let, $d(N^n x, N^n y) \leq k_n d(x, y)$.

Then, $\sup d(N^n x, N^n y) \leq k_n \sup d(x, y)$.

This is equivalent to say that,

$\mu(N^n A) \leq k_n \mu(A)$.

However, since $\sum_{n=1}^{\infty} k_n < \infty$, then $k_n \to 0$ for $n \to \infty$.

Then from Theorem 4.10, N has at least one fixed point.

Suppose that p and q are two fixed points for N. Then,

$d(p, q) = d(N^n p, N^n q) \leq k_n d(p, q)$.

Using that $k_n \to 0, n \to +\infty$. We obtain $0 \leq d(p, q) \leq 0$.

Thus $p = q$ and the fixed point is unique.

Theorem 4.13 Let A be a bounded subset of a Banach space $(X, \|.\|)$ and $N: A \to A$ satisfies:

$$\mu(N^n A) \leq \eta(\mu(A))\mu(A), \tag{4.5}$$

where, either $\eta: \mathbb{R}_+ \to [0,1[$ is a decreasing function or $\eta: \mathbb{R}_+ \to [1, \infty[$ is a function such that $\lim_{n\to\infty} \eta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Then, N has a fixed point in A.

Proof Let $A_{n+1} = conv(N^n A_n)$ such that $(A_n)_{n\in\mathbb{N}}$ is a sequence of nonempty closed, bounded and convex subsets of a Banach space X. It is easy to see that $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence.

In further, by using condition (4.5), we get

$\mu(A_{n+1}) \leq \eta(\mu(A_n))\mu(A_n)$.

Suppose that $\eta: \mathbb{R}_+ \to [1, \infty[$ is a function such that $\lim_{n\to\infty} \eta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

In further, since (A_n) is a decreasing sequence

$$1 \leq \frac{\mu(A_n)}{\mu(A_{n+1})} \leq \frac{1}{\eta(\mu(A_n))} \leq 1. \tag{4.6}$$

By taking limit, we obtain

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(A_{n+1}) \text{ and } \lim_{n\to\infty} \frac{1}{\eta(\mu(A_n))} = 1.$$

Hence, $\lim_{n\to\infty} \eta(\mu(A_n)) = 1$ which implies that $\lim_{n\to\infty} \mu(A_n) = 0$. Thus, A_∞ is compact and N has at least one fixed point in A.

Now, suppose that $\eta: \mathbb{R}_+ \to [1, \infty[$ is a decreasing function.

From inequality (4.6). we have

$$0 \leq \frac{\mu(A_{n+1})}{\mu(A_n)} \leq \eta(\mu(A_n)).$$

Repeating this process and using the fact that η is a decreasing function, we obtain

$$0 \leq \frac{\mu(A_n)}{\mu(A_{n-1})} \leq \eta(\mu(A_{n-1})) \leq \eta(\mu(A_n))$$

$$\vdots$$

$$0 \leq \frac{\mu(A_1)}{\mu(A_0)} \leq \eta(\mu(A_0)) \leq \eta(\mu(A_n)).$$

In further, we have

$$\begin{aligned} 0 \leq \frac{\mu(A_{n+1})}{\mu(A_0)} &= \frac{\mu(A_{n+1})}{\mu(A_n)} \cdot \frac{\mu(A_n)}{\mu(A_{n-1})} \cdots \frac{\mu(A_1)}{\mu(A_0)} \\ &\leq \eta(\mu(A_n)) \cdot \eta(\mu(A_n)) \cdots \eta(\mu(A_n)) \\ &= [\eta(\mu(A_n))]^n . \end{aligned}$$

Since $\eta(\mu(A_n)) \in [0,1[$, then $\lim_{n\to\infty} [\eta(\mu(A_n))]^n = 0$ and so $\lim_{n\to\infty} \frac{\mu(A_{n+1})}{\mu(A_0)} = 0$.

Since $\mu(A_0)$ is a finite constant, then $\lim_{n\to\infty} \mu(A_n) = 0$.

Consequently, A_∞ is compact and N has a fixed point in A_∞.

Theorem 4.14 Let A be a nonempty closed, bounded and convex subspace of a Banach space $(X, \|.\|)$ and $N: A \to A$ be a mapping such that for any $n \geq 1$ we have $N^n\big(conv\ (A)\big) \subseteq conv\ (N^n A)$

and

$$\mu(N^n A) \leq \varphi_n\big(\mu(A)\big), \tag{4.7}$$

where $\varphi_n: [0, \infty[\ \to [0, \infty[$ are continuous and $\varphi_n \to 0$ when $n \to 0$ uniformly. Then, N has at least one fixed point in A.

Proof Let $A_{n+1} = conv(NA_n)$ such that $(A_n)_{n\in\mathbb{N}}$ is a sequence of nonempty closed, bounded and convex subsets of a Banach space X . Then as we did before, we get

$$\mu(A_n) \leq \mu(N^n A_0).$$

From condition (4.7), we obtain

$$\mu(A_n) \leq \varphi_n\big(\mu(A_0)\big).$$

By taking limits, we get

$$0 \leq \lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \varphi_n\big(\mu(A_0)\big) = 0.$$

Thus, $\lim_{n\to\infty} \mu(A_n) = 0$. Hence, A_∞ is compact and N has at least one fixed point in A.

4.2 Extension of Darbo Fixed Point Theorem to Frechet Spaces

In the following using Tychonoff theorem, Darbo fixed point theorem is extended for Frechet spaces.

Theorem 4.15 Let A be a nonempty closed bounded and convex subset of a Frechet space F. If $N: A \to A$ is a continuous mapping such that

$$\mu(NA) \leq \Phi\big(\mu(A)\big),$$

where $\Phi: [0, \infty[\ \to [0, \infty[$ is a nondecreasing function and $\lim_{n\to\infty} \Phi^n(t) = 0,\ \forall t > 0.$

Then, N has at least one fixed point.

Proof Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed bounded and convex subsets of a Frechet space F such that $A_{n+1} = conv(NA_n)$.

Then, it is easy to see that

$A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_0$.

Additionally by using properties of the MNC, we get

$$\mu(A_{n+1}) = \mu(conv(NA_n)) = \mu(NA_n)$$

$$\leq \Phi(\mu(A_n)) = \Phi(\mu(conv(NA_{n-1}))) = \Phi(\mu(NA_{n-1}))$$

$$\leq \Phi^2(\mu(A_{n-1})).$$

Repeating this process *n*-times, we get

$$\mu(A_{n+1}) \leq \Phi^{n}(\mu(A_0)).$$

By taking the limit we obtain,

$$\lim_{n\to\infty} \mu(A_{n+1}) \leq \lim_{n\to\infty} \Phi^{n}(\mu(A_0)) = 0.$$

Thus, $0 \leq \lim_{n\to\infty} \mu(A_{n+1}) \leq 0$ this implies $\lim_{n\to\infty} \mu(A_{n+1}) = 0$.

Then by Schauder-Tychonoff theorem, we conclude that *N* has a fixed point.

Corollary 4.16 Let N be a nonempty closed bounded and convex subset of a Frechet space F. If $N: A \to A$ is a continuous mapping such that

$$\mu(NA) \leq k\mu(A) \ \ for \ \ k \in [0,1[,$$

Then, *N* has a fixed point.

Proof It can be seen easily by taking $\Phi(t) = kx$ in Theorem 4.15.

Remark 4.17 If *F* is a Banach space in Corollary 4.16, then we obtain the Darbo fixed point theorem.

From the above theorem, the following alternative can be proved.

Theorem 4.18 Let *A* be a nonempty closed, bounded and convex subset of a Frechet space *F* and let $N: A \to X$ be a continuous mapping such that

$$\mu(NA) \leq \Phi(\mu(A)),$$

where $\Phi: [0, \infty[\to [0, \infty[$ is a nondecreasing function and $\lim_{n\to\infty} \Phi^n(t) = 0, \ \forall t > 0$.

Then, either

1) N has a fixed point in A,
2) or there is a point $x \in \partial A$ such that $x = \lambda Nx$ for some $0 < \lambda < 1$.

Proof Without loss of generality, assume that $A = \{x: p(x) \leq r\}$ where $r > 0$ is a constant and let $\varphi: F \to A$ defined by

$$\varphi(x) = \begin{cases} x, & x \in A \\ r\frac{x}{p(x)}, & x \in F \setminus A \end{cases}$$

It's clear that φ is continuous, hence $\varphi(N)$ is continuous. In further, for $x \in F \setminus A$, then

$\varphi(x) = r\frac{x}{p(x)} = \lambda x + (1 - \lambda)0$, with $\lambda = \frac{r}{p(x)}$.

Then, $\varphi(A) \subseteq conv(A \cup \{0\})$.

Although for $x \in A$, we have $\varphi(x) = x$, hence $\varphi(A) \subseteq A \cup \{0\}$.

Thus, we conclude that in both cases we have $\varphi(A) \subseteq conv(A \cup \{0\})$.

Now, by using properties of MNC we have for any bounded $S \subseteq A$,

$$\begin{aligned} \mu(\varphi(NS)) &\leq \mu(conv(N(S) \cup \{0\})) \\ &= \mu(N(S)) \\ &\leq \Phi(\mu(S)). \end{aligned}$$

Then, $\varphi(N)$ has at least one fixed point in A (i.e.) there exist at least one $x \in A$ such that $\varphi(N(x)) = x$.

From the definition of φ, there are two cases.

First case: when $p(Nx) \leq r$, so $Nx \in A$, then $\varphi(Nx) = Nx = x$, thus N has at least one fixed point.

The second case, when $p(Nx) > r$ i.e. $Nx \notin A$ then,

$\varphi(Nx) = r\frac{N(x)}{p(N(x))} = x \Rightarrow r = p(x)$.

Hence, $x \in \partial A$ and $x = \lambda Nx$ with $\lambda = \frac{r}{p(Nx)}$.

Corollary 4.19 Let A be a nonempty closed, bounded and convex subset of a Frechet space F. Assume that $N: A \to A$ is a continuous mapping such that for every bounded $\Omega \subseteq A$,

$\mu(N\Omega) \leq k\mu(\Omega),\ k \in [0,1[$.

Then, either

1) N has a fixed point in A

2) or there is a point $x \in \partial A$ such that $x = \lambda N x$ for some $0 < \lambda < 1$.

4.3 Nonlinear Impulsive Differential Equation with Nonlocal Conditions

In this section as applications of the previous results, existence principles are presented for the following nonlinear impulsive differential equation with nonlocal conditions in a real Banach space X denoted by the following form

$$x'(t) = Ax(t) + f\big(t, x(t)\big),\ t \in \mathbb{R}_+ \text{ and } t \neq t_i, \tag{4.8}$$

$$x(0) = g(x), \tag{4.9}$$

$$\Delta x(t_i) = I_i\big(x(t_i)\big),\ 0 = t_0 < t_1 < \cdots \text{ and } \lim_{i\to\infty} t_i = \infty. \tag{4.10}$$

where $f: \mathbb{R}_+ \times PC(\mathbb{R}_+, X) \to X$ is a suitable function and $\{A(t)\}_{t\geq 0}$ is a family of linear closed (not necessarily bounded) operators from X into X that generate a strongly continuous evolution system of operators $\{U(t,s)\}_{0\leq s\leq t<+\infty}$ in a Banach space X and $g: PC(\mathbb{R}_+, X) \to X$ is an X −valued function.

Denote,

$$PC(\mathbb{R}_+, X) = \left\{ \begin{matrix} x: \mathbb{R}_+ \to X, \quad x(t)\ is\ continuous\ for\ t \neq t_i\ \ and\ left\ continuous \\ at\ t = t_i\ and\ the\ right\ limit\ x(t_i^+)\ exists\ for\ i = 1,2,\dots \end{matrix} \right\}$$

Definition 4.20 The function $x(.): \mathbb{R}_+ \to X$ is a mild solution of the evolution system (4.8)-(4.10) if for each $t \in \mathbb{R}_+$, x satisfies the following integral equation

$$x(t) = U(t,0)g(x) + \int_0^t U(t,s) f\big(s, x(s)\big) ds + \sum_{0<t_i<t} U(t,t_i) I_i\big(x(t_i)\big).$$

Assume the following hypotheses,

1) The function $f(.,.): \mathbb{R}_+ \times X \to X$ is an equicontinuous and Caratheodory function, that is, for each $x \in X$ the function $f(., x): \mathbb{R}_+ \to X$ is measurable and the function $f(t,.): X \to X$ is continuous for a.e $t \in \mathbb{R}_+$.

2) There exists nondecreasing continuous and integrable function $p \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that,

$$\big|f\big(t, x(t)\big)\big| \leq p(t)\Psi(\|x(t)\|),$$

for all $x \in F$ and a.e $t \in \mathbb{R}_+$.

3) There exists $p \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that for any bounded $A \subset PC\big([0, t_{n_0+1}], X\big)$,

$\mu(f(t,A)) \le q(t)\mu(A)$ for a.e. $t \in \mathbb{R}_+$.

4) The mapping $g: PC(\mathbb{R}_+, X) \to X$ is continuous and there is a positive constant α and $n_0 \in \mathbb{N}$ such that for any bounded $A \subset PC([0, t_{n_0+1}], X)$,

$\mu(g(A)) \le \alpha\mu(A)$ for $a.e. t \in \mathbb{R}_+$.

5) The mapping $I: X \to X$ is continuous and there is a positive constants β_i such that for any bounded $A \subset PC([0, t_{n_0+1}], X)$,

$\mu(I_i(A)) \le \beta_i\mu(A)$ for $a.e. t \in \mathbb{R}_+$.

6) There exists $R > 0$ such that for any $x \in B_R = \{x \in PC(\mathbb{R}_+, X) : \|x\| \le R\}$

$$M\left[\sup_{x\in B_R}\|g(x)\| + \Psi(R)\int_0^t |p(s)|ds + \sum_{0<t_i<t}\sup_{x\in B_R}\left\|I_i(x(t_i))\right\|\right] \le R,$$

where M is a positive constant that satisfies $\|U(t,s)x\| \le M\|x\|$.

Theorem 4.21 Assume that $(1-6)$ hold and that there is no point $x \in \partial A$ such that $x = \lambda Nx$ for some $0 < \lambda < 1$.

Then the nonlocal initial problem $(4.8) - (4.10)$ has at least one impulsive mild solution in the Frechet space $PC(\mathbb{R}_+, X)$.

Proof To solve problem $(4.8) - (4.10)$ we transform it to the following fixed-point problem,

$$Nx(t) = U(t,0)g(x) + \int_0^t U(t,s)f(s,x(s))ds + \sum_{0<t_i<t} U(t,t_i)I_i(x(t_i)), \quad \forall t \in \mathbb{R}_+.$$

Clearly, the fixed points of operator N coincide with mild solutions of the problem $(4.8) - (4.10)$. To show that $N: PC(\mathbb{R}_+, X) \to PC(\mathbb{R}_+, X)$ admits fixed points we need to verify that N satisfy the conditions of Corollary 4.19.

Firstly, we show that N is a self mapping, let for $x \in B_R$,

$$\|Nx(t)\| \le \|U(t,0)g(x)\| + \int_0^t \left\|U(t,s)f(s,x(s))\right\|ds + \sum_{0<t_i<t}\left\|U(t,t_i)I_i(x(t_i))\right\|.$$

Since $\{U(t,s)\}$ is strongly continuous, then

$$\begin{aligned}\|Nx(t)\| &\le M\|g(x)\| + M\int_0^t \left\|f(s,x(s))\right\|ds + M\sum_{0<t_i<t}\left\|I_i(x(t_i))\right\| \\ &\le M\sup_{x\in B_R}\|g(x)\| + M\Psi(R)\int_0^t \|p(s)\|ds + M\sum_{0<t_i<t}\left\|I_i(x(t_i))\right\|. \\ &\le M\left[\sup_{x\in B_R}\|g(x)\| + \Psi(R)\int_0^t |p(s)|ds + \sum_{0<t_i<t}\sup_{x\in B_R}\left\|I_i(x(t_i))\right\|\right]\end{aligned}$$

$\leq \; R.$

Therefore, N maps the closed ball B_R into itself.

Now let show that N is continuous, let $x, y \in X$ and $\delta > 0$ such that $\|x(t) - y(t)\| < \delta$, then

$$\|Nx(t) - Ny(t)\| \leq M\|g(x(t)) - g(y(t))\| + M\int_0^t \|f(s, x(s)) - f(s, y(s))\|ds$$

$$+M \textstyle\sum_{0<t_i<t}\|I_i(x(t_i)) - I_i(y(t_i))\|.$$

Since $g,\ f$ and I are continuous then there exists ε such that,

$\|g(x) - g(y)\| \leq \frac{\varepsilon}{3M}, \quad \|f(s, x(s)) - f(s, y(s))\| \leq \frac{\varepsilon}{3M}$

and $\|I_i(x(t_i)) - I_i(y(t_i))\| \leq \frac{\varepsilon}{3M}$.

Therefore, we get $\|Nx(t) - Ny(t)\| \leq \varepsilon$. Thus, N is continuous.

Finally, let verify the set contraction condition.

Let A be a bounded subset of $PC([0, t_{n_0+1}], X)$, we have

$$\mu(NA) \leq M\mu(g(A)) + M\mu\left(\int_0^t f(s, A)ds\right) + M\mu\left(\textstyle\sum_{0<t_i<t} I_i(A)\right). \tag{4.11}$$

Since A is bounded and f is continuous, we get $f(t, A)$ is bounded.

Then using Lemma 2.76, we obtain

$$\mu\left(\int_0^t f(s, A)ds\right) \leq 2\mu\left(\int_0^t f(s, A)ds\right). \tag{4.12}$$

Substituting (4.12) in (4.11),

$$\mu(NA) \leq M\mu(g(A)) + 2M\int_0^t \mu(f(s, A)ds) + 2M \textstyle\sum_{0<t_i<t} \mu(I_i(A)).$$

From conditions $(3-5)$, follows that

$$\mu(NA) \leq \alpha M\mu(A) + 2M\int_0^t q(s)\mu(A)ds + 2M\sum_{0<t_i<t} \beta_i\mu(A)$$

$$\leq \left[\alpha M + 2M\int_0^t q(s)ds + 2M\textstyle\sum_{0<t_i<t}\beta_i\right]\mu(A).$$

Consequently for $\left[\alpha M + 2M\int_0^t q(s)ds + 2M\sum_{0<t_i<t}\beta_i\right] \leq 1$, in view of Corollary 4.19, either N has a fixed point in $\bar{A}$ or there is a point $x \in \partial A$ such that $x = \lambda Nx$ for

some $0 < \lambda < 1$.

Since we assumed there is no point $x \in \partial A$ such that $x = \lambda N x$ ($0 < \lambda < 1$). Then, problem $(4.8) - (4.10)$ has at least one solution.

CHAPTER 5

ON COMMON FIXED POINT THEOREMS FOR SET CONTRACTION COMMUTING MAPPINGS

One of the powerful results of the fixed point theory is a theorem due to Meir and Keeler [99]. They defined a new class of contraction operators which includes the contraction mappings as a special case and proved a very interesting theorem more general than Banach contraction theorem.

Then Park and Bae [100] extended this new class of mappings in order to study the existence of common fixed points.

Recently, motivated by Meir and Keeler Aghajani et al. [101] presented a new class of condensing operators and proved a theorem which presents a very nice generalization of Darbo's fixed point theorem.

In this chapter, theorems that guarantee the existence of common fixed points for commuting set contraction mappings and generalize the work of Jungck [102] are proved. Additionally, motivated by Park and Bae [100] new classes of set contraction mappings are introduced and the existence of common fixed points for these classes of mappings is investigated, the results obtained extend many works existing in the literature.

As applications, existence of common fixed points for mappings that satisfy conditions of integral type is given. Finally, the solvability of a new type of integral equation is studied.

5.1 On common Fixed Points for Commuting Set Contraction Mappings

For starters, the following theorem that guarantees the existence of common fixed points for commuting set contraction mappings.

Theorem 5.1 Let $H: X \to X$ be a continuous self-mapping defined on X. If there exists a self-mapping $N: X \to X$ such that N commutes with H and

$$\mu(NA) \leq \Phi\big(\mu(HA)\big), \tag{5.1}$$

where $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and upper semicontinuous function such that for $\Phi(\mathrm{t}) < t$ for all $t > 0$.

Then, N and H have at least one common fixed point and the set of common fixed points is compact.

Proof Let $(A_n)_{n\in\mathbb{N}}$ be a closed, bounded and convex sequence of subset of X. Let define the following iteration $A_{n+1} = conv(HA_n)$. Then, it is easy to see that

$A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_0$.

Moreover, let

$$\mu(A_{n+1}) = \mu(conv(NA_n)) = \mu(NA_n)$$

$$\leq \Phi\big(\mu(HA_n)\big). \tag{5.2}$$

Since H is a self-mapping, $HA_n \subseteq A_n$ which implies

$\mu(HA_n) \leq \mu(A_n)$.

Using the fact that Φ is nondecreasing, we get

$$\Phi\big(\mu(HA_n)\big) \leq \Phi\big(\mu(A_n)\big). \tag{5.3}$$

Substuting (5.3) in (5.2), we get

$\mu(A_{n+1}) \leq \Phi\big(\mu(A_n)\big)$.

Repeating this process *n*-times, we get

$\mu(A_{n+1}) \leq \Phi^{\mathrm{n+1}}\big(\mu(A_0)\big)$.

By taking the limit we obtain,

$\lim_{n\to\infty} \mu(A_{n+1}) \leq \lim_{n\to\infty} \Phi^{\mathrm{n}}\big(\mu(A_0)\big) = 0$.

Thus, $0 \leq \lim_{n\to\infty} \mu(A_{n+1}) \leq 0$ this implies $\lim_{n\to\infty} \mu(A_{n+1}) = 0$.

Using Lemma 2.38, we have $\lim_{\mathrm{n}\to\infty} \Phi^{\mathrm{n+1}}\big(\mu(\mathrm{A_0})\big) = 0$. Hence, $\lim_{\mathrm{n}\to\infty} \mu(\mathrm{A_{n+1}}) = 0$.

In view of Definition 2.74, $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is compact.

Since N is a continuous mapping on a compact A_∞, then by Schauder's theorem, H has at least one fixed point.

Suppose that x^* is the fixed point of H, that is, $Hx^* = x^*$, then $HNx^* = NHx^* = x^*$, hence $Hx^*(= x^*)$ is also fixed point of N.

Consequently, N and H have at least one common fixed point.

Now let show that the set of common fixed points is compact. Suppose

$\mathcal{F} = \{x \in X,\ Nx = Hx = x\}$ and $\mu(\mathcal{F}) \neq 0$, then

$\mu(\mathcal{F}) = \mu(N\mathcal{F}) \leq \Phi\big(\mu(H\mathcal{F})\big) < \mu(N\mathcal{F}) = \mu(\mathcal{F})$.

Contradiction !! Thus, $\mathcal{F}$ is compact.

Corollary 5.2 Let N and H given as in Theorem 5.1 such that

$\mu(NA) \leq k\mu(HA) \quad for\ k \in [0,1[$.

Then, N and H have at least one common fixed point and the set of common fixed points is compact.

Proof It suffices to take $\Phi(t) = kt$ in Theorem 5.1.

Theorem 5.3 Let N and H given as in Theorem 5.1. If instead of condition (5.1) we have the following inequality

$\mu(NA) < \Phi\big(\mu(HA)\big)$,

then N and H have at least one common fixed point in A and the set of common fixed points is compact.

Proof Let for a given $x_0 \in X$, the set

$S = \{S \cup \{x_0\} /\ S\ is\ closed\ and\ convex\ subset\ of\ X\}$.

Denote $A = \bigcap S$. Suppose that, A is not compact, that is, $\mu(A) \neq 0$.

It is clear that A is non empty (since $x_0 \in A$).

Now, define

$R = conv(NA) \cup \{x_0\}$.

Obviously, R is closed and convex subset of X that contains x_0, then $R \in S$.

Thus, $A \subseteq R$.

Moreover, $R \subseteq A \cup \{x_0\} \subseteq A$ (since N is a self-mapping, then $conv(NA) \subseteq A$). Consequently, $R = A$.

Let, $\mu(A) = \mu(R) = \mu(conv(NA) \cup \{x_0\})$.

Using Definition 2.74, we get

$\mu(A) = \mu(NA) < \Phi\big(\mu(HA)\big) < \mu(HA) \leq \mu(A)$.

Then, A should be compact. If N (or H) is continuous, then by Schauder's theorem we conclude that N (or H) has at least one fixed point on A.

Suppose that x^* is a fixed point of N, that is, $Nx^* = x^*$, then $NHx^* = HNx^* = x^*$. Hence $Nx^*(= x^*)$ is a fixed point of H.

Similarly if x^* is a fixed point of H.

Finally, N and H have at least one common fixed point on A and as we did in proof of Theorem 5.1, it can be easily shown that the set of common fixed points is compact.

Corollary 5.4 Let N and H given as in Theorem 5.1. If we have instead of condition (5.1) the following inequality

$\mu(NA) < \mu(HA)$,

then N and H have at least one common fixed point in A and the set of common fixed points is compact.

Proof Obviously since $\Phi(t) < t, \ \forall t > 0$.

Corollary 5.5 Let N and H given as in Theorem 5.1. If we have instead of condition (5.1) the following inequality

$\mu(NA) < k\mu(HA) + (1-k)\mu(A) \quad for\ k \in [0,1[$.

Then, N and H have a common fixed point and the set of common fixed points is compact.

Proof Obvious.

Definition 5.6 Let N and H be two self-mappings on X, N is an (ε,δ) H -set-contraction iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leq \mu(HA) < \varepsilon + \delta$ implies that $\mu(NA) < \varepsilon$.

Definition 5.7 Let N and H be two self-mappings on X, N is said to be a generalized (ε, δ) H-set-contraction iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leq \Phi\big(\mu(HA)\big) < \varepsilon + \delta$ implies that $\mu(NA) < \varepsilon$.

Remark 5.8 The case $H = I$ was introduced by Aghajani et al. in [5] under the name of N Meir-Keeler condensing operator.

Theorem 5.9 If N is a continuous self-mapping of X and H is a (ε, δ) H-set-contraction which commutes with N. Then, N and H have at least one common fixed point in X and the set of fixed points is compact.

Proof Interestingly, although the class of (ε, δ) H-set-contraction mappings is larger than that of set-contraction mappings, one may prove Theorem 5.9 by means of Theorem 5.3. Indeed, since

$\varepsilon \leq \Phi\big(\mu(HA)\big) < \varepsilon + \delta$ implies that $\mu(NA) < \varepsilon$.

Then, obviously

$\mu(NA) < \Phi\big(\mu(HA)\big)$.

Then by Theorem 5.3, N and H have at least one fixed point and the set of fixed points is compact.

Theorem 5.10 If N is a continuous self-mapping of X and H is a generalized (ε, δ) H-set-contraction which commutes with N. Then, N and H have at least one common fixed point in X and the set of fixed points is compact.

Proof Same as proof of Theorem 5.9.

Corollary 5.11 [5] If N is a continuous self-mapping of X and it is a Meir-Keeler condensing operator. Then, N has at least one fixed point in X.

5.2 Application to Commuting Set Contraction Mappings of Integral Type

In the following, integral version of the fixed point theorems presented above is established. The proofs of these results are given only in sketches since they are based on arguments similar to the ones used previously.

Theorem 5.12 Let $N, H: A \to A$ be commuting mappings such that either N or H is continuous. If there exists a nondecreasing and an upper semi-continuous function $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Phi(t) < t$ for all $t > 0$ and for which we have

$$\int_0^{\mu(NA)} \varphi(t)dt \leq \Phi\left(\int_0^{\mu(HA)} \varphi(t)dt\right). \tag{5.2}$$

Then, $\mathcal{F} = \{x \in A: \quad Sx = Tx = x\}$ is nonempty and compact.

Corollary 5.13 Let $H: A \to A$ be a continuous mapping such that

$$\int_0^{\mu(HA)} \varphi(t)dt \leq \mathrm{k} \int_0^{\mu(A)} \varphi(t)dt \quad \text{for } k \in [0,1[.$$

Then, H has at least one fixed point and the set of fixed points is compact.

Theorem 5.14 Let N and H given as in Theorem 5.11 such that instead of condition (5.2) we have the following inequality

$$\int_0^{\mu(NA)} \varphi(t)dt \leq k \int_0^{\mu(HA)} \varphi(t)dt + (1-k) \int_0^{\mu(A)} \varphi(t)dt \;\; \text{for } k \in [0,1[.$$

Then, N and H have a common fixed point and the set of common fixed points is compact.

Definition 5.15 Let N and H two self mappings on X, N is an (ε, δ) H-set-contraction of integral type iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \int_0^{\mu(HA)} \varphi(t)dt < \varepsilon + \delta \text{ implies that } \int_0^{\mu(NA)} \varphi(t)dt < \varepsilon.$$

Definition 5.16 Let N and H two self mappings on X, N is said to be a generalized (ε, δ) H-set-contraction of integral type iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \Phi\left(\int_0^{\mu(HA)} \varphi(t)dt\right) < \varepsilon + \delta \text{ implies that } \int_0^{\mu(NA)} \varphi(t)dt < \varepsilon.$$

Theorem 5.17 Let H be a continuous self mapping on A and N is a generalized (ε, δ) H-set contraction of integral type which commutes with H, then H and N have at least one common fixed point in X and the set of fixed points is compact.

5.3 Nonlinear Integral Equation of Integral Type

Let introduce the following Volterra integral equation

$$\int_0^{x(t)} \varphi(r)dr = \int_0^{f\left(t, x(t), \int_0^t K(s,t)x(s)ds\right)} \varphi(r)dr. \tag{5.4}$$

In the following as application of the previous results, we characterize solutions of integral equation (5.4) defined on the Banach space $C([0,1])$ consisting of all defined, bounded and continuous functions on the interval $C([0,1])$.

We endow the space $C([0,1])$ with the usual norm

$\|x\| = \sup\{|x(t)|:\ t > 0\}$.

Let X be a nonempty and bounded subset of the space $C([0,1])$. Fix $x \in C([0,1])$, $T \geq 0$ and $\varepsilon > 0$. Let us denote

$\omega_0(A) = \lim_{\varepsilon \to 0} \sup\{\omega\ (x, \varepsilon): x \in A\}$

where $A \in \mathcal{P}_b(C\ [0,1])$ and $\omega\ (x, \varepsilon) = sup\{|x(t) - x(s)|: t, s \in [0,1], |t - s| \leq \varepsilon\ \}$.

According to [14] the above defined $\omega_0(A)$ is a regular MNC in the space $C\ [0,1]$.

Then under the following hypotheses:

1) The function $K: [0,1] \times [0,1] \to \mathbb{R}$ is continuous and there exists

   $M = \sup\{|K(t,s)|, s, t \in [0,1]\} < 1$.

2) The function $f: [0,1]^3 \to \mathbb{R}$ is continuous and there exist bounded functions $a(t)$, $b(t)$ defined on $[0,1]$ such that

   $|f(t, x_1, y) - f(t, x_2, y)| \leq a(t)|x_1 - x_2|$,

   $|f(t, x, y_1) - f(t, x, y_2)| \leq b(t)|y_1 - y_2|$.

3) There exists a positive constant α for which f satisfies the following condition

   $f(t, x(t), 0) \leq \alpha$.

We have the following theorem.

Theorem 5.18 Assuming the hypotheses (1-3) hold. Then, Equation (5.4) has at least one solution in $C\ [0,1]$.

Proof In order to study the existence of solution for equation (5.4) we study the existence of fixed points of the following operator

$Fx(t) = f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right)$.

Firstly, we verify that F is a self-mappings. To do this, we fix $x \in B$, then

$$|Fx(t)| = \left|f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right)\right|$$

$$\leq \left|f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right) - f(t, x(t), 0)\right| + |f(t, x(t), 0)|$$

$$\leq \left|f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right) - f(t, x(t), 0)\right| + |f(t, x(t), 0)|$$

$$\leq \quad b(t)\left|\int_0^1 K(s,t)x(s)ds\right| + \alpha$$

$$\leq \quad b(t)\left|\int_0^1 K(s,t)x(s)ds\right| + \alpha$$

$$\leq \quad b(t)M\|x\| + \alpha.$$

Since $b(t)M < 1$, we can found r_0 that satifies the inequality $r \leq \frac{\alpha}{1-b(t)M}$.

Thus, $FB_{r_0} \subset B_{r_0}$ and F is a self mapping.

Next, we verify that F is continuous on B_{r_0}. To do this, we fix $\delta > 0$ and take arbitrary $x, y \in B_{r_0}$ such that $\|x-y\| \leq \delta$. Then for $t \geq 0$,

$$\begin{aligned}
|Fx(t)-Fy(t)| &= \left|f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right) - f\left(t, y(t), \int_0^1 K(s,t)y(s)ds\right)\right| \\
&\leq \left|f\left(t, x(t), \int_0^1 K(s,t)x(s)ds\right) - f\left(t, x(t), \int_0^1 K(s,t)y(s)ds\right)\right| \\
&+ \left|f\left(t, x(t), \int_0^1 K(s,t)y(s)ds\right) - f\left(t, y(t), \int_0^1 K(s,t)y(s)ds\right)\right| \\
&\leq a(t)|x(t)-y(t)| + b(t)\int_0^1 |K(s,t)||x(s)-y(s)|ds \\
&\leq [a(t) + Mb(t)]\|x-y\|.
\end{aligned}$$

Now for $x \in B_{r_0}$, let

$$\int_0^{|Fx(t_1)-Fx(t_2)|} \varphi(r)dr = \int_0^{\left|f\left(t_1, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_2)x(s)ds\right)\right|} \varphi(r)dr$$

$$\leq \int_0^{\substack{\left|f\left(t_1, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right)\right| \\ +\left|f\left(t_2, x(t_1), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_1)x(s)ds\right)\right| \\ +\left|f\left(t_2, x(t_2), \int_0^1 K(s,t_1)x(s)ds\right) - f\left(t_2, x(t_2), \int_0^1 K(s,t_2)x(s)ds\right)\right|}} \varphi(r)dr$$

$$\leq \int_0^{\omega(f,\varepsilon)+a(t)\omega(x,\varepsilon)+b(t)\omega(K,\varepsilon)\int_0^1 |x(s)|ds} \varphi(r)dr,$$

where,

$\omega(f,\varepsilon) = \sup\{|f(t_1,.,.) - f(t_2,.,.)| : t_i \in [0,1], |t_2 - t_1| \leq \varepsilon\}$

and

$\omega(K,\varepsilon) = \sup\{|K(t_1,.,.) - K(t_2,.,.)| : t_i \in [0,1], |t_2 - t_1| \leq \varepsilon\}$.

Since f and K are continuous on $[0,1] \times B_{r_0} \times B_{r_0}$, $[0,1] \times [0,1]$ resp. Then, they are uniformly continuous on $[0,1] \times B_{r_0} \times B_{r_0}$, $[0,1] \times [0,1]$ resp.

Hence,

$\lim_{\varepsilon \to 0} \omega(f, \varepsilon) = \lim_{\varepsilon \to 0} \omega(K, \varepsilon) = 0.$

Consequently,

$$\int_0^{\omega(Fx,\varepsilon)} \varphi(r)dr \leq \int_0^{\sup_{t\in[0,1]}|a(t)|\omega(x,\varepsilon)} \varphi(r)dr.$$

By making a change of variable, we get

$$\int_0^{\omega(Fx,\varepsilon)} \varphi(r)dr \leq \sup_{t\in[0,1]}|a(t)| \int_0^{\omega(x,\varepsilon)} \varphi\left(r \sup_{t\in[0,1]}|a(t)|\right)dr.$$

Since $\sup_{t\in[0,1]}|a(t)| \leq 1$ and φ is increasing then it is obviously that

$$\varphi\left(r \sup_{t\in[0,1]}|a(t)|\right) \leq \varphi(r).$$

Hence,

$$\int_0^{\omega(Fx,\varepsilon)} \varphi(r)dr \leq \sup_{t\in[0,1]}|a(t)| \int_0^{\omega(x,\varepsilon)} \varphi(r)dr.$$

By taking limits, we get

$$\int_0^{\omega_0(Fx)} \varphi(r)dr \leq \sup_{t\in[0,1]}|a(t)| \int_0^{\omega_0(x)} \varphi(r)dr.$$

Then, using Corollary 5.13 the equation (5.4) has at least one solution in $C[0,1]$.

As an example, let solve the following integral equation on $C[0,1]$.

$$\int_0^{x(t)} \varphi(r)dr = \int_0^{-\frac{1}{8}e^{-t}+x(t)+\frac{1}{8}\int_0^1 e^{s-t}x(s)ds} \varphi(r)dr, \qquad (5.5)$$

where $t \in [0,1]$.

We notice that by taking $\varphi(r) = \sqrt{r}$, $K(s,t) = \frac{1}{4}e^{s-t}$ and

$$f\left(t, x(t), y(t)\right) = -\frac{1}{8}e^{-t} + x(t) + \frac{1}{8}\int_0^1 e^{s-t}x(s)ds,$$

we get the integral equation,

$$\int_0^{x(t)} \varphi(r)dr = \int_0^{f\left(t,x(t),\int_0^1 K(s,t)x(s)ds\right)} \varphi(r)dr.$$

Since, for any $t, s \in [0,1]$,

$$|K(s,t)| = \frac{1}{4}e^{s-t} \leq \frac{e^s}{4} \leq \frac{e}{4} \approx 0.68.$$

In further, we know that $x(t) \in C[0,1]$, then

$f(t, x(t), 0) = -\frac{1}{8}e^{-t} + x(t) \leq \frac{7}{8} = \alpha.$

It is easy to see that $a(t) = 1$ and $a(t) = \frac{1}{2}$, since

$\left|f\left(t, x(t), y(t)\right) - f\left(t, u(t), y(t)\right)\right| = |x(t) - u(t)|,$

$\left|f\left(t, x(t), y(t)\right) - f\left(t, x(t), v(t)\right)\right| = |y(t) - v(t)|.$

Then by Theorem 5.18, the esquation (5.5) has at least one solution in $C[0,1]$. We notice that, $\int_0^{e^{-t}} \sqrt{r}dr = \frac{2}{3}e^{-\frac{3}{2}t}$ is one of the solutions of equation (5.5) and the following is a representation graphic of the solution:

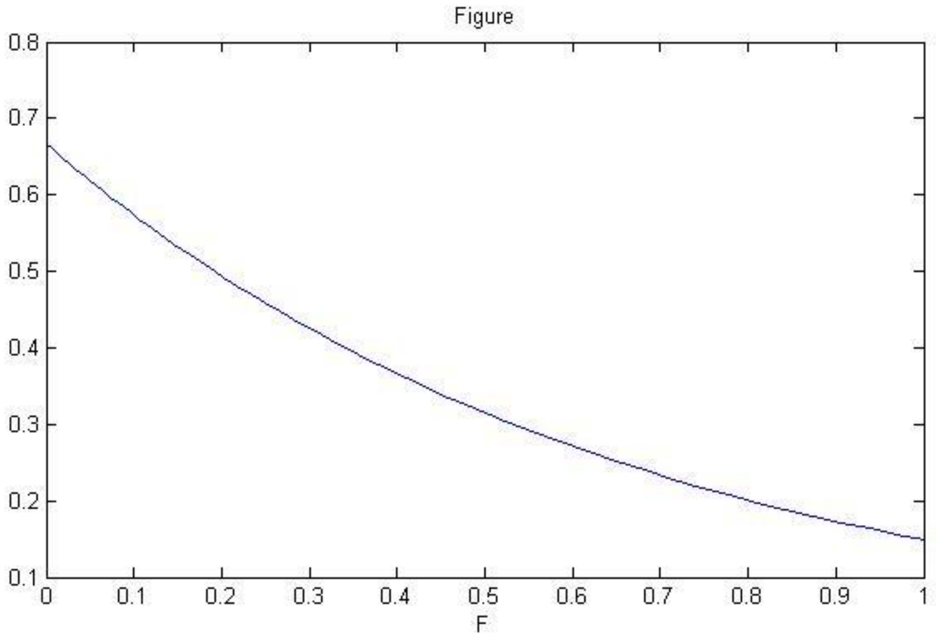

Figure 5.1 Representation graphic of the solution

CHAPTER 6

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS WITHOUT THE COMPACTNESS CONDITION

In this chapter, a version of Meir-Keeler theorem for condensing multivalued mappings is given. In further, a version of Caristi theorem is presented for multivalued mappings and some related results. Moreover, a theorem of fixed point for power multivalued set contraction mappings is proved. Finally, to indicate the applicability of the obtained results a nonlocal differential evolution equation is solved.

6.1 Theorems for Multivalued Meir-Keeler Set Contraction Mappings

Definition 6.1 A multivalued mapping N is said to be Meir-Keeler condensing multivalued mapping if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(A) < \varepsilon + \delta \ \Rightarrow \mu(NA) < \varepsilon.$$

Remark 6.2 The condensing multivalued mappings of Meir-Keeler type are more general than condensing mappings. Indeed, let N be a condensing mapping, i.e.,

$\mu(NA) \leq k\mu(A)$ for any bounded $A \subset X$.

Suppose that for $\delta = \left(\frac{1}{k} - 1\right)\varepsilon$, we have $\varepsilon \leq \mu(A) < \varepsilon + \delta$.

Then,

$$\mu(NA) \leq k\mu(A) < \varepsilon + k\left(\frac{1}{k} - 1\right)\varepsilon = \varepsilon.$$

Thus, N is a Meir-Keeler condensing multivalued mapping.

The inverse is not true.

Theorem 6.3 Let X be a Banach space and A be a nonempty closed, bounded and convex subset of X. Let $N: X \to \mathcal{P}_{cl,cv}(X)$ be a multivalued upper semicontinuous mapping such that for any bounded $W \subseteq A$, we have

$\varepsilon \leq \mu(W) < \varepsilon + \delta \ \ \Rightarrow \mu(NW) < \varepsilon.$

Then, W has at least one fixed point in A.

Proof Obviously, if we have $\varepsilon \leq \mu(W) < \varepsilon + \delta \ \ \Rightarrow \mu(NW) < \varepsilon.$

Then,

$\mu(NA) < \mu(A).$

Thus by Theorem 3.4, N has at least one fixed point.

Corollary 6.4 Let X be a Banach space and $N: X \to \mathcal{P}_{cl,cv}(X)$ be a multivalued mapping with convex values, closed graph and bounded range such that for any bounded $A \subset X$, we have

$\mu(NA) \leq k\mu(A)$ for $0 \leq k < 1$.

Then, N has at least one fixed point in A.

6.2 Fixed Point Theorem for Multivalued Set Contraction Mappings of Caristi Type

In the following an existence theorem for multivalued set contraction mappings of Caristi type.

Theorem 6.5 Let X be a Banach space and A be a nonempty closed, bounded and convex subset of X. Let $N: X \to \mathcal{P}_{cl,cv}(X)$ be a multivalued upper semi-continuous mapping such that for any bounded $W \subseteq A$, we have

$$\Psi\big(\mu(NW)\big) \leq \Psi\big(\mu(W)\big) - \varphi\big(\mu(W)\big), \tag{6.1}$$

where μ is an arbitrary MNC and $\Psi, \varphi: \mathbb{R}_+ \to \mathbb{R}_+$ are given functions such that φ is lower semi-continuous and Ψ is continuous on $\mathbb{R}_+$. Moreover, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then, N has at least one fixed point in A.

Proof Define the sequence $W_0 = W$ and $W_{n+1} = \overline{conv}(NW_n)$, clearly $(W_n)_{n\in\mathbb{N}}$ is a nonempty closed, bounded, convex sequence and

$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n.$

Since the sequence $\left(\mu(W_n)\right)_{n\in\mathbb{N}}$ is decreasing and bounded below ($\mu(W_n) > 0$), then $\left(\mu(W_n)\right)_{n\in\mathbb{N}}$ is a convergent sequence. Put $\lim_{n\to\infty} \mu(W_n) = l$.

In further, using properties of the MNC we have,

$\mu(W_{n+1}) = \mu\left(\overline{conv}(NW_n)\right) = \mu(NW_n)$.

Then, in view of condition (6.1), we have

$$\Psi\left(\mu(W_{n+1})\right) = \Psi\left(\mu(NW_n)\right)$$
$$\leq \Psi\left(\mu(W_n)\right) - \varphi\left(\mu(W_n)\right).$$

By taking the limit sup, we get

$$\limsup_{n\to\infty} \Psi\left(\mu(W_{n+1})\right) \leq \limsup_{n\to\infty} \Psi\left(\mu(W_n)\right) - \liminf_{n\to\infty} \varphi\left(\mu(W_n)\right).$$

Since Ψ is continuous and φ is lower semi-continuous, we get

$\Psi(l) \leq \Psi(l) - \varphi(l)$.

Hence we obtain that $\varphi(l)$ must be null, which means that $l = 0$. Thus,

$$0 = \limsup_{n\to\infty} \mu(W_n) = \liminf_{n\to\infty} \mu(W_n) = \lim_{n\to\infty} \mu(W_n).$$

Hence using Definition 2.74, we get $W_\infty = \bigcap_{n=0}^{\infty} W_n$ is compact. Then, N has at least one fixed point.

6.3 Fixed Points for Multivalued Power Set Contraction Mappings

The following theorem guarantees the existence of fixed points for multivalued power set contraction mappings.

Theorem 6.6 Let A be a nonempty closed, bounded and convex subset of a Banach space X and $N: X \to \mathcal{P}_{cl,cv}(X)$ be a k-set contraction mapping on A. Then, N^n (for an integer $n > 0$) is a k^n-set contraction on A.

Proof: Let A be a nonempty closed, bounded and convex subset of X, then for any bounded $W \subseteq A$, then we have

$$\mu(N^n W) = \mu\left(N(N^{n-1}W)\right)$$
$$\leq k\, \mu(N^{n-1}W)$$

$$\leq k^2\, \mu(N^{n-1}W)$$

$$\vdots$$

$$\leq k^n\, \mu(W).$$

Since $0 \leq k < 1$, hence $0 \leq k^n < 1$ and so N^n is also a k-set contraction mapping.

Remark 6.7 As we shown in the single valued case the inverse is not true. That is, if N^n is a k-set contraction mapping then N may not be a k-set contraction mapping.

Theorem 6.8 Let A be a nonempty closed, bounded and convex subspace of a Banach space X and $N: X \to \mathcal{P}_{cl,cv}(X)$ be an upper semi-continuous multivalued mapping such that for any $n \geq 1$ we have $N^n\big(conv(W)\big) \subseteq conv(N^n W)$ and

$$\mu(N^n W) \leq k_n \mu(W) \tag{6.2}$$

where W is any bounded subset of A and $k_n \to 0$, $n \to +\infty$. Then, there exists at least one x such that $x \in Nx$.

Proof Let the iteration $W_0 = W$ and $W_{n+1} = \overline{conv}(NW_n)$, clearly $(W_n)_{n\in\mathbb{N}}$ is a nonempty closed, bounded, convex sequence and

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n.$$

Then, by using the properties of the MNC, we get

$$\mu(W_n) = \mu\big(\overline{conv}(NW_{n-1})\big) = \mu(NW_{n-1})$$

$$\leq \mu\Big(N\big(\overline{conv}(NW_{n-2})\big)\Big)$$

$$\leq \mu\big(\overline{conv}N(NW_{n-2})\big)$$

$$= \mu(N^2 W_{n-2}).$$

Repeating this process many times we get

$$\mu(W_n) \leq \mu(N^n W_0).$$

Using inequality (6.2), we get

$$\mu(W_n) \leq \mu(N^n W_0) \leq k_n \mu(W_0).$$

By taking the limit, we get $\lim_{n\to\infty} \mu(W_n) = 0$ which implies that $W_\infty = \bigcap_{n=0}^{\infty} W_n$ is compact. Hence, N has at least one fixed point in $W_\infty \subseteq A$.

6.4 Application to Differential Evolution Inclusions

The multivalued fixed point theorems of this section can have some nice applications to differential and integral inclusions as an example we choose to provide an application for Theorem 6.3.

One can notice that other applications can be given by changing the contractive condition which the mappings are supposed to satisfy.

Let following evolution differential inclusion with a nonlocal condition

$$y'(t) \in A(t)y(t) + F\big(t, y(t)\big),\ t \in J := [0, b] \tag{6.3}$$

$$y(0) = \varphi(y), \tag{6.4}$$

where F is an upper Caratheodory multimap, $\varphi: C(J, X) \to X$ is a given X-valued function. $\{A(t): t \in J\}$ is a family of linear closed unbounded operators on X with domain $D\big(A(t)\big)$ independent of t that generates an evolution system of operators

$\{U(t, s): t, s \in \Delta\}$ with $\Delta = \{(t, s) \in J \times J: 0 \le s \le t \le b\}$.

The main work for this section is to study the existence of mild solutions for this non-local inclusion.

Define the set

$S_F(y) = \{f \in L^1(J, X): f(t) \in F\big(t, y(t)\big)\}$.

Definition 6.9 The function $y \in C(J, X)$ is say to be a mild solution of the evolution system $(6.3) - (6.4)$ if it satisfies the following integral equation

$y(t) = U(t, 0)\varphi(y) + \int_0^t U(t, s)f(s)ds,$

for all $t \in J$ and $f \in S_F(y)$.

Let assume the following hypotheses which are needed thereafter:

1) $\{A(t): t \in J\}$ is a family of linear operators. $A(t): D(A) \subset X \to X$ generates an equicontinuous evolution system $\{U(t, s): (t, s) \in \Delta\}$.
2) The multifunction $F: J \times C(J, X) \to \mathcal{P}_{cl,cv}(X)$ is an upper Carathéodory and $\varphi: C(J, X) \to X$ is continuous. Moreover, if we have for any $\varepsilon > 0$ there exists $\delta > 0$ such that

   $\varepsilon \le \mu(W) < \varepsilon + \delta$ for any bounded $W \subseteq A$ implies

$\mu(\varphi(W)) < \frac{\varepsilon}{2M}$ and $\mu(F(t,W)) < \frac{\varepsilon}{2Mt}$ for any $t \in J$.

3) There exists a constant $r > 0$ such that

$M[\|\varphi(y)\| + \{\|f(t)\|_1\colon\ f \in S_F(y),\ y \in W\}] \leq r,$

where $W = \{y \in C(J,X)\colon \|y(t)\| \leq r\ for\ any\ t \in J\}$.

Theorem 6.10 Under the assumptions $(1-3)$ the non local problem $(6.3)-(6.4)$ has at least one mild solution in the space $C(J,X)$.

Proof To solve problem $(6.3)-(6.4)$ we transform it to the following fixed point problem.

Consider the multivalued operator $N\colon C(J,X) \to \mathcal{P}(C(J,X))$ defined by

$$N(y) = \left\{h \in C(J,X)\colon\ h(t) = U(t,0)\varphi(y) + \int_0^t U(t,s)f(s)ds,\ \ f \in S_F(y)\right\}.$$

We can notice that fixed points of the operator N are the mild solutions of problem $(6.3)-(6.4)$.

Clearly for each $y \in C(J,X)$, the set $S_F(y)$ is nonempty. Indeed from Theorem 2.58, F is an upper Carathéodory multimap defined on the separable space $C(J,X)$, then it has at least one measurable selection.

To prove that N has a fixed point, we need to satisfy all conditions of one of the above theorems, as example we choose Theorem 6.3.

Let $W = \{y \in C(J,X)\colon \|y(t)\| \leq r\ for\ any\ t \in J\}$. Obviously, W is closed, bounded and convex.

To show that $NW \subseteq W$, we need first to prove that the family

$$\left\{\int_0^t U(t,s)f(s)ds : f \in S_F(y)\ and\ y \in W\right\},$$

is equicontinuous for $t \in J$ that is all the functions are continuous and they have equal variation over a given neighborhood.

In view of hypothesis (1), we have that functions in the set $\{U(t,s)\colon t,s \in \Delta\}$ are equicontinuous i,e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t-\tau| < \delta$ implies $\|U(t,s) - U(\tau,s)\| < \varepsilon$ for all $U(t,s) \in \{U(t,s)\colon t,s \in \Delta\}$.

Then given some $\varepsilon > 0$, let $\delta = \frac{\varepsilon\prime}{\varepsilon\|f\|_\infty}$ such that $|t-\tau| < \delta$, we have

$$\left|\int_0^t U(t,s)f(s)ds - \int_0^\tau U(\tau,s)f(s)ds\right| \le \int_\tau^t |U(t,s) - U(\tau,s)||f(s)|ds.$$

Regarding the fact that $\{U(t,s): t,s \in \Delta\}$ is equicontinuous then

$$\left|\int_0^t U(t,s)f(s)ds - \int_0^\tau U(\tau,s)f(s)ds\right| \le \varepsilon\|f\|_\infty|t-\tau|$$

$$\le \varepsilon\|f\|_\infty \frac{\varepsilon\prime}{\varepsilon\|f\|_\infty} = \varepsilon'.$$

Hence, $\left\{\int_0^t U(t,s)f(s)ds : \ f \in S_F(y) \ and \ y \in W\right\}$ is equicontinuous for $t \in J$.

Now, let show that $NW \subseteq W$. Let for $t \in J$,

$$|h(t)| = \left|U(t,0)\varphi(y) + \int_0^t U(t,s)f(s)ds\right|$$

$$\le |U(t,0)\varphi(y)| + \int_0^t |U(t,s)f(s)|\,ds$$

$$\le M\|\varphi(y)\| + M\|f\|_1$$

$$\le M[\|\varphi(y)\| + \|f\|_1] \le r.$$

Consequently $NW \subseteq W$.

In further, it is easy to see that N has convex valued.

Moreover to prove that N has a closed graph. Suppose that $y_n \to y$ and $h_n \to h$ such that $h_n(t) \in N(y_n)$ and let show that $h(t) \in N(y)$.

Then, there exists a sequence $f_n \in S_F(y_n)$ such that

$$h_n(t) = U(t,0)\varphi(y_n) + \int_0^t U(t,s)f_n(s)ds\,.$$

Consider the linear operator $\Phi: L^1(J,X) \to C(J,X)$ defined by

$$\Phi f(t) = \int_0^1 U(t,s)f(s)ds.$$

Clearly, Φ is linear and continuous. Then using Theorem 2.59, we have $\Phi \circ S_F(y)$ is a closed graph operator.

In further, we have

$$h_n(t) - U(t,0)\varphi(y_n) \in \Phi \circ S_F(y).$$

Since $y_n \to y$ and $h_n \to h$, then

$$h_n(t) - U(t,0)\varphi(y_n) \in \Phi \circ S_F(y_n).$$

In other words, there exists a function $f \in S_F(y)$ such that

$h(t) = U(t,0)\varphi(y) + \int_0^t U(t,s)f(s)ds.$

Therefore N has a closed graph, hence N has closed values on $C(J,X)$.

Let W be a bounded subset of A such that

$\varepsilon \leq \mu(W) < \varepsilon + \delta.$

We know that the family $\left\{\int_0^t U(t,s)f(s)ds : f \in S_F\big(W(t)\big)\right\}$ is equicontinuous.

Hence by Lemma 2.77, we have

$$\begin{aligned}\mu\left(\int_0^t U(t,s)f(s)ds : f \in S_F\big(W(t)\big)\right) &\leq \int_0^t \mu\left(U(t,s)f(s)ds : f \in S_F\big(W(t)\big)\right)\\ &\leq M\int_0^t \mu\left(f(s)ds : f \in S_F\big(W(t)\big)\right)\\ &\leq Mt\mu\left(F\big(t,W(t)\big)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mu\big(NW(t)\big) &= \mu\left(U(t,0)\varphi\big(W(t)\big) + \int_0^t U(t,s)f(s)ds, f \in S_F\big(W(t)\big)\right)\\ &\leq \mu\left(U(t,0)\varphi\big(W(t)\big)\right) + \mu\left(\int_0^t U(t,s)f(s)ds, f \in S_F\big(W(t)\big)\right)\\ &\leq M\mu\left(\varphi\big(W(t)\big)\right) + Mt\mu\left(F\big(t,W(t)\big)\right).\end{aligned}$$

In view of hypothesis (2), we get

$\mu\big(NW(t)\big) \leq M\frac{\varepsilon}{2M} + Mt\frac{\varepsilon}{2Mt} = \varepsilon.$

Therefore, for $\varepsilon \leq \mu \leq \varepsilon + \delta$ we obtained $\mu\big(NW(t)\big) \leq \varepsilon$.

Thus regarding Theorem 6.3, N has at least one fixed point, hence the problem $(6.3) - (6.4)$ has at least one solution.

CHAPTER 7

RESULTS AND DISCUSSION

In this thesis, most theorems of the fixed point theory were surveyed and meaningful generalizations were obtained.

For starters, combining the concept of MNC and fixed point theory, different types of theorems on existence of fixed points were proved. The results given can be considered as generalizations for a lot of works existing in the literature. Furthermore, in order to illustrate and emphasize the importance of these results, they were used to solve some integral and differential equations. However, these results are not the only generalizations that could be extracted. For example, the contractive conditions could be weakened to set contraction mappings of integral type.

Moreover, existence of fixed points for commutative set contraction mappings was established. Also, some new classes of set contraction mappings were deduced and then the existence of their fixed points was guaranteed. As application, a new type of integral equation was introduced and its solvability was investigated. As results, it is natural to wonder what happens for non-commutative set contraction mappings and under what conditions, one can guarantee the existence of their fixed points.

Finally, employing the MNC, different types of existence theorems were shown for multivalued set contraction mappings. The obtained results were used to solve some differential inclusion with nonlocal conditions. One could use these results to investigate the existence of coupled, tripled and n-tupled fixed points for multivalued set contraction mappings.

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PUBLISHMENTS

Papers

1. Karakaya V., Bouzara N.H, Dogan K., Atalan Y., (2014). "Existence of Tripled Fixed Points for a Class of Condensing Operators in Banach Spaces", The Scientific World Journal, 2014:1-9.
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2. Bouzara N.H., Karakaya V., Doğan K., Atalan Y., (2014). "Existence of Fixed points for Continuous Operators in Banach Spaces Using Measure of Noncompactness and Under an Integral Condition", ICRAPAM, 6-9 November 2014, Antalya, Turkey.
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4. Bouzara N.H., Karakaya V., (2015). "Application of Equilibria and Fixed Point Theory in Solving Differential Equations", ICFPTA, 20-24 July 2015, Istanbul, Turkey.
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9. Bouzara N.H., Karakaya V., (2016). “On different types of fixed point theorems for multivalued mappings via measure of noncompactness”, International Conference On Analysis and its Applications, 12-15 July 2016, Kirsehir, Turkey.
10. Sekman D., Bouzara N.H., Karakaya V., (2016). “On Existing Of Fixed Point For Multivalued Mapping Via Measure Of Noncompactness”, International Conference On Analysis and its Applications, 12-15 July 2016, Kirsehir, Turkey.