REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

A STUDY OF LEBESGUE CONSTANTS IN BARYCENTRIC RATIONAL AND MULTIVARIATE POLYNOMIAL INTERPOLATION

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İSTANBUL, 2014

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This study was supported by Research Grant 2214/B from the Scientific and Technological Research Council of Turkey (TUBITAK) for 12 months.

ACKNOWLEDGMENTS

This is a joint doctoral study between the Universiteit Antwerpen (Belgium) and Yıldız Technical University (Turkey).

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Department of Mathematics

Ph.D. Thesis

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The Lebesgue constant is a valuable numerical instrument for linear interpolation, because it indicates how the interpolant of a function compares to the best linear approximant of that function. Furthermore, if the interpolant is computed by making use of the Lagrange basis functions, then the Lebesgue constant also expresses the conditioning of the interpolation problem at hand. Many publications have been devoted to the search for optimal interpolation points, optimal in the sense that these points lead to a minimal Lebesgue constant for interpolation problems on the interval [-1,1].

In this thesis, the best results obtained in univariate polynomial interpolation are generalized to univariate rational interpolation. In addition, this generalization provides a very practical and useful result in the case of barycentric rational interpolation, where simple equidistant interpolation points apparently yield very slowly increasing Lebesgue constants.

The literature demonstrates a direct link between the orthogonality of polynomials and optimal interpolation points for polynomial interpolation. In this thesis, this connection is further explored for the case of linear interpolation, using rational functions with a predetermined denominator (preassigned poles) on the one hand and multivariate polynomial functions on the unit disk on the other hand.

Keywords: Lebesgue constants, condition number, polynomial interpolation, barycentric rational interpolation, interpolation points, preassigned poles, orthogonal polynomials.

YILDIZ TECHNICAL UNIVERSITY

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ÖZET

A STUDY OF LEBESGUE CONSTANTS IN BARYCENTRIC RATIONAL AND MULTIVARIATE POLYNOMIAL INTERPOLATION

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Lebesgue sabiti (Lebesgue constant) lineer özellikli interpolasyon için, bir fonksiyonun interpolasyonu ile o fonksiyonun en iyi lineer yaklaşımının kıyaslanması bakımından çok değerli bir nümerik enstrümandır. Dahası, eğer interpolasyon Lagrange bazları kullanılarak hesaplanıyorsa Lebesgue sabiti interpolasyon yönteminin koşullanmasını ifade eder. Bu bağlamda optimal interpolasyon noktalarının araştırılması pek çok bilimsel yayının konusu olmuştur. Burada optimallik [-1,1] aralığında Lebesgue sabitini minunuma götüren noktaları ifade etmektedir.

Bu tezde, tek değişkenli polinom tipi interpolasyon yöntemi için elde edilmiş sonuçlar tek değişkenli rasyonel tipi interpolasyon yöntemine genelleştirilmiştir. Buna ilaveten, eşit aralıklı interpolasyon noktaları belirgin bir şekilde çok yavaş büyüyen Lebesgue sabitlerine sahip olması dolayısıyla, bu genelleme barisentrik rasyonel interpolasyon yöntemi için çok kullanışlı ve pratik sonuç sağlar.

Literatür polinom tipi interpolasyon için optimal interpolasyon noktaları ile polinomların ortogonalliği arasında doğrudan bir ilişki olduğunu gösterir. Bu tezde bu ilişki durumu lineer özellikteki interpolasyon yöntemleri için, bir yandan paydası önceden belirlenmiş tekil noktalar (preassigned poles) kullanılarak rasyonel fonksiyonlar için, diğer yandan ise birim disk üzerinde çok değişkenli polinom tipi fonksiyonlar için araştırılmıştır

Anahtar Kelimeler: Lebesgue sabitleri, koşul sayısı, polinom interpolasyon, barisentrik rasyonel interpolasyon, interpolasyon noktaları, önceden belirlenmiş tekil noktalar, ortogonal polinomlar.

YILDIZ TEKNİK ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ

SAMENVATTING

A STUDY OF LEBESGUE CONSTANTS IN BARYCENTRIC RATIONAL AND MULTIVARIATE POLYNOMIAL INTERPOLATION

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De Lebesgue constante is een waardevol numeriek instrument bij lineaire interpolatie, omdat ze aangeeft hoe de interpolant van een functie vergelijkt met de beste lineaire approximant van die functie. Als de interpolant tevens berekend wordt met behulp van de Lagrange basisfuncties, drukt de Lebesgue constante ook nog eens de conditionering uit van het interpolatieprobleem in kwestie. Er zijn al zeer veel publicaties gewijd aan de zoektocht naar optimale interpolatiepunten, optimaal in de zin dat de punten een minimale Lebesgue constante opleveren voor interpolatieproblemen geformuleerd in het standaard interval [-1, 1].

In deze thesis worden de beste resultaten, behaald in univariate veelterminterpolatie, veralgemeend naar univariate rationale interpolatie. Deze veralgemening levert daarenboven een zeer praktisch bruikbaar resultaat op, omdat blijkbaar bij barycentrische rationale interpolatie eenvoudige equidistante interpolatiepunten zeer traaggroeiende Lebesgue constanten opleveren.

Uit de wetenschappelijke literatuur blijkt ook de directe link tussen orthogonaliteit van veeltermen en optimale interpolatiepunten voor veelterminterpolatie. Deze link wordt in de thesis verder uitgediept voor de interpolatie, met enerzijds rationale functies met vooraf vastgelegde noemer (of polen) en anderzijds multivariate veeltermfuncties (op de gesloten eenheidsschijf).

Keywords: Lebesgueconstanten, conditiegetal gegeven, veelterminterpolatie, barycentrische rationale interpolatie, interpolatiepunten, vooraf vastgelegde polen, orthogonale veeltermen.

CHAPTER 1

INTRODUCTION

1.1 Literature Review

The Lebesgue constant is a valuable numerical instrument for linear interpolation, because it provides a measure of how close the interpolant of a function is to the best linear approximant of the function. Moreover, if the interpolant is computed by using the Lagrange basis, then the Lebesgue constant also expresses the conditioning of the interpolation problem. In addition, many publications have been devoted to the search for optimal interpolation points, in the sense that these points lead to a minimal Lebesgue constant for the interpolation problems on the interval [-1, 1]. An explicit formula for the x_j that minimize the Lebesgue constant is not known, and if no further constraints are imposed on the interpolation points, then the solution is not even unique. But it is proved in [1, 2, pp. 110-121] that the minimal growth of the Lebesgue constant, in terms of the number of interpolation points $n+1$, is given by

$$\frac{2}{\pi}\left(\ln(n+1)+\gamma+\ln\left(\frac{4}{\pi}\right)\right) \sim \frac{2}{\pi}\ln(n+1)+0.52125\ldots$$

with γ the Euler constant.

Several node sets $\{x_0,\ldots,x_n\}$ come close to realizing this minimal growth, among which the Chebyshev zeroes [3, 4, 5] and the Fekete points [6]. The node set known in closed form that approximates the optimal node set best is probably the so-called extended Chebyshev node set given by

$$x_j = -\frac{\cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}{\cos\left(\frac{\pi}{2(n+1)}\right)}, \qquad j=0,\ldots,n.$$

The division by $\cos(\pi/(2n+2))$ guarantees that $x_0 = -1$ and $x_n = 1$. The growth of the Lebesgue constant for the extended Chebyshev nodes is bounded by [5, 7]

$$\Lambda_n(x_0,\ldots,x_n) < \frac{2}{\pi}\log(n+1) + 0.5829\ldots, \qquad n \geq 4,$$

which is only slightly larger than the minimal growth. At the same time, it is known that the Lebesgue constant Λ_n for equidistant interpolation points grows exponentially [8, 9].

A rough analysis of the growth of the Lebesgue constant in the case of barycentric rational interpolation at equidistant interpolation points, was made in [10] and [11], leading to the conclusion that it only grows logarithmically.

The minimal growth of the Lebesgue constant for bivariate polynomial interpolation is different for different bivariate domains. For instance, on the square the minimal order of growth is $O(\ln^2(n+1))$ and this order is achieved for the configurations of interpolation points given in [12] and [13]. On the disk the minimal order of growth is quite different, namely $O(\sqrt{n+1})$, as proved in [14]. No configurations of interpolation points obeying this order of growth are known. On the simplex the minimal order of growth is not even known. Instead, in [15] some (non closed form) configurations of interpolation points are obtained from the solution of a minimization problem. There is clearly a lot of interest in the problem.

1.2 Objective of the Thesis

The search for sets of good interpolation points is highly motivated by the fact that, due to the finite precision of digital computers, valid results can only be expected when the interpolation problem is well-conditioned. In addition, the conditioning of polynomial interpolation and of rational interpolation with preassigned poles is measured by the respective Lebesgue constants.

Although near-optimal choices for basis and interpolation points are known in the univariate case, little or nothing is known in the multivariate case. And when generalizing the problem to rational interpolation (with a prescribed denominator to stick to a linear problem statement), only partial results have been discovered. In this thesis we tackle both these problems, the latter in more detail than the former.

1.3 Hypothesis of the Thesis

The choice of the polynomial basis and the location of the interpolation points play an important numerical role in univariate polynomial interpolation. The best results obtained in univariate polynomial interpolation can be generalized to univariate rational interpolation with preassigned poles discussed in Chapters 3 to 4. If the choice of the polynomial basis and the location of the interpolation points play an important numerical role in univariate polynomial interpolation, they do so even more in the multivariate case discussed in Chapter 5. In this thesis, we also point out the many links of the close connection between the orthogonality of polynomials and optimal interpolation points under discussion with the existing literature.

CHAPTER 2

STATE OF THE ART IN UNIVARIATE POLYNOMIAL INTERPOLATION

The classical problem of polynomial interpolation through data $f_0, \ldots, f_n$ given at interpolation points $x_0, \ldots, x_n$, can be expressed in several polynomial bases, each giving rise to a different linear system of interpolation conditions. Among others, we mention the standard monomial basis, the Newton basis for use with divided differences, the Lagrange basis for a simple explicit formula, or a choice of an orthogonal polynomial basis. Although the choice of the basis does not make a difference from a mathematical point of view, it influences the conditioning of the problem when computing the interpolating polynomial numerically.

But the nature of the interpolation points always plays an important role, both mathematically and numerically. Suffices to mention the well-known Runge phenomenon to understand this. The quality of the polynomial interpolant depends heavily on the location of the interpolation points. The general recommendation is to have significantly more interpolation points towards the boundary of the interval. In addition, the conditioning of the interpolation problem may vary greatly with the location of the points as well.

In Section 1 we introduce the univariate polynomial interpolation problem, for which we give two useful error formulas in Section 2. The conditioning of polynomial interpolation is analysed in detail in Section 3, in case of the Lagrange basis, and in Section 4 for other bases. For the Lagrange form the condition number is the Lebesgue constant. For other representations we inspect the condition number of the (generalized) Vandermonde matrix.

2.1 Problem statement

In the classical (polynomial) theory of interpolation, several forms can be used to write down the polynomial interpolation problem. This section introduces two of them: the Lagrange form, to be used in the definition of the Lebesgue constant in Section 2.3, and the (generalized) Vandermonde system, to be used in the definition of the condition number in Section 2.4.

For convenience, without loss of generality, we work with the interval $[-1,1]$, although what we have to say about polynomial interpolation may be applied to any finite interval by making a linear change of variable.

In this study, we assume that the values of a function are known a priori at some points. The interpolating polynomial is then constructed from this information.

2.1.1 Lagrange form

For $n \in \mathbb{N}$, let

$$X = \{x_j : j = 0, 1, \ldots, n\} \tag{2.1}$$

be a set of $n+1$ distinct interpolation points (or nodes) on the real interval $[-1,1]$ such that

$$-1 \leq x_0 < x_1 < \cdots < x_n \leq 1. \tag{2.2}$$

Let the function $f \in C([-1,1])$. When approximating f by an element from a finite-dimensional vector space $\mathscr{V}_n = \text{span}\{\phi_0, \phi_1, \ldots, \phi_n\}$ and if $\phi_0, \phi_1, \ldots, \phi_n$ are a Chebyshev system with $\phi_i \in C([-1,1])$ for $0 \leq i \leq n$, it is well known that there exists a unique generalized polynomial

$$p_n(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

such that

$$p_n(x_j) = \sum_{i=0}^{n} a_i \phi_i(x_j) = f(x_j), \qquad j = 0, \ldots, n. \tag{2.3}$$

Let $\mathscr{P}_n$ be the $(n+1)$ - dimensional vector space of polynomials of degree at most n,

$\mathscr{P}_n = \text{span}\{1, x, \ldots, x^n\}$.

The operator that associates with f its polynomial interpolant $p_n(x)$ can be expressed as

$$P_n[x_0,\ldots,x_n] : C([-1,1]) \to \mathscr{P}_n :$$
$$f(x) \to p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x) \tag{2.4}$$

where the basic Lagrange polynomials $\ell_i(x)$ are given by

$$\ell_i(x) = \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}. \tag{2.5}$$

The polynomials $\ell_i(x)$ have the property

$$\ell_i(x_j) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, \ldots, n.$$

2.1.2 Vandermonde system

An alternative method (and a natural approach) to the interpolation problem in (2.3) is to consider it directly in matrix form as $\Phi(x_0,\ldots,x_n)a = y$, where $y_i = f(x_i)$, or in detail as

$$\begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}. \tag{2.6}$$

The $n \times n$ matrix Φ appearing here is called the generalized Vandermonde matrix. The $\det\Phi(x_0,\ldots,x_n) \neq 0$ for any set of distinct points $x_0, x_1, \cdots, x_n$ in $[-1,1]$, if $\phi_0, \phi_1, \ldots, \phi_n$ are a Chebyshev system [16] on the interval $[-1,1]$.

In (2.6), replacing ϕ_i by x^i gives the special case

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} \tag{2.7}$$

which has a unique solution for the real coefficients a_j. These coefficients can be computed by solving a structured linear system of equations, the so-called Vandermonde system, with a coefficient matrix given by

$$V_n(x_0,\cdots,x_n) = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}. \tag{2.8}$$

2.2 Interpolation error

2.2.1 Explicit formula

When we interpolate the function $f \in C^{(n+1)}([-1,1])$ over the interval $[-1,1]$ at the $(n+1)$ distinct points x_j, the error associated with $f(x)$ and its polynomial interpolant $p_n(x)$ at a non-interpolation point x, can be expressed as

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \tag{2.9}$$

where the point $\xi_x \in (x_0, x_n)$ and $f^{(n+1)}(\xi_x)$ is the $(n+1)$ -st derivative of $f(x)$ evaluated at ξ_x. The point ξ_x depends on the function f, the points x_j and the evaluation point x. In most cases, the value of ξ_x is not known exactly and the following estimate becomes important [17, pp. 56-57]:

$$||f - p_n||_\infty \le \max_{x\in[-1,1]} \left(\frac{|f^{(n+1)}(x)|}{(n+1)!} \right) \max_{x\in[-1,1]} \prod_{j=0}^{n} |x - x_j|. \tag{2.10}$$

Here and throughout this chapter, except where indicated otherwise, $\|\cdot\|$ denotes the maximum norm (∞ -norm or uniform norm) on $[-1,1]$ defined by

$$\|f\|_\infty = \max_{-1\le x\le 1} |f(x)|, \quad f \in C([-1,1]).$$

In many cases, of course, the actual error is much smaller than predicted by (2.10). This inequality indicates that any influence on the interpolation error can only be expected from controlling the location of the points x_j because the behavior of f (and its derivatives) is fixed by the problem. Therefore, it is natural to look for x_j such that $\|(x-x_0)\cdots(x-x_n)\|_\infty$ is minimized.

It is well known that

$$\|(x-x_0)\cdots(x-x_n)\|_\infty$$

is minimized on $[-1,1]$ with minimum value 2^{-n}, by choosing $x_0,\ldots,x_n$ as the zeros of the $(n+1)$ - th degree Chebyshev polynomial of the first kind $T_{n+1}(x) = \cos((n+1)\arccos x)$, in other words [18]

$$\prod_{j=0}^{n}(x-x_j) = \frac{1}{2^n}T_{n+1}(x).$$

Then we obtain the following upper bound for the maximum error

$$\|f-p_n\|_\infty \le \left(\frac{1}{2^n(n+1)!}\right)\max_{x\in[-1,1]}\left|f^{(n+1)}(x)\right|.$$

If we choose equidistant interpolation points $x_0=-1, x_n=1, x_i=-1+\frac{2i}{n}$, then we obtain the following estimate [19, pp. 266-267]

$$\frac{(2n)!}{2^{2n+1}n!} \le \|(x-x_0)\cdots(x-x_n)\|_\infty.$$

On comparing 2^{-n} and $\frac{(2n)!}{2^{2n+1}n!}$, using Stirling's formula for large n, it follows that

$$2^{-n} \div \frac{(2n)!}{2^{2n+1}n!} = \frac{n}{\sqrt{2}}(0.6796\ldots)^n.$$

This indicates that the ratio of the maximum error factors decreases at least exponentially.

2.2.2 Mini-max polynomial approximation

A best polynomial approximation p_n^* in terms of the maximum norm is called a mini-max approximation, because

$$\|f-p_n^*\|_\infty \le \|f-p_n\|_\infty, \quad p_n \in \mathscr{P}_n.$$

Mini-max approximations are important because of the famous Weierstrass approximation theorem [20] stating that every continuous function over a closed interval can be approximated to arbitrary accuracy by polynomials.

Theorem 2.2.1 (Weierstrass) *Given $f \in C([-1,1])$ and $\varepsilon > 0$, there exist n and a polynomial p_n of degree n, such that*

$$\|f - p_n\|_\infty = \max_{-1\le x\le 1} |f(x) - p_n(x)| < \varepsilon.$$

The first constructive proof of this theorem was given by Bernstein (1912). A detailed explanation of this proof can be found in [17, pp. 108-111] (or in [21, pp. 66-69]). Additionally, it is known that for every continuous function f over a closed interval, a mini-max approximation of a given degree n exists and is unique [17, pp. 140-146].

The mini-max polynomial approximation is characterized completely by the equioscillation property. The error curve takes on extreme values in at least $n+2$ points with alternating signs. As an example, since $2^{-n}T_{n+1}(x)$ equioscillates on $n+2$ points belonging to $[-1,1]$, it follows that $x^{n+1} - 2^{-n}T_{n+1}(x)$ is the mini-max polynomial approximation of degree at most n for x^{n+1} on $[-1,1]$.

The equioscillation property indicates that there exist $n+1$ points $x_0^*, \dots, x_n^*$ in $[-1,1]$ where p_n^* interpolates f. If we apply these points for the error formula (2.9) combined with the upper bound (2.10), then we obtain the two-sided estimate

$$\frac{1}{2^n}\frac{\min\left|f^{(n+1)}(x)\right|}{(n+1)!} \le \|f - p_n^*\|_\infty \le \frac{1}{2^n}\frac{\max\left|f^{(n+1)}(x)\right|}{(n+1)!}, \quad -1 \le x \le 1.$$

Another bound for the interpolation error is given by

$$||f - p_n||_\infty \le (1 + ||P_n||)\,||f - p_n^*||_\infty, \qquad ||P_n|| = \sup_{||f||_\infty \le 1} ||P_n f||_\infty$$

where $P_n := P_n[x_0, \dots, x_n]$ is the linear operator defined by (2.4) and p_n^* is the best uniform polynomial approximation to f. It is easy to show how to obtain this inequality. From the uniqueness of the interpolating polynomial, we have $p_n = P_n(f)$ and $p_n^* = P_n(p_n^*)$. Now by employing this, it follows that

$$\begin{aligned}\|f-p_n\|_\infty &= \|f-p_n^*+p_n^*-p_n\|_\infty \\ &\le \|f-p_n^*\|_\infty + \|P_n(p_n^*-f)\|_\infty \\ &\le (1+||P_n||)\,\|f-p_n^*\|_\infty .\end{aligned}$$

As the computation of the best approximation is more complicated than that of the interpolant (2.3), there is interest in interpolating polynomials that are near-best approximants. Indeed, in most cases, the small difference between the approximation errors of p_n^* and p_n is easily compensated by increasing the degree of the interpolation polynomial. Therefore, in practice, interpolation polynomials are often used instead of best approximating polynomials.

2.3 Lebesgue Function and Constant

2.3.1 Definition and Properties

Recall from (2.4) that

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

where $p_n(x)$ is the Lagrange form for the polynomial that interpolates f in the interpolation points $x_0, x_1, \cdots, x_n$ defined by (2.2) and the basic Lagrange polynomials $\ell_i(x)$ are defined by (2.5).

For fixed n and given $x_0, \cdots, x_n$, the Lebesgue function is defined by

$$L_n(x) := L_n(x_0, \cdots, x_n; x) = \sum_{i=0}^{n} |\ell_i(x)|$$

and the Lebesgue constant is defined by

$$\Lambda_n := \Lambda_n(x_0, \cdots, x_n) = \max_{-1\le x\le 1} \sum_{i=0}^{n} |\ell_i(x)|.$$

It is clear that both $L_n(x)$ and Λ_n depend on the location of the interpolation points x_j (and also the degree n) but not on the function values $f(x_i)$. Note that the operator norm of P_n

defined by (2.4) is equal to the ∞ -norm of its Lebesgue function:

$$\|P_n\|_\infty = \Lambda_n = \max_{-1\leq x\leq 1} L_n(x).$$

Here and in the following, with the set X defined by (2.1), we sometimes write $L_n(X;x) := L_n(x_0,\cdots,x_n;x)$ and $\Lambda_n(X) := \Lambda_n(x_0,\cdots,x_n)$ to simplify the notations.

The following presents some basic properties of Lebesgue functions for Lagrange interpolation (see e.g., [22, 23]):

(i) For any set X, with $n \geq 2$, $L_n(X;x)$ is a piecewise polynomial satisfying $L_n(X;x) \geq 1$ with equality only at the interpolation points x_j, $j = 0,\ldots,n$.

(ii) On each subinterval (x_{j-1}, x_j) for $1 \leq j \leq n$, $L_n(X;x)$ has precisely one local maximum that is denoted by $\lambda_j(X)$. If the endpoints -1 and $+1$ are not interpolation points, i.e. $-1 < x_0$ and $x_n < 1$, then there are two other subintervals and thus, two other local maxima that are at -1, and $+1$. We denote the latter two local maxima by $\lambda_0(X)$ and $\lambda_{n+1}(X)$.

(iii) The greatest and the smallest local maxima of $L_n(X;x)$ are denoted correspondingly by $\mathscr{M}_n(X)$ and $m_n(X)$ and let $\delta_n(X)$ denote the maximum deviation among the local maxima $\delta_n(X) = \mathscr{M}_n(X) - m_n(X)$. We also denote the position of the Lebesgue constant (by taking one of the greatest local maxima) by $x^*(X)$ for the set of interpolation points X.

(iv) The equality $L_n(X;x) = L_n(X;-x), x \in [-1,1]$ holds if and only if $x_{n-j} = -x_j$, $j = 0,\ldots,n$.

(v) The Lebesgue constant is invariant under the linear transformation $t_j = \hat{a}x_j + \hat{b}$, $j = 0,\ldots,n$, $(\hat{a} \neq 0)$. Interpolation sets that include the endpoints of the interval as interpolation points are called canonical interpolation sets. Let $\hat{X}$ denote a canonical interpolation set. In particular, we may construct a set $\hat{X}$, obtained from X by mapping $[x_0, x_n]$ onto $[-1,1]$ by the unique linear transformation $t_i = \hat{a}x_i + \hat{b}, i = 0,\ldots,n$ where $\hat{a}$ and $\hat{b}$ are determined by $-1 = \hat{a}x_0 + \hat{b}$, $1 = \hat{a}x_n + \hat{b}$. Here the set $\hat{X}$ is also called the canonicalization of the set X. It can be seen that the Lebesgue constant for $\hat{X}$ is [22, 24, pp. 104-105]

$$\Lambda_n(t_0, \cdots, t_n) = \max_{x_0 \le x \le x_n} L_n(x_0, \cdots, x_n; x) \le \max_{-1 \le x \le 1} L_n(x_0, \cdots, x_n; x).$$

We use these properties in the sequel.

2.3.2 Importance of Lebesgue constants

One motivation for investigating the Lebesgue constant is that another upper bound for the interpolation error (2.9) is given by (see, e.g., [25, 2, 24])

$$\|f - p_n\|_\infty \le (1 + \Lambda_n) \|f - p_n^*\|_\infty \tag{2.11}$$

where p_n^* is the best polynomial approximation to f on $[-1, 1]$ and therefore Λ_n quantifies how much larger the interpolation error $\|f - p_n\|_\infty$ is compared to the smallest possible error $\|f - p_n^*\|_\infty$ in the worst case. In this study, we call this inequality the Lebesgue inequality or the second error formula. As a simple consequence of this inequality, it is obvious that $p_n \to f$ whenever the factor $\Lambda_n \|f - p_n^*\|_\infty \to 0$. Namely, the Lebesgue inequality indicates that for the interpolation of a fixed function f on $[-1, 1]$, convergence can be expected only if f is smooth enough such that $\|f - p_n^*\|_\infty$ decreases as $n \to \infty$, faster than Λ_n increases.

Another motivation for investigating the Lebesgue constant is that the Lebesgue constant also expresses the conditioning of the polynomial interpolation problem in the Lagrange basis. Let $\tilde{p}_n(x)$ denote the polynomial interpolant of degree n for the perturbed function $\tilde{f}$ in the same interpolation points:

$$\tilde{p}_n(x) = \sum_{i=0}^{n} \tilde{f}(x_i) \ell_i(x).$$

Since $||p_n||_\infty \ge \max_{i=0,\ldots,n} |f(x_i)|$ we have

$$\begin{aligned} \frac{||p_n - \tilde{p}_n||_\infty}{||p_n||_\infty} &\le \frac{\max_{x \in [-1,1]} \sum_{i=0}^{n} |f(x_i) - \tilde{f}(x_i)| |\ell_i(x)|}{\max_{i=0,\ldots,n} |f(x_i)|} \\ &\le \Lambda_n(x_0, \ldots, x_n) \frac{\max_{i=0,\ldots,n} |f(x_i) - \tilde{f}(x_i)|}{\max_{i=0,\ldots,n} |f(x_i)|}. \end{aligned} \tag{2.12}$$

This indicates that if we are able to choose interpolation points such that Λ_n is small, then we have found the Lagrange interpolant that is less sensitive to errors in the function values. For this reason, numerical interpolation in floating-point arithmetic will generally be useless, even for smooth functions f, whenever the Lebesgue constant Λ_n is larger than the inverse of the machine precision, which is typically about 10^{16}.

2.3.3 Some specific sets of points

This subsection gives a summary of some results for particular sets of interpolation points for which the behavior of the Lebesgue function has been investigated [23, 7, 26] and the references therein for more detailed results.

Equidistant nodes E:

There are many studies on the behavior of the Lebesgue function corresponding to the set of equidistant points, although this set is a bad choice for polynomial interpolation owing to the Runge phenomenon.

For the set of equidistant points

$$E = \left\{ x_j = -1 + \frac{2j}{n}, j = 0, 1, \ldots, n \right\} \tag{2.13}$$

the Lebesgue constant $\Lambda_n(E)$, grows exponentially with the asymptotic estimate [8, 9]

$$\Lambda_n(E) \simeq \frac{2^{n+1}}{en\,(\log n + \gamma)}, \quad n \to \infty \tag{2.14}$$

where

$$\gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right) = 0.577\ldots$$

is Euler's constant (or the Euler-Mascheroni constant). Also, an asymptotic expansion that improves (2.14) (with unknown explicit general formula for the series coefficients) is found in [27].

For $\Lambda_n(E)$, the upper and lower bounds

$$\frac{2^{n-2}}{n^2} < \Lambda_n\,(E) < \frac{2^{n+3}}{n}, \quad n \geq 1 \tag{2.15}$$

have been suggested [28]. In [8], an upper bound is given for the smallest local maxima

$m_n(E)$:

$$m_n(E) < \frac{2}{\pi}(\log(n+2)+\log 2+\gamma). \tag{2.16}$$

From (2.15) together with (2.16), it follows that $\Lambda_n(E)$ and $m_n(E)$ are of different orders of magnitude and hence, the maximum deviation $\delta_n(E)$ of the local maxima tends to infinity exponentially fast. As Figure 2.1 (left) illustrates, the Lebesgue function $L_n(E;x)$ has wild oscillations near the endpoints, like the Runge phenomenon in the error curve. The local maxima of $L_n(E;x)$ decrease strictly from the outside towards the midpoint of the interval $[-1,1]$ [29]. This behavior suggests that the location of the Lebesgue constant is in the first subinterval (or due to symmetry in the last subinterval). Numerical observation shows that the location of the Lebesgue constant occurs near the midpoint of the last (or first) subinterval, i.e., $x^*(E) \approx (n-1)/n$ for the interval $[-1,1]$.

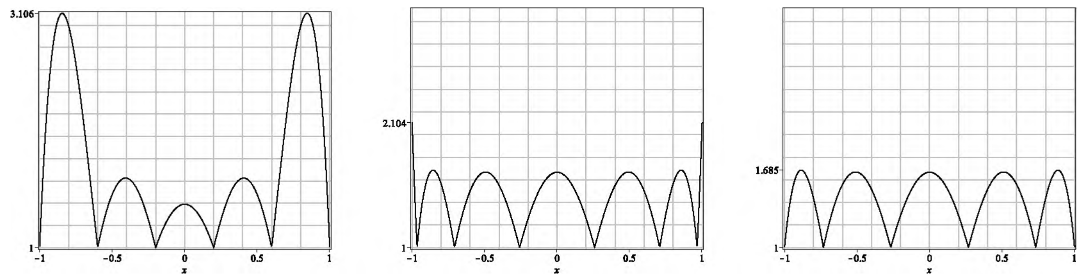

Figure 2.1 Graphs of $L_5(E;x)$ (left), $L_5(T;x)$ (center), $L_5(\hat{T};x)$ (right).

From the Lebesgue inequality (2.11), it is known that equidistant points with this very fast growth of the Lebesgue constant give very poor approximations as n increases. Indeed, numerical experiments show that for degree $n \geq 65$, the Lebesgue constant $\Lambda_n(E)$ reaches the inverse of the machine precision.

Chebyshev nodes of the first kind T:

The literature describes numerous investigations for the behavior of the Lebesgue function corresponding to the set of Chebyshev nodes. They are a very good choice of points for polynomial interpolation and as was discussed in Section 2.2, they give the smallest upper bound for the first error formula (2.9) of polynomial interpolation. As mentioned in

Section 2.2, they are the zeros of the $(n+1)$ - th degree Chebyshev polynomial of the first kind and their explicit formula is known by (2.17), below. The set of Chebyshev points

$$T = \left\{ x_j = -\cos\left(\frac{\pi}{2}\frac{(2j+1)}{(n+1)}\right), j = 0, 1, \ldots, n \right\} \tag{2.17}$$

are distributed more densely towards the endpoints of the interval $[-1, 1]$, as illustrated in Figure 2.2 for $n = 32$.

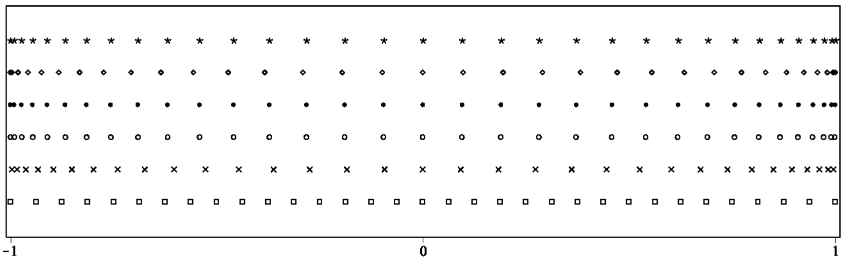

Figure 2.2 Graphs of sets of 33 nodes; $(*)\check{T}$, $(\diamond)\bar{U}$, $(\bullet)\hat{T}$, $(\circ)T$, $(\times)U$, $(\square)E$ from top to bottom.

The Lebesgue constant $\Lambda_n(T)$, for polynomial interpolation, grows logarithmically with the asymptotic expression [5]

$$\Lambda_n(T) = \frac{2}{\pi}\left(\log(n+1) + \gamma + \log\frac{8}{\pi}\right) + \alpha_n, \quad 0 < \alpha_n < \frac{\pi}{72(n+1)^2}, \quad n \geq 1$$

from which, the upper and lower bounds

$$\frac{2}{\pi}\log(n+1) + 0.9625\ldots < \Lambda_n(T) < \frac{2}{\pi}\log(n+1) + 0.9734\ldots, \quad n \geq 1 \tag{2.18}$$

can be deduced.

For $\Lambda_n(T)$, an asymptotic series expansion, which is valid for all finite n, is given by [30, 31, 5]

$$\Lambda_n(T) = \frac{2}{\pi}\left(\log(n+1) + \gamma + \log\frac{8}{\pi}\right) + \sum_{v=1}^{\infty} \frac{\mathscr{A}_v}{(n+1)^{2v}}, \quad n \geq 1$$

where the coefficients $\mathscr{A}_v$ have alternating signs and can be calculated as

$$\mathscr{A}_v = (-1)^{v-1}\frac{4}{\pi}\frac{1-2^{1-2v}}{(2\pi)^{2v}}(2v-1)!\,\zeta(2v)\left(1 + \sum_{j=v+1}^{\infty}\frac{\zeta(2j)}{(2)^{2j-1}}\begin{pmatrix} 2j-1 \\ 2v-1 \end{pmatrix}\right),$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the Riemann zeta function.

Using the little-o notation defined by $\varepsilon(n) = o(e(n))$ when $\varepsilon(n)/e(n) \to 0, n \to \infty$, Brutman showed [32] that

$$m_n(T) = \Lambda_{\lfloor \frac{n}{2} \rfloor}(T) + o(1), \quad n \geq 2$$

from which the lower bound

$$\frac{2}{\pi}\log(n+1) + \underbrace{\frac{2}{\pi}\left(\log\frac{4}{\pi} + \gamma\right)}_{0.521251...} < m_n(T)$$

is obtained. Later, this bound was improved [33] as follows

$$\frac{2}{\pi}\left(\log(n+1) + \log\frac{4}{\pi} + \gamma\right) + \frac{\pi}{18(n+1)^2} - \frac{49\pi^3}{10800(n+1)^4} < m_n(T). \tag{2.19}$$

A comparison of (2.18) and (2.19) shows that the deviation between any two local maxima of the Lebesgue function $L_n(T;x)$ does not exceed 0.4522. This result was improved in [5] to

$$\delta_n(T) = \mathscr{M}_n(T) - m_n(T) \leq \frac{2}{\pi}\log 2 = 0.44127.$$

As Figure 2.1 (center) suggests, the local maxima of $L_n(T;x)$ are decreasing strictly from the outside towards the midpoint of the interval $[-1,1]$, which was proven in [32]. The figure also shows that the location of the Lebesgue constant occurs at ± 1, i.e. $x^*(T) = \pm 1$ [4, 34].

We know from the first error formula (2.9) that the Chebyshev points are a good choice for polynomial interpolation. Now, this slow growth of the Lebesgue constant confirms that they are also a good choice for the second error formula (2.11), which becomes

$$\|f - p_n\|_\infty \leq \left(\frac{2}{\pi}\log(n+1) + 2\right)\|f - p_n^*\|_\infty$$

for the Chebyshev nodes. For example, for $n = 100$, the interpolation error based on the

Chebyshev points is

$$\|f-p_n\|_\infty \le \left(4.9\times 10^{0}\right)\|f-p_n^*\|_\infty,$$

i.e., even in the worst case, the interpolation error $\|f-p_n\|_\infty$ is only 4.9 times larger than the smallest possible error. For comparison, if we choose equidistant points for the same degree, then the upper bound for the interpolation error is

$$\|f-p_n\|_\infty \le \left(1.8\times 10^{27}\right)\|f-p_n^*\|_\infty.$$

Extended Chebyshev nodes $\hat{T}$:

The extended Chebyshev nodes $\hat{T}$ are defined by

$$\hat{T}=\left\{x_j=-\cos\left(\frac{\pi}{2}\frac{(2j+1)}{(n+1)}\right)\Big/\cos\left(\frac{\pi}{2\,(n+1)}\right), j=0,1,\ldots,n\right\} \tag{2.20}$$

where the division by $\cos\left(\pi/2\,(n+1)\right)$ guarantees that $x_0=-1$ and $x_n=1$, the set $\hat{T}$ is obtained from the set T by the linear transformation, which maps $[x_0,x_n]$ onto $[-1,1]$. Therefore, the set $\hat{T}$ is the canonicalization of the Chebyshev set T (see Figures 2.3 and 2.2).

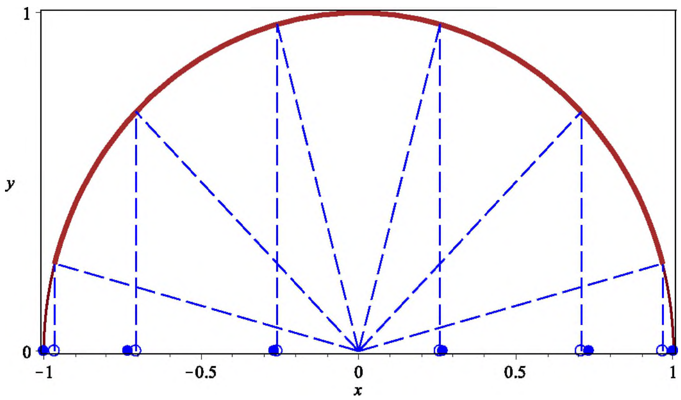

Figure 2.3 6 Chebyshev (○) and extended Chebyshev (●) nodes.

From the monotonicity result for the local maxima of $L_n(T;x)$ and the property (v) given in Section 2.3.1 [22, 34],

$$\max_{x_0<x<x_n} L_n(x_0,\cdots,x_n;x) \le \max_{-1\le x\le 1} L_n(x_0,\cdots,x_n;x)$$

it follows that the Lebesgue constant $\Lambda_n(\hat{T})$ is equal to the second local maximum $\lambda_1(T)$, (or $\lambda_n(T)$) [32, 35, 34]. By using this characterization, an asymptotic expression for the Lebesgue constant for the extended Chebyshev nodes is given by [5]

$$\Lambda_n(\hat{T}) = \frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{8}{\pi}\right) - \frac{4}{3\pi} + \beta_n,$$
$$0 < \beta_n < \frac{0.01}{\log\left((n+1)/4\right)}, \quad n \geq 4. \quad (2.21)$$

Hence one can derive the upper and lower bounds,

$$\frac{2}{\pi}\log(n+1)+0.5381\ldots < \Lambda_n(\hat{T}) < \frac{2}{\pi}\log(n+1)+0.5829\ldots, \quad n \geq 4.$$

Also, an asymptotic expansion of β_n (with unknown explicit general formula for the series coefficients, in contrast to the Chebysev nodes) can be found in [35].

As for the maximum deviation $\mathscr{M}_n(\hat{T}) - m_n(\hat{T})$ of the extended Chebyshev nodes, the following estimate is given [5],

$$\delta_n(\hat{T}) = \mathscr{M}_n(\hat{T}) - m_n(\hat{T}) \leq 0.0196, \quad n \geq 70.$$

From (2.21) together with (2.19), it follows that this maximum deviation converges to

$$\lim_{n\to\infty} \delta_n(\hat{T}) = 0.016858\ldots.$$

Chebyshev extrema $\bar{U}$:

The Chebyshev extrema $\bar{U}$ are the zeros of the polynomial $\left(1-x^2\right)T_n'(x)$ and are given in explicit form as

$$\bar{U} = \left\{x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0,1,\ldots,n\right\}.$$

The Lebesgue constant $\Lambda_n(\bar{U})$ for polynomial interpolation is [4, 36]:

$$\Lambda_n(\bar{U}) = \begin{cases} \Lambda_{n-1}(T), & n \text{ odd} \\ \Lambda_{n-1}(T) - \alpha_n, \quad 0 < \alpha_n < \frac{1}{n^2}, & n \text{ even.} \end{cases}$$

It is shown in [32] that the smallest local maxima $m_n(\bar{U})$ are bounded (in contrast to the case of the Chebyshev nodes T) by

$$m_n(\bar{U}) < 1.57079\ldots.$$

Thus, as in the case of the set E, $\Lambda_n(\bar{U})$ and $m_n(\bar{U})$ are of different orders of magnitude and the maximum deviation of the local maxima $\delta_n(\bar{U})$ tends to infinity logarithmically.

As was proven in [32], the local maxima of $L_n(\bar{U};x)$ increase strictly monotonically from the outside towards the midpoint of the interval $[-1,1]$. This behavior suggests that the Lebesgue function $L_n(\bar{U};x)$ achieves its maximum value on the subinterval $\left(x_{n/2},\, x_{(n+2)/2}\right)$ (or its mirror) for even degrees and on the subinterval $\left(x_{(n-1)/2},\, x_{(n+1)/2}\right)$ for odd degrees. Numerical observation indicates that the location of the Lebesgue constant occurs at $x^*(\bar{U}) \approx \frac{\pi}{2n}$ (or its mirror) for (large) even degrees and at $x^*(\bar{U}) = 0$ for odd degrees.

Chebyshev nodes of the second kind U:

The Chebyshev nodes of the second kind U are the zeros of the $(n+1)$ -th degree Chebyshev polynomial of the second kind

$$U_{n+1}(x) = \frac{\sin((n+2)\arccos(x))}{\sin(\arccos(x))}$$

and are given in closed form by

$$U = \left\{x_j = -\cos\left(\frac{(j+1)\pi}{n+2}\right), \quad j = 0,1,\ldots,n\right\}.$$

For the Lebesgue constant, it is known that $\Lambda_n(U) = O(n)$ [37, pp. 335-339]. In [32], an exact expression for $\Lambda_n(U)$ is given by

$$\Lambda_n(U) = n+1,$$

and a lower bound for $m_n(U)$ is given by

$$\frac{2}{\pi}\log(n+1) + 0.3259\ldots < m_n(U).$$

Thus, as in the cases of the sets E and $\bar{U}$, $\Lambda_n(U)$ and $m_n(U)$ are of different orders of magnitude. In this case, the maximum deviation of the local maxima $\delta_n(U)$ has a linear growth.

Note that these interpolation points can be obtained from the zeros of the polynomial $\left(1-x^2\right)T'_{n+2}(x)$ by deleting the zeros ± 1.Thus, it follows that the Lebesgue constants are sensitive to the deletion of the endpoints.

Fekete nodes F:

The nodes F are the zeros of the polynomial $\left(1-x^2\right)Q'_n(x)$, where Q_n is the Legendre polynomial of degree n. There is no explicit expression for these nodes.

It is well known that Fekete points maximize the Vandermonde determinant, $|V(x_0,\cdots,x_n)|$ defined by (2.8) on the interval $[-1,1]$. Since the basic Lagrange polynomials $\ell_i(x)$ may be expressed with $V(x_0,\cdots,x_n)$ in the form

$$\ell_i(x) = \frac{|V(x_0,\cdots,x_{i-1},x,x_{i+1},\cdots,x_n)|}{|V(x_0,\cdots,x_n)|},$$

we have $\|\ell_i(x)\|_\infty \le 1$, for $0 \le i \le n$ and thus, the corresponding Lebesgue constant is bounded by (at most) the dimension of the interpolation space,

$$\Lambda_n(F) = \max_{-1\le x\le 1}\sum_{i=0}^{n}|\ell_i(x)| \le n+1.$$

Moreover [38], the Fekete points minimize $\max\limits_{-1\le x\le 1}\sum_{i=0}^{n}(\ell_i(x))^2$ and for these points $\max\limits_{-1\le x\le 1}\sum_{i=0}^{n}(\ell_i(x))^2 = 1$. From this, by applying the Cauchy-Schwartz inequality,

$$\Lambda_n(F) \le \sqrt{n+1}.$$

This upper bound, however, is very pessimistic. In [6], an improved upper bound for $\Lambda_n(F)$ is given by

$$\Lambda_n(F) \le c\log(n+1)$$

with the positive constant c not determined. In addition [7], based on numerical experiments, the estimate

$$\Lambda_n(F) \le \frac{2}{\pi}\log(n+1) + 0.685$$

was conjectured. Accordingly, this confirms the conjecture in [22] that

$$\Lambda_n(\hat{T}) < \Lambda_n(F) < \Lambda_n(T), \quad n \ge 3.$$

Optimal nodes X^*:

The set of interpolation points is said to be optimal if it minimizes the Lebesgue constant. We denote the set of optimal nodes by X^* (or the Lebesgue-optimal point set in $[-1,1]$):

$$\Lambda_n(X^*) = \min_X \Lambda_n(X).$$

Owing to the second error formula (2.11) and also formula (2.12) (for sensitivity to perturbations in the function values), it is desirable to minimize the Lebesgue constant. However, the set of optimal nodes on the interval $[-1,1]$ is known explicitly only for degrees less than four [39], although their characterization is known from the Bernstein-Erdös conjectures.

In 1931, Bernstein [40] conjectured that the greatest local maxima of the Lebesgue function is minimal when $L_n(x)$ equioscillates,

$$\lambda_1(X^*) = \lambda_2(X^*) = \cdots = \lambda_n(X^*).$$

Later, Erdös [41, 42] added to this conjecture that there is a unique canonical set $\hat{X}^*$ for which the smallest local maxima achieve its maximum. This is the case when the local maximum values are equal, or in other words

$$m_n(X) \le m_n(X^*) = \Lambda_n(X^*) = \mathscr{M}_n(X^*) \le \mathscr{M}_n(X), \quad \text{for every set } X. \tag{2.22}$$

These conjectures were proven by Kilgore [43, 44] and by de Boor and Pinkus [45]. They showed that for degree n, the optimal canonical interpolation set is unique, symmetric and that its Lebesgue function must necessarily equioscillate. By using these basic properties of the optimal nodes, a numerical procedure based on a nonlinear Remez search and exchange algorithm is given to compute the optimal nodes for polynomial interpolation on $[-1,1]$ [46]. Moreover, many authors [7, 47] have investigated (near) optimal point sets (in different norms) defined by the solution of certain optimization problems.

The first sharp estimate for the optimal Lebesgue constant is given by Vértesi [1]. By constructing the following modification of the Chebyshev nodes, asymptotically optimal upper and lower bounds are given [1, 48, 2, pp. 110-121]. Let us denote this set by

$$\check{T} = \left\{ x_j = -\frac{\cos\left(\frac{\pi}{2}\frac{(2j+1)}{(n+1)}\right)}{\cos\left(\frac{\pi}{2(n+1)}\left(1+\frac{1}{4\log(n+1)}\right)\right)}, \quad j = 1,\ldots,n-1, \quad x_n = -x_0 = 1 \right\}.$$

The Lebesgue constant $\Lambda_n(\check{T})$ satisfies

$$c\left(\frac{\log\log(n+1)}{\log(n+1)}\right)^2 > \Lambda_n(\check{T}) - \frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{4}{\pi}\right)$$
$$> \begin{cases} \frac{\pi}{18(n+1)^2} + O\left(\frac{1}{(n+1)^4}\right) & n \quad \text{odd} \\ -\frac{2}{\pi(n+1)} + O\left(\frac{1}{(n+1)^2}\right) & n \quad \text{even} \end{cases}$$

where c is a an undetermined positive constant.

An application of the Erdös inequality (2.22) combined with the lower bound for $m_n(T)$ (2.19) and the upper bound for $\Lambda_n(\check{T})$, gives

$$\frac{\pi}{18(n+1)^2} + O\left(\frac{1}{(n+1)^4}\right) < \Lambda_n(X^*) - \frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{4}{\pi}\right)$$
$$< \mathrm{c}\left(\left(\frac{\log\log(n+1)}{\log(n+1)}\right)^2\right).$$

From this, one can deduce that the precise growth formulas for $\Lambda_n(X^*)$ and $\Lambda_n\left(\check{T}\right)$ are, respectively,

$$\Lambda_n(X^*) = \frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{4}{\pi}\right) + o(1)$$

and

$$\Lambda_n\left(\check{T}\right) = \frac{2}{\pi}\left(\log(n+1)+\gamma+\log\frac{4}{\pi}\right) + o(1).$$

As $\Lambda_n\left(\check{T}\right)$ and $\Lambda_n(X^*)$ have the same asymptotic growth, one can conclude that the set $\check{T}$ has asymptotically minimal Lebesgue constants.

At this point, some remarks are useful. The precise growth formula for $\Lambda_n(\hat{T})$,

$$\Lambda_n(\hat{T}) = \frac{2}{\pi}\log(n+1) + 0.5381\ldots + o(1)$$

can be derived from (2.21). On comparing $\Lambda_n(\hat{T})$ and $\Lambda_n\left(\check{T}\right)$, it can be seen that the set $\check{T}$ is better than the set $\hat{T}$ in minimizing Lebesgue constant. Indeed, numerical results confirm this (see Table 2.1 and also Figure 2.4). The maximum deviation of the nodal set

$\check{T}$ converges to (see [48, (39)])

$$\lim_{n\to\infty} \delta_n\left(\check{T}\right) = 0.089245\ldots.$$

Table 2.1 The values of the maximum deviations and Lebesgue constants for sets $\check{T}$, $\hat{T}$ and X^*.

n	set $\check{T}$		set $\hat{T}$		set X^*
	$\delta_n\left(\check{T}\right)$	$\Lambda_n(\check{T})$	$\delta_n\left(\hat{T}\right)$	$\Lambda_n(\hat{T})$	$\Lambda_n\left(X^*\right)$
10	0.050 781	2.056 087	0.019 471	2.068 744	2.051 706
20	0.056 995	2.463 129	0.019 340	2.479 193	2.460 788
40	0.061 827	2.887 067	0.018 952	2.904 441	2.885 809

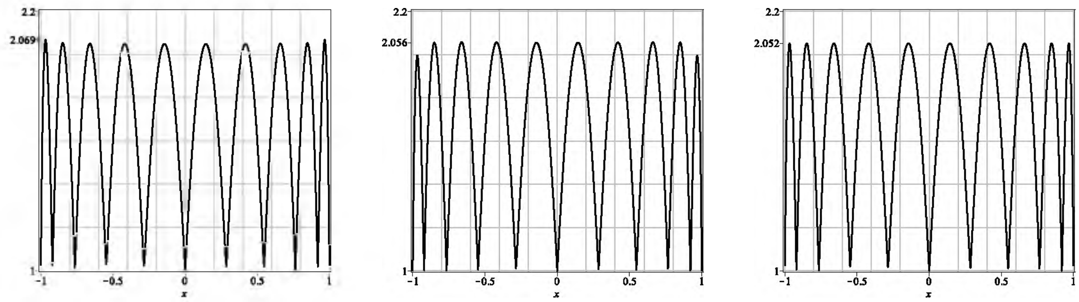

Figure 2.4 Graphs of $L_{10}\left(\hat{T};x\right)$, $L_{10}\left(\check{T};x\right)$, $L_{10}\left(X^*;x\right)$ from left to right.

The nodal sets studied in this section can be ordered with respect to their maximum deviation $\delta_n(X) = \mathscr{M}_n - m_n(X)$ and their Lebesgue constant $\Lambda_n(X)$, in the following way

$$\delta_n(E) > \delta_n(U) > \delta_n(\bar{U}) > \delta_n(T) > \delta_n\left(\check{T}\right) > \delta_n\left(\hat{T}\right) > \delta_n\left(X^*\right) = 0$$

and

$$\Lambda_n(E) > \Lambda_n(U) > \Lambda_n(T) > \Lambda_n(\bar{U}) > \Lambda_n(F) > \Lambda_n\left(\hat{T}\right) > \Lambda_n\left(\check{T}\right) > \Lambda_n\left(X^*\right).$$

2.4 Vandermonde Condition number

Recall (2.8),

$$V_n(x_0,\cdots,x_n) = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

which is the coefficient matrix of the Vandermonde system (2.7)

$$\sum_{i=0}^{n} a_i x_j^i = f(x_j), \quad j = 0,\dots n$$

where $x_0, x_1, \cdots, x_n$ are the interpolation points.

The condition of the Vandermonde system (2.7) in terms of a suitable matrix norm (e.g., the p -norm, $p = 1,2,\infty$ or the *Frobenius* -norm) is given by the condition number

$$\kappa_p(V_n) := \kappa_p(V_n; x_0,\cdots,x_n) = \|V_n(x_0,\cdots,x_n)\|_p \left\|V_n^{-1}(x_0,\cdots,x_n)\right\|_p$$

In the following, sometimes we use $\kappa_p(V_n) := \kappa_p(V_n; x_0,\cdots,x_n)$ to simplify the notations.

It is obvious that $\kappa_p(V_n)$ depends on the location of the interpolation points x_j and the norm (and also the number of points n). But any two condition numbers, e.g., $\kappa_{p_1}(\cdot)$ and $\kappa_{p_2}(\cdot)$ are equivalent in that positive constants c_1 and c_2 (depending only on n) can be found for which

$$c_1 \kappa_{p_1}(A) \le \kappa_{p_2}(A) \le c_2 \kappa_{p_1}(A), \quad A \in \mathbb{R}^{n\times n}. \tag{2.23}$$

For example, $\kappa_1(\cdot)$ and $\kappa_\infty(\cdot)$ satisfy

$$\frac{1}{n^2}\kappa_1(A) \le \kappa_\infty(A) \le n^2 \kappa_1(A).$$

Let the vector $\tilde{a}$ denote the coefficients $\tilde{a}_i$ for the interpolant of the vector $\tilde{y}$ of the perturbed function values $\tilde{f}(x_j)$ in the same interpolation points on the interval $[-1,1]$:

$$\sum_{i=0}^{n} \tilde{a}_i x_j^i = \tilde{f}(x_j), \quad j = 0,\dots n, \qquad V_n \tilde{a} = \tilde{y}.$$

Then

$$\|a-\tilde{a}\|_p \leq \left\|V_n^{-1}\right\|_p \|y-\tilde{y}\|_p.$$

Combining this with $\|y\|_p \leq \|V_n\|_p \|a\|_p$ yields

$$\frac{\|a-\tilde{a}\|_p}{\|a\|_p} \leq \kappa_p(V_n) \frac{\|y-\tilde{y}\|_p}{\|y\|_p}.$$

From this, the condition number indicates by how much the relative error in the solution of the coefficients $a_0, a_1 \cdots, a_n$ is, compared with the relative change in the function values $f(x_j)$. If the condition number is very large, then the solution a of (2.7) is considered unreliable and the system is said to be ill-conditioned. Note that owing to the equivalence property given above, if a system is ill-conditioned in the p_1-norm, then it is also ill-conditioned in the p_2-norm. Therefore, the smaller the condition numbers $\kappa_p(V_n)$, the better the condition of the Vandermonde system.

For $x_j \in [-1,1]$, we find in [49, 50] that for the Vandermonde matrix in the 1-norm,

$$\kappa_\infty\left(V_n^T\right) = \kappa_1(V_n) \leq \max_{0\leq i\leq n} \sum_{j=0}^{n} x_j^i \max_{0\leq i\leq n} \prod_{j=0, j\neq i}^{n} \frac{1+|x_j|}{|x_i - x_j|}$$

with equality if the x_j all lie in $[0,1]$ or $[-1,0]$. Thus, in the latter case, condition numbers $\kappa_1(V_n)$ can be computed directly without requiring matrix inversion. Moreover, any results of condition numbers $\kappa_p(V_n)$ in the case $x_j \in [0,1]$ hold in the case $x_j \in [-1,0]$ as well.

2.4.1 Condition number for some specific sets of points

Here we summarize some results for the condition number of the Vandermonde matrix of size $n+1$. It is well known that Vandermonde matrices with real nodes are ill-conditioned, even for not very high orders [51, pp. 428]. The ill-conditioning is a consequence of the monomials. For large n, the monomials are less distinguishable from one another and this causes the columns of the Vandermonde matrix to become nearly linearly dependent in this case.

Let us denote $f(n) \sim g(n), n \to \infty$ when $\lim_{n\to\infty} f(n)/g(n) = 1$. For the Chebyshev nodes on $(-1,1)$ defined by (2.17), the condition number of the Vandermonde matrix of size

$n+1$ in the 1-norm satisfies [50]

$$\kappa_1(V_n;T) \sim \frac{3^{\frac{3}{4}}}{4}\left(1+\sqrt{2}\right)^{n+1}, \quad n \to \infty. \tag{2.24}$$

For equidistant nodes on $[-1,1]$, it satisfies [50]

$$\kappa_1(V_n;E) \sim \frac{\sqrt{2}}{\pi}\left(\sqrt{2}e^{\frac{\pi}{4}}\right)^{n}, \quad n \to \infty. \tag{2.25}$$

For nonnegative equidistant nodes on $[0,1]$, it satisfies [52]

$$\kappa_1(V_n;E) \sim \frac{2\sqrt{2}}{\pi}8^n, \quad n \to \infty. \tag{2.26}$$

Comparing (2.24) and (2.25), it is easy to see that the growth rate of the condition number for Chebyshev nodes is smaller than that for the equidistant nodes. Comparing (2.25) and (2.26) reveals that the rate of growth can be reduced by using symmetric nodes.

2.4.2 Optimally conditioned Vandermonde matrices

We introduce the notation $\kappa_p^*(V_n;[\alpha,\beta]) := \inf_{x_0,\ldots,x_n\in[\alpha,\beta]} \kappa_p(V_n)$, where $[\alpha,\beta]$ denotes the real interval. While the exact growth formula of the minimal condition number is not known, for any Vandermonde matrix of size $n+1$ the lower bound for the condition number is of exponential order of growth [53]. More precisely, for symmetric real nodes ($x_j = -x_{n-j}, j=0,1,\ldots,n$) it is proven that

$$\kappa_1^*(V_n;\mathbb{R}) \geq 2^{(n+1)/2}, \quad n \geq 1$$

and for nonnegative ones ($x_j \geq 0$), it is proven that

$$\kappa_1^*\left(V_n;\mathbb{R}^+\right) \geq 2^n, \quad n \geq 1.$$

It is also shown in [53, 54] that these lower bounds can be improved slightly.

In [55], lower and upper bounds for the condition number $\kappa_2^*(V_n;[-1,1])$ of the Vandermonde matrix for the interval $[-1,1]$ are given by

$$\frac{\sqrt{2}\left(1+\sqrt{2}\right)^{n-1}}{\sqrt{n+1}} \leq \kappa_2^*(V_n;[-1,1]) \leq (n+1)\sqrt{2}\left(1+\sqrt{2}\right)^{n-1} \quad n \geq 1.$$

For the interval $[0,1]$, they are given by

$$\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1+\sqrt{2}\right)^{-2n}}{2\sqrt{n+1}} \leq \kappa_2^*\left(V_n;[0,1]\right) \leq (n+1)\,\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1+\sqrt{2}\right)^{-2n}}{2}, n \geq 1.$$

These are asymptotically optimal condition number estimations for real Vandermonde matrices. Related results involving condition numbers in the 1-norm, using Chebyshev nodes mapped onto the real interval $[\alpha,\beta]$ with $-\alpha=\beta$ or $\alpha\beta \geq 0$, are given by [56]

$$\min_{-\alpha=\beta} \kappa_1^*\left(V_n;[\alpha,\beta]\right) \leq \frac{3^{3/4}}{2}\left(\frac{2}{\pi}\right)^{\sqrt{2}/4} \frac{\left(1+\sqrt{2}\right)^{n+1}}{2\,(n+1)^{\sqrt{2}/4}}, \qquad -\alpha=\beta>0,$$

$$\min_{0=\alpha<\beta} \kappa_1^*\left(V_n;[\alpha,\beta]\right) \leq \frac{\sqrt{2}\left(1+\sqrt{2}\right)^{2(n+1)}}{4\left((n+1)\,\pi\right)^{\sqrt{2}/4}}, \qquad 0=\alpha<\beta.$$

For fixed n the value for β that minimizes the aforementioned condition numbers is always close to 1 [57]. Asymptotically, the best interpolation interval with respect to $\kappa_1(V_n)$ is in the symmetric case $[-1,1]$ and in the non-symmetric (for nonnegative nodes) case $[0,1]$ [56]. From the exponential growth of $\kappa_1^*\left(V_n;[\alpha,\beta]\right)$ together with equivalence property (2.23), it follows that

$$\lim_{n\to\infty}\left(\min_{x_j} \kappa_p^*\left(V_n;\mathbb{R}\right)\right)^{1/(n+1)} = \left(1+\sqrt{2}\right),$$

$$\lim_{n\to\infty}\left(\min_{x_j\geq 0} \kappa_p^*\left(V_n;\mathbb{R}^+\right)\right)^{1/(n+1)} = \left(1+\sqrt{2}\right)^2.$$

From these formulas, it can be concluded that for the best possible conditioned Vandermonde matrix V_n, without restriction on the nodes x_j, $\kappa_p^*\left(V_n;\mathbb{R}\right)$ still goes to ∞ as fast as $\left(1+\sqrt{2}\right)^{n+1}$ and that subject to all being nonnegative nodes, $\kappa_p^*\left(V_n;\mathbb{R}^+\right)$ goes to ∞ as fast as $\left(1+\sqrt{2}\right)^{2(n+1)}$. It was proven in [53] that subject to all x_j being nonnegative, the best possible condition number goes to ∞ at least as fast as 2^n, whereas with symmetric nodes, it goes to ∞ at least as fast as $\left(\sqrt{2}\right)^{n+1}$.

The problem of finding the location of nodes minimizing the 1- condition number of the Vandermonde matrices has been solved analytically in [58] for symmetric nodes for $n \leq 5$ (for nonnegative ones for $n \leq 2$). In addition, assuming that the optimally conditioned V_n is unique, Gautschi showed that the optimally conditioned V_n must have symmetric nodes (with respect to the origin).

Table 2.2 The condition numbers $\kappa_\infty\left(\tilde{V}_n\right)$ for the monomial, Chebyshev and Legendre basis.

n	Basis	$\kappa_\infty\left(\check{V}_n;E\right)$	$\kappa_\infty\left(\check{V}_n;T\right)$
4	Monomial	5.3333 E+1	4.6951 E+1
	Chebyshev	6.6667 E+0	4.7336 E+0
	Legendre	1.2190 E+1	9.6909 E+0
9	Monomial	2.0562 E+3	6.7024 E+3
	Chebyshev	7.5880 E+1	8.7616 E+0
	Legendre	1.1243 E+2	2.8058 E+1
19	Monomial	1.7511 E+9	6.3678 E+7
	Chebyshev	3.4032 E+4	1.6857 E+1
	Legendre	3.8688 E+4	8.0575 E+1
39	Monomial	1.2084 E+19	4.1684 E+15
	Chebyshev	1.9360 E+10	3.3064 E+1
	Legendre	2.1448 E+10	2.2984 E+2
79	Monomial	5.6019 E+38	1.1415 E+23
	Chebyshev	1.2224 E+22	6.5484 E+1
	Legendre	1.3143 E+22	6.5300 E+2

Owing to the exponential growth of the condition number when the number of interpolation points increases, the Vandermonde system (2.7) becomes ill-conditioned. Changing the monomial basis to an orthogonal basis is necessary to improve the conditioning of the system.

2.4.3 Vandermonde-like matrices

In (2.6), substituting ϕ_i by polynomials $p_i(x)$ of exact degree i, $i = 0, 1, \ldots, n$, gives a special case for the matrix Φ,

$$\breve{V}_n(x_0,\cdots,x_n) = \begin{pmatrix} p_0(x_0) & p_1(x_0) & p_2(x_0) & \cdots & p_n(x_0) \\ p_0(x_1) & p_1(x_1) & p_2(x_1) & \cdots & p_n(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_0(x_n) & p_1(x_n) & p_2(x_n) & \cdots & p_n(x_n) \end{pmatrix},$$

which is called a Vandermonde-like matrix. These $p_i(x)$ are often orthogonal polynomials, such as Chebyshev polynomials, Legendre polynomials and Jacobi polynomials, and often the nodes x_j are the zeros of $p_{n+1}(x)$ in the orthogonal family. Regarding the conditioning of $\breve{V}_n(x_0,\cdots,x_n)$, the Vandermonde-like matrices with Chebyshev polynomials and points are perfectly conditioned [59] with respect to the Frobenius norm and their condition numbers satisfy $\kappa_F\left(\breve{V}_n\right) = n$.

Therefore, changing the monomial basis to an orthogonal basis improves the conditioning of the system. We show this in Table 2.2 and in Figure 2.5 for increasing values of n, for the sets E and T defined by (2.13) and (2.17), respectively.

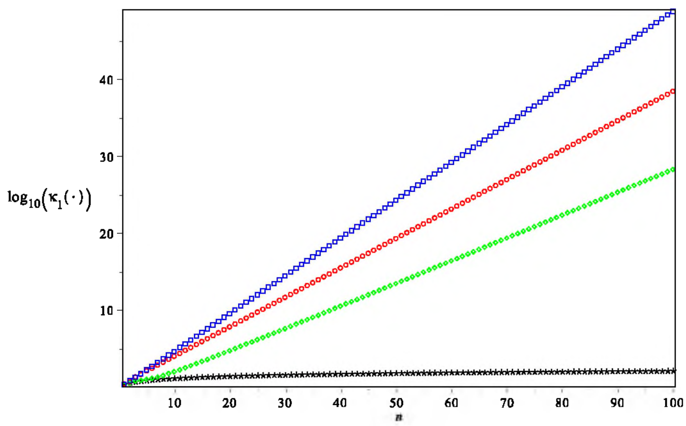

Figure 2.5 Condition numbers of Vandermonde matrices $(\square)\kappa_1(V_n;E)$, $(\circ)\kappa_1(V_n;T)$ using the monomial basis. Condition numbers of Vandermonde-like matrices $(\diamond)\kappa_1\left(\breve{V}_n;E\right)$, $(\star)\kappa_1\left(\breve{V}_n;T\right)$ using the Chebyshev basis. Note the logarithmic scale.

CHAPTER 3

RATIONAL INTERPOLATION WITH FIXED POLES

3.1 Problem statement

When moving to rational interpolation, the conclusions in Chapter 2 do not hold anymore. For instance, rational interpolation using the Chebyshev nodes may yield worse results than using equidistant interpolation points. As an example we mention $f(x) = \arctan(3x)$ on $[-1,1]$ with the numerator and denominator degrees of the rational interpolant respectively equal to 5 and 4. In addition, the approximation and interpolation problems become nonlinear unless one considers the case of a priori fixed poles as we do in this chapter. So let $q_m(x) = \prod_{k=0}^{m-1}(1 - x/\xi_k)$ with $\xi_k \notin [-1,1]$ and interpolate

$$p_n(x_j) = f(x_j)q_m(x_j), \qquad j = 0,\ldots,n \tag{3.1}$$

with $p_n(x) \in \text{span}\{1,\ldots,x^n\}$. In the sequel we restrict ourselves to polynomials $q_m(x)$ having real coefficients, in other words having poles that are real or appear in complex conjugate pairs.

3.2 Minimizing the interpolation error bound

With $x_j \in [-1,1]$ and $\xi_k \notin [-1,1]$ the rational interpolation error is bounded above by

$$\left\|f - \frac{p_n}{q_m}\right\|_\infty \leq \max_{x\in[-1,1]}\left(\frac{|(fq_m)^{(n+1)}(x)|}{(n+1)!}\right)\max_{x\in[-1,1]}\prod_{j=0}^{n}\frac{|x-x_j|}{|q_m(x)|}. \tag{3.2}$$

The factor $(x-x_0)\cdots(x-x_n)/q_m(x)$ has minimal absolute value if the x_j are the Chebyshev-Markov nodes [60]. As defined as follows [61], these are also the zeroes of the orthogonal rational function $\mathscr{T}_{n+1}(x)$ with numerator of degree $n+1$, denominator equal to $q_m(x)$ and satisfying [62]

$$\int_{-1}^{1} \mathscr{T}_{n+1}(x)\frac{p_k(x)}{q_m(x)}\frac{dx}{\sqrt{1-x^2}} = 0, \qquad k = 0,\ldots,n.$$

If $n \geq m$ then we first complement the set of poles ξ_k with $\xi_m = \ldots = \xi_n = \infty$. Consider the Joukowski transform

$$J : \mathbb{C} \to \mathbb{C} : z \to J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

For $x = J(z)$ also $x = J(1/z)$ and so we restrict the inverse of the Joukowski transform to $|z| \leq 1$. Now take ζ_k, $0 \leq k \leq n$ such that $\xi_k = J(\zeta_k)$ and define

$$B_0(z) = 1, \qquad B_k(z) = \frac{z-\zeta_{k-1}}{1-\overline{\zeta}_{k-1}z}B_{k-1}(z), \qquad k = 1,\ldots,n,$$

$$\mathscr{T}_0(x) = \sqrt{\frac{1}{\pi}},$$

$$\mathscr{T}_{n+1}(x) = \sqrt{\frac{1-|\zeta_n|^2}{2\pi}}\left(\frac{z\overline{B}_n(\overline{z})}{1-\zeta_n z} + \frac{1}{(z-\zeta_n)B_n(z)}\right).$$

This orthogonal Chebyshev rational function has the preassigned poles $\xi_k \notin [-1,1]$ and so is different from the classical Chebyshev rational function with coinciding poles in -1: $\mathscr{T}_{n+1}(x)$ is of the form $p_{n+1}(x)/q_m(x)$.

If the poles ξ_k are real or appear in complex conjugate pairs, then the zeroes of $\mathscr{T}_{n+1}(x)$ are indeed real, simple and belong to the open interval $(-1,1)$ [62].

3.3 Minimizing the Lebesgue constant

The rational interpolant can also be seen as an element of span$\{1/q_m(x),\, x/q_m(x),\ldots,\, x^n/q_m(x)\}$. Since $\xi_k \notin [-1,1], 0 \leq k \leq m-1$ these functions form a Chebyshev system and hence the existence of the unique best approximant and of the interpolant are both guaranteed. The operator R_n that associates with f the rational interpolant p_n/q_m with preassigned poles is linear and so we can define the Lebesgue constant

$$M_n := M_n(x_0,\ldots,x_n;\xi_0,\ldots,\xi_{m-1}) = ||R_n||,$$

$$M_n = \sup_{||f||_\infty \leq 1} ||R_n f||_\infty = \max_{x\in[-1,1]} \sum_{i=0}^{n} \frac{|q_m(x_i)\ell_i(x)|}{|q_m(x)|}.$$

In [63] the authors determine the location of the poles $\xi_k, 0 \le k \le n-1$ that minimize the Lebesgue constant M_n for given interpolation points $x_j, 0 \le j \le n$. In [10] the asymptotic behavior of the Lebesgue constant M_n is given for equidistant nodes x_j and

$$q_n(x) = \sum_{i=0}^{n} (-1)^i \prod_{j=0, i \neq j}^{n} (x - x_j) \tag{3.3}$$

also defined in terms of the nodes. In both studies $m = n$ and $q_n(x)$ has real coefficients. When using rational interpolants with preassigned poles, none of the above situations is very practical. The location and the number of the poles is usually determined by the nature of the function f that one is modelling. Hence optimal interpolation points need to be found in terms of the poles and not vice versa.

Another practical drawback is the following. The values for M_n obtained in [63] are optimal in the sense that they are minimal for the considered $(x_0, \ldots, x_n; \xi_0, \ldots, \xi_{n-1})$ combination: changing either the poles or the interpolation points may increase M_n. Hence these values provide the rational analogue of the minimal growth behavior in the polynomial case. Note that neither these optimal poles $\xi_1, \ldots, \xi_{n-1}$ nor the minimal value for M_n are known by an explicit formula. All are obtained from the solution of a hefty optimization problem.

Our aim is to present a node set that doesn't suffer from the mentioned drawbacks: we give interpolation points that are nearly optimal for given arbitrary poles outside the interval of interpolation instead of vice versa, and our points can easily be obtained from a generalized eigenvalue problem [62]. We make use of the formulas from Section 3.2

If the preassigned finite poles ξ_k are real or appear in complex conjugate pairs, then for $n+1 \ge m$ the zeroes of $\mathscr{T}_{n+1}(x)$ are real, simple and belong to $(-1,1)$ [61]. These zeroes are the rational counterpart of what the Chebyshev nodes are in the polynomial case and hence are suitable interpolation points for (3.1). And as with other orthogonal functions, they can be obtained from a generalized eigenvalue problem. Unless there is a pole ξ_k at a very small distance of the interval $[-1,1]$, the maximum value of the Lebesgue function

$$\frac{\sum_{i=0}^{n} |q_m(x_i)\ell_i(x)|}{|q_m(x)|}$$

is not obtained near the endpoints of the interval. Hence extending the points as in the polynomial case to place x_0 in -1 and x_n in $+1$ usually makes no sense.

In Figure 3.1 we compare

- (full line) the nearly optimal Lebesgue constant $\Lambda_n(x_0,\ldots,x_n)$ for polynomial interpolation using the extended Chebyshev nodes (2.20),
- ($\circ$) the Lebesgue constant $M_n(x_0,\ldots,x_n;\xi_0,\ldots\xi_{n-1})$ for the Chebyshev nodes $x_j = -\cos((2j+1)\pi/(2n+2))$ with $q_n(x)$ and the ξ_k given by (3.3),
- ($\square$) the Lebesgue constant $M_n(x_0,\ldots,x_n;\xi_0,\ldots\xi_{n-1})$ for equidistant interpolation points $x_j = -1+j/n$ and with $q_n(x)$ and ξ_k determined by (3.3),
- ($+$) the optimal Lebesgue constant obtained in [63] for the case of equidistant interpolation points and optimally associated poles ξ_k,
- and our approach ($\star$), where we take the ξ_k from the same polynomial (3.3) to be comparable, but take the interpolation points for (3.1) from $\mathcal{T}_{n+1}(x) = 0$.

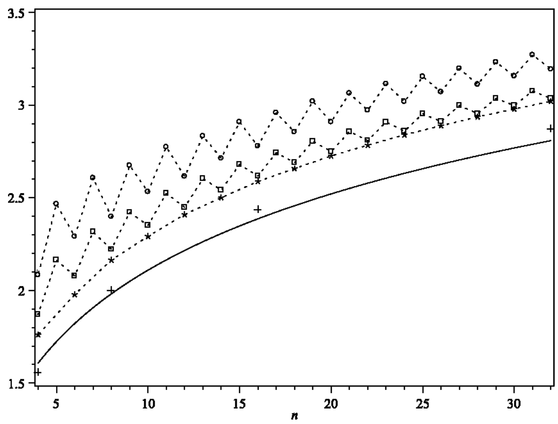

Figure 3.1

In Figure 3.2 we present, from left to right, the Lebesgue functions for $n = 10$ associated with the Lebesgue constants indicated by $+$, $\star$, $\square$ respectively.

Note that the rational interpolants with preassigned poles all generate Lebesgue constants that are very comparable to the one from the (almost) optimal polynomial interpolant. This comes in addition to the well-known ease of rational interpolation to fit steep changes

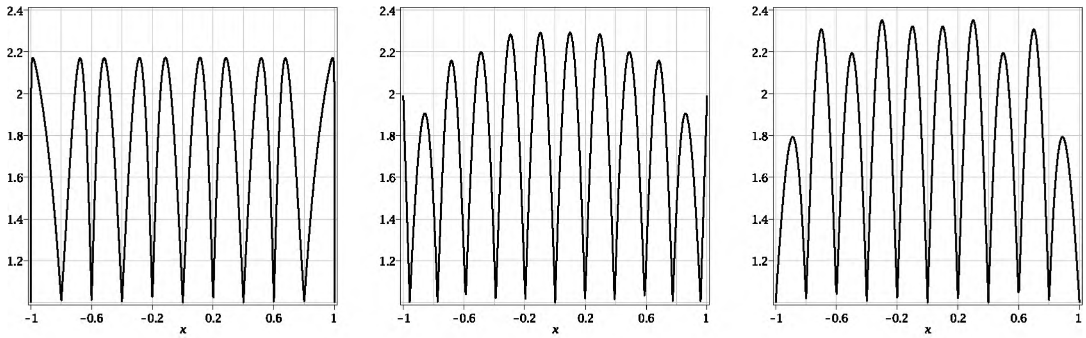

Figure 3.2

and asymptotic bahaviour and its tendency to oscillate less inbetween interpolation points than polynomials. And the new technique allows to determine good interpolation points for any set of preassigned poles $\{\xi_0, \ldots, \xi_{m-1}\}$, also for $m < n$ and not only for those determined by (3.3). Also, the new technique leads to smaller Lebesgue constants M_n than the ones from [10].

The latter is better illustrated in the Tables 3.1 and 3.2 where we show the variation between

- on the one hand the Lebesgue constants of the linear rational interpolation (3.1) using equidistant ($M_n^{\square}$), Chebyshev (M_n°) or extended Chebyshev nodes ($M_n^{\bullet}$), and
- on the other hand the Lebesgue constant from our technique ($M_n^{\star}$) that takes the interpolation points from $\mathscr{T}_{n+1}(x) = 0$.

In Table 3.1 we placed two poles at ± 1.001 and we choose the remaining poles randomly in $[-50, -1[\cup]1, 50]$. For $M_n^{\star}$ we extended the zeroes to put x_0 in -1 and x_n in $+1$. In Table 3.2 all poles were complex conjugate pairs with real part in $[-1, 1]$ and imaginary parts ± 0.01. Here we did not use extended nodes for $M_n^{\star}$. The displayed results are typical. The rate of growth is different between the situation illustrated in Table 3.1 and the one illustrated in Table 3.2. In the former the extended Chebyshev nodes maintain a rather modest rate of growth while the Chebyshev nodes generate a clearly faster growth and the equidistant nodes cause an explosion of the Lebesgue constant. In the latter both Chebyshev sets perform equally bad. We stress that in [10] and [63] the poles ξ_k are preassigned but dictated by the interpolation procedure. In our approach the poles are

Table 3.1

$M_{10}^{\star}$	2.491×10^{0}	$M_{20}^{\star}$	3.006×10^{0}
$M_{10}^{\bullet}$	2.017×10^{1}	$M_{20}^{\bullet}$	7.743×10^{1}
M_{10}°	3.586×10^{2}	M_{20}°	4.846×10^{2}
$M_{10}^{\square}$	4.943×10^{2}	$M_{20}^{\square}$	5.354×10^{5}

Table 3.2

$M_{10}^{\star}$	3.515×10^{0}
$M_{10}^{\bullet}$	2.714×10^{4}
M_{10}°	2.971×10^{4}
$M_{10}^{\square}$	1.702×10^{9}

freely preassigned and the interpolation points are adapted as in Section 3.2. In this more general setting our method offers a clear advantage.

In Figure 3.3 we graph the Lebesgue functions for $n = 10$ associated with the Lebesgue constants $M_{10}^{\star}, M_{10}^{\bullet}, M_{10}^{\square}$ of Table 3.1 respectively.

3.4 Maximizing the determinant of the Haar system

The rational interpolant p_n/q_m is a linear combination of the $\phi_i = x^i/q_m(x)$, $0 \leq i \leq n$ and therefore can be expressed as

$$\frac{p_n}{q_m}(x) = \sum_{i=0}^{n} f(x_i)\lambda_i(x),$$

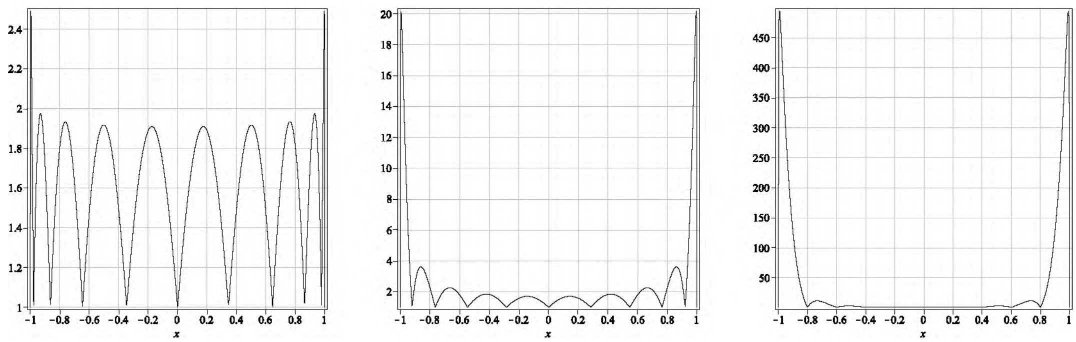

Figure 3.3

where the basic rational interpolants $\lambda_i(x)$ equal the quotient of determinants

$$\lambda_i(x) = \frac{|H(x_0,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n)|}{|H(x_0,\ldots,x_n)|},$$

$$H(x_0,\ldots,x_i\ldots,x_n) = \begin{pmatrix} 1/q_m(x_0) & \ldots & x_0^n/q_m(x_0) \\ \vdots & & \vdots \\ 1/q_m(x_i) & \ldots & x_i^n/q_m(x_i) \\ \vdots & & \vdots \\ 1/q_m(x_n) & \ldots & x_n^n/q_m(x_n) \end{pmatrix}. \tag{3.4}$$

The rational function $\lambda_i(x)$ satisfies $\lambda_i(x_j) = \delta_{ij}$ and further equals $q_m(x_i)\ell_i(x)/q_m(x)$. Maximizing the value of $|H(x_0,\ldots,x_n|$ is an unsolved problem that may provide another explicitly knowns node set.

CHAPTER 4

SHARP BOUNDS FOR LEBESGUE CONSTANTS OF BARYCENTRIC RATIONAL INTERPOLATION

4.1 Lebesgue constants for barycentric rational interpolation

When $m = n$ in (3.1), the operator that associates with f its rational interpolant p_n/q_n satisfying (3.1) is still linear and given by

$$R_n : C([-1,1]) \to V_n : f(x) \to \frac{p_n(x)}{q_n(x)} = \sum_{i=0}^{n} f(x_i) \frac{q_n(x_i)\ell_i(x)}{q_n(x)}.$$

In the same way as in Section 2.3, we obtain that the error in rational interpolation with preassigned poles is bounded above by

$$\left|\left| f - \frac{p_n}{q_n} \right|\right|_\infty \leq (1 + ||R_n||) \left|\left| f - \frac{p_n^*}{q_n} \right|\right|_\infty, \quad ||R_n|| = \max_{x\in[-1,1]} \sum_{i=0}^{n} \frac{|q_n(x_i)\ell_i(x)|}{|q_n(x)|},$$

where p_n^* is the best polynomial approximant of degree n to fq_n. Here $M_n := M_n(x_0, \ldots, x_n; \xi_1, \ldots, \xi_n) = ||R_n||$ is the Lebesgue constant of rational interpolation in the points $x_0, \ldots, x_n$ with preassigned poles at $\xi_1, \ldots, \xi_n$. The function

$$M_n(x) := M_n(x_0, \ldots, x_n; \xi_1, \ldots, \xi_n; x) = \sum_{i=0}^{n} \frac{|q_n(x_i)\ell_i(x)|}{|q_n(x)|}$$

is called the Lebesgue function of rational interpolation with predetermined poles.

In [64] the behaviour of M_n is investigated in case the x_j are the extended Chebyshev-Markov nodes for some predetermined $q_n(x)$. The notion extended is again to be understood in the way as in (2.20). It is important to note that $\mathscr{T}_{n+1}(x)$ is the rational function with monic numerator of degree $n+1$ and denominator $q_m(x)$ having minimal norm $||\cdot||_\infty$ on $[-1,1]$. So $\mathscr{T}_{n+1}(x)$ minimizes the bound (3.2) in the same way as $T_{n+1}(x)$ minimizes (2.10).

In [63] the poles ξ_k are determined in order to minimize M_n in the case of equidistant interpolation points x_j. So there the location of the poles is adapted to the given equidistant interpolation points, while in [64] the location of the interpolation points is optimized for given poles. It depends on the numerical application of course, whether it is more important to have equidistant data available than to make use of predetermined poles that dictate the shape and the behaviour of the interpolant.

Here we want to give sharp bounds on the growth of the Lebesgue constant M_n in the case of $n+1$ equidistant interpolation points x_j and n poles fixed by either [65]

$$q_n(x) = \sum_{i=0}^{n} (-1)^i \prod_{j=0, j\neq i}^{n} (x - x_j) \tag{4.1}$$

as in Section 4.2, or by [66]

$$s_n^{(d)}(x) = \sum_{i=0}^{n} (-1)^i \sigma_i \prod_{j=0, j\neq i}^{n} (x - x_j), \qquad \sigma_i = \sum_{j=\max(i-d,0)}^{\min(i,n-d)} \binom{d}{i-j},$$

$$n \geq 2d, \quad d = 1, 2, \ldots \tag{4.2}$$

as in Section 4.3. It is well-known that neither the polynomial $q_n(x)$ [65] nor the polynomial $s_n^{(d)}(x)$ [66] have zeroes on the real line. Hence in both cases $\xi_k \notin [-1,1]$.

A first analysis of M_n for equidistant interpolation points and poles preassigned by (4.1) or (4.2) is given in [10] and [11] respectively. We denote the former Lebesgue constant by

$$M_n^{(0)} := M_n(x_0, \ldots, x_n; q_n(\xi_k) = 0)$$

and the latter by

$$M_n^{(d)} := M_n(x_0, \ldots, x_n; s_n^{(d)}(\xi_k) = 0), \qquad d \geq 1.$$

In both cases we denote the Lebesgue function by $M_n(x)$, as it is clear from the context in which case we are.

4.2 Precise growth formula in case of Berrut's rational interpolation

For $q_n(x)$ in $||R_n||$ given by (4.1), the expression for the Lebesgue function $M_n(x)$ can be simplified to

$$M_n(x) = \frac{\sum_{i=0}^{n} 1/|x-x_i|}{|\sum_{i=0}^{n}(-1)^i/(x-x_i)|}. \tag{4.3}$$

In [10] crude lower and upper bounds are given for $M_n^{(0)}$:

$$\frac{2}{\pi+\frac{4}{n}}\ln(n+1) \leq M_n^{(0)} \leq 2+\ln(n).$$

We illustrate these in Figure 4.1, where $M_n^{(0)}$, for subsequent values of n, is indicated with the symbol □.

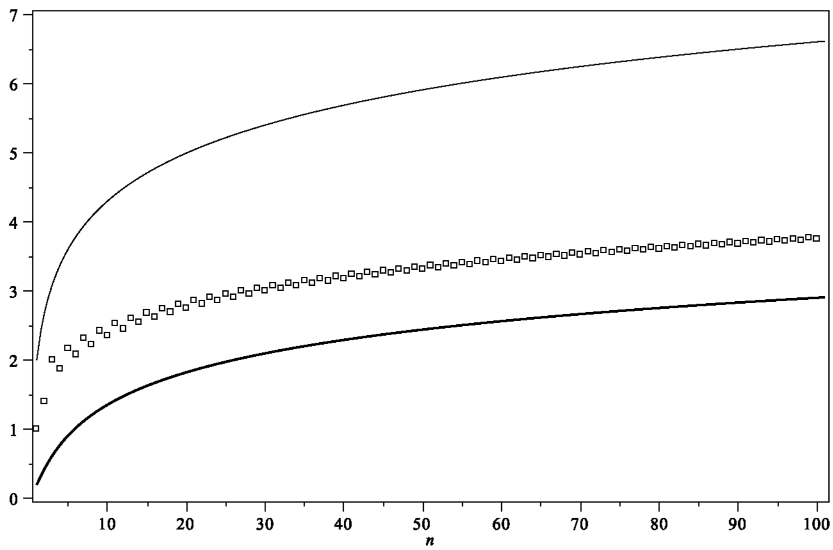

Figure 4.1 Bounds for $M_n^{(0)}$ as in [10].

As proved in Section 4.4, the growth rate of $M_n^{(0)}$ is given more precisely by

$$\frac{2\,(\ln(n+1)+\ln 2+\gamma)}{\pi+\frac{4}{n+3}} \leq M_n^{(0)} \simeq \frac{2\left(\ln(n+1)+\ln 2+\gamma+\frac{1}{24n}\right)}{\pi-\frac{4}{n+2}}. \tag{4.4}$$

This is the exact asymptotic growth of the Lebesgue constant $M_n^{(0)}$. The new bounds are illustrated in Figure 4.2.

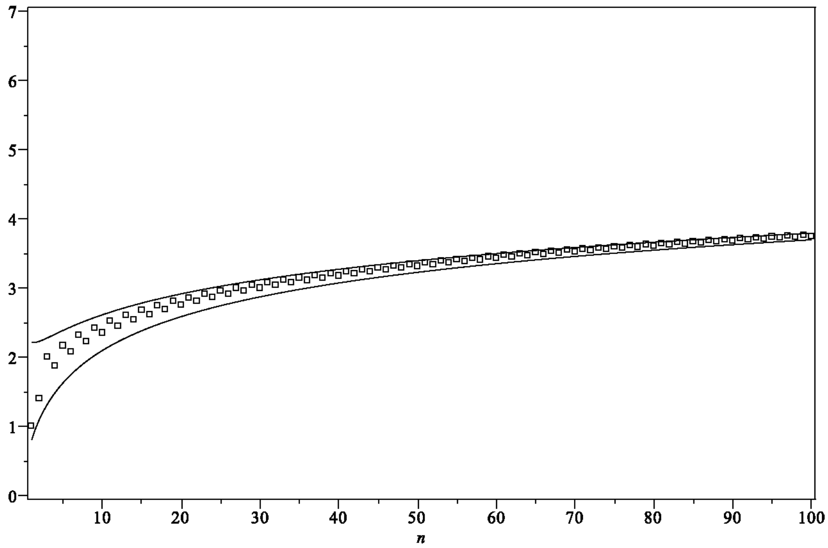

Figure 4.2 Sharpened bounds for $M_n^{(0)}$.

The fine proof is based on numerical experiments carried out in exact arithmetic up to $n = O(10^{1000})$! The advantage of exact arithmetic here (besides the absence of rounding errors) is that, in computer algebra software, there is a nice compact expression for the evaluation of $M_n(x)$ halfway between two neighbouring interpolation points in terms of the digamma function $\Psi(x)$. This expression allows us to evaluate it for very high values of n. When making the detailed analysis, the true problem to obtain an accurate bound becomes clear. The maximum value of the Lebesgue function $M_n(x)$ is not taken near a fixed location, independent of n, like the midpoint or the endpoints of the interval: for n even, the location of the positive maximum is a function of $\log_{10} n$. More precisely : it changes with n mod 4 and moves up with $\log_{10}\lfloor n/4 \rfloor$ when n is even! To illustrate this, we show the value of $M_n(x)$ near the many local maxima (see also Figure 4.5). For $n = 4 \times 10^{100}$ and $x = (20i+9)/n, i = 0, \ldots, 9$ the values can be found in Table 4.1: a global maximum is (not even at, but) near $x = 149/n$ (we also show the values of $M_n(x)$ at $x = 145/n$ and $x = 153/n$ for comparison). For $n = 4 \times 10^{1000} + 2$ and $x = (200i+67)/n, i = 0, \ldots, 7$ the values can be found in Table 4.2: a global maximum is very near $x = 1467/n$ (compare with the value of $M_n(x)$ at $x = 1463/n$ and $x = 1471/n$).

Table 4.1 Local near-optima of $M_n(x), n = 4 \times 10^{100}$, 209 digits

$M_n(189/n)$	148.2784002 (189 digits) 1347141070
$M_n(169/n)$	148.2784002 (189 digits) 1371588542
$M_n(153/n)$	148.2784002 (189 digits) 1379687364
$M_n(149/n)$	148.2784002 (189 digits) 1380120520
$M_n(145/n)$	148.2784002 (189 digits) 1379917056
$M_n(129/n)$	148.2784002 (189 digits) 1372737004
$M_n(109/n)$	148.2784002 (189 digits) 1349437993
$M_n(89/n)$	148.2784002 (189 digits) 1310223488
$M_n(69/n)$	148.2784002 (189 digits) 1255093489
$M_n(49/n)$	148.2784002 (189 digits) 1184047995
$M_n(29/n)$	148.2784002 (189 digits) 1097087007
$M_n(9/n)$	148.2784002 (189 digits) 0994210525

Table 4.2 Local near-optima of $M_n(x), n = 4 \times 10^{1000} + 2$, 2012 digits

$M_n(1471/n)$	1467.562478 (1987 digits) 700047035547169
$M_n(1467/n)$	1467.562478 (1987 digits) 700047058426038
$M_n(1463/n)$	1467.562478 (1987 digits) 700047017642930
$M_n(1267/n)$	1467.562478 (1987 digits) 699967033348511
$M_n(1067/n)$	1467.562478 (1987 digits) 699727853327892
$M_n(867/n)$	1467.562478 (1987 digits) 699329518364181
$M_n(667/n)$	1467.562478 (1987 digits) 698772028457378
$M_n(467/n)$	1467.562478 (1987 digits) 698055383607483
$M_n(267/n)$	1467.562478 (1987 digits) 697179583814496
$M_n(67/n)$	1467.562478 (1987 digits) 696144629078418

4.3 Growth formulas in case of Floater and Hormann's rational interpolation

For $q_n(x)$ in $||R_n||$ given by (4.2), the expression for the Lebesgue function $M_n(x)$ simplifies to

$$M_n(x) = \frac{\sum_{i=0}^{n} \sigma_i/|x-x_i|}{|\sum_{i=0}^{n}(-1)^i \sigma_i/(x-x_i)|}. \tag{4.5}$$

Taking a closer look at the $\sigma_i, i = 0, \ldots, n$, we see that

$$\sigma_i = \begin{cases} \sum_{j=0}^{i} \binom{d}{j}, & i \le d, \\ 2^d, & d \le i \le n-d, \\ \sigma_{n-i}, & i \ge n-d. \end{cases}$$

The observation in [11] that with n odd for $d=1$, the maximum of the Lebesgue function occurs at the origin, is imprecise. We illustrate this in Figure 4.3: with $n=11, d=1$ the Lebesgue function $M_n(x)$ achieves its maximum near $\pm 2/11$. A correcter statement is that for $n \bmod 4 = 1$ the maximum is at $x^* = 0$ and for $n \bmod 4 = 3$ the maximum is near $\pm 2/n$. For $d=1$ with n even, the maximum is near $1/n$.

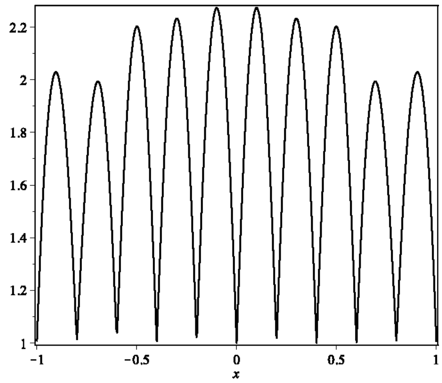

Figure 4.3 Graph of $M_{11}(x)$.

But the same sharp lower and upper bound estimates as given in (4.4) apply to $M_n^{(1)}$ with $d=1$ in (4.2). An illustration is given in Figure 4.4, where $M_n^{(1)}$ is indicated by $\diamond$.

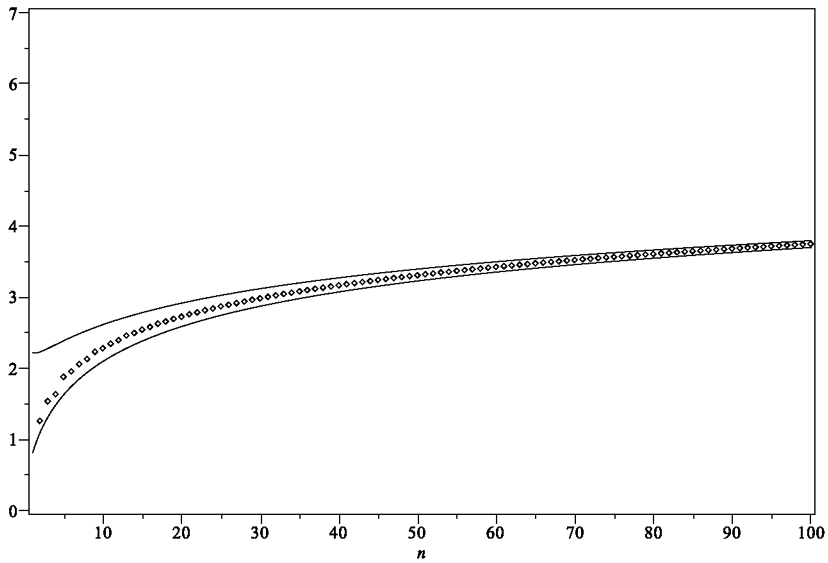

Figure 4.4 Sharpened bounds for $M_n^{(1)}$.

When $d > 1$, then an improved (but not yet fine) upper bound is given by:

$$M_n^{(d)} \leq 2^{d-1} \frac{2}{\pi - \frac{4}{n+2}} \left(\ln(n+1) + \ln 2 + \gamma + \frac{1}{24n} \right), \qquad d > 1. \tag{4.6}$$

The results mentioned in Section 4.3 for $d \geq 1$ can be proved in a similar way as we prove (4.4) in Section 4.4.

4.4 Proof of sharp growth estimates for $M_n^{(0)}$

Let the interpolation points x_j be equidistant, $x_j = -1 + 2j/n, j = 0, \ldots, n$ and let the poles $\xi_1, \ldots, \xi_n$ lie outside $[-1, 1]$. The Lebesgue function $M_n(x)$ given by (4.3) takes the (minimum) value 1 at the interpolation points $x_j, j = 0, \ldots, n$ and has n local maxima inbetween each pair of consecutive interpolation points. It is clear that the Lebesgue function $M_n(x)$ is symmetric with respect to the origin: $M_n(-x) = M_n(x)$. The graph of $M_n(x)$ essentially takes 4 different shapes, depending on the value of n, and the proof of the growth rate distinguishes these 4 different cases. In Figure 4.5 we show $M_n(x)$ for $n = 4k, 4k+1, 4k+2, 4k+3$ with $k = 1$.

As we prove further down and as is clearly visible from Figure 4.5, the position of a global maximum x^* of $M_n(x)$ changes with n mod 4 and is (except for $n = 4k+3$) located near (not precisely at!) a midpoint of two interpolation points (note that $2/n$ is the distance between two consecutive interpolation points). Also, because of the symmetry of $M_n(x)$, whenever x^* is a maximum, so is $-x^*$. We focus on the positive argument of the maximum.

For small $n \geq 3$ (for $n = 1, x^* = 0$ and for $n = 2, x^* \approx {}^1/_2$) the statement can be made rather precise:

$$\begin{aligned}
&n \bmod 4 = 0 : x^* \approx \frac{1}{n}, \qquad n \leq 24,\\
&n \bmod 4 = 1 : x^* \approx \frac{2}{n},\\
&n \bmod 4 = 2 : x^* \approx \frac{3}{n}, \qquad n \leq 718,\\
&n \bmod 4 = 3 : x^* = 0.
\end{aligned} \tag{4.7}$$

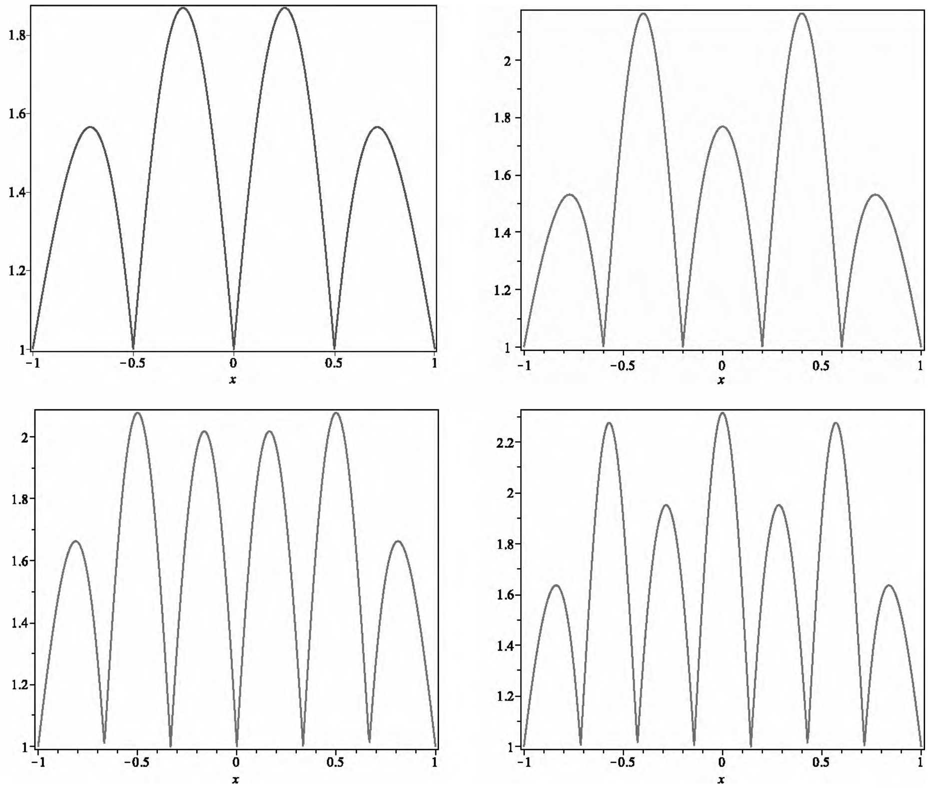

Figure 4.5 Graphs of $M_4(x), M_5(x), M_6(x), M_7(x)$ from left to right and top to bottom.

And more generally, for $k = \lfloor n/4 \rfloor > 1$:

$$\begin{aligned}
&n \bmod 4 = 0 : x^* \in]x_{2k}, x_{2k+\lceil \log_{10} k \rceil + 2}[= \left]0, \frac{2(\lceil \log_{10} k \rceil + 2)}{n}\right[, \\
&n \bmod 4 = 1 : x^* \approx \frac{2}{n}, \\
&n \bmod 4 = 2 : x^* \in]x_{2k+1}, x_{2k+\lceil \log_{10} k \rceil + 1}[= \left]0, \frac{2\lceil \log_{10} k \rceil}{n}\right[, \\
&n \bmod 4 = 3 : x^* = 0.
\end{aligned} \tag{4.8}$$

To determine the location of the maximum we further make use of some simple rules.

Rules

$$\frac{N}{D} < \frac{A}{B} \Rightarrow \frac{N}{D} < \frac{N+A}{D+B}, \qquad N, D, A, B > 0, \tag{4.9a}$$

$$D \leq N \Rightarrow \frac{N+C}{D+C} \leq \frac{N}{D}, \qquad N, D, C > 0, \tag{4.9b}$$

$$D \leq N, B < D \Rightarrow \frac{N+A}{D+A} \leq \frac{N+B}{D-B}, \qquad N, D, A, B > 0, \tag{4.9c}$$

$$D \leq N, B < A \Rightarrow \frac{N+A}{D+A} \leq \frac{N+B}{D+B}, \qquad N, D, A, B > 0. \tag{4.9d}$$

To prove the estimates, once the location of a maximum is known, we also need a lemma [35] and bounds on the partial sums of the Leibniz series.

Lemma

$$\sum_{k=0}^{n} \frac{1}{2k+1} < \frac{1}{2}\ln(n+1) + \ln 2 + \frac{\gamma}{2} + \frac{1}{48(n+1)^2}. \tag{4.10}$$

Series

$$\frac{\pi}{4} - \frac{1}{2n+3} < \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} < \frac{\pi}{4} + \frac{1}{2n+3}.$$

Now let's start by proving (4.8). In order to simplify the computations, we make a change of variable, from $x \in [-1,1]$ to $y \in [0,1]$ by $y := (x+1)/2$. This way we are dealing only with positive values in the subsequent sums. The interpolation points x_i are then mapped to equidistant points y_i at a distance $1/n$ of each other. Because there is no risk of ambiguity, when consistently using y-values with evaluations expressed in the transformed variable and x-values with evaluations expressed in the original variable, the same notation M_n is used for the Lebesgue function in the variable x and the function after the transformation of x to y.

We now investigate the value of the Lebesgue function $M_n(y)$ at the midpoints $\hat{y}_i = (y_{i-1} + y_i)/2, i = 1, \ldots, n$ which are local near-maxima (the values displayed in Table 4.1 are for instance $M_n(\hat{y}_{2k+\ell})$ for $\ell = 10i + 5, i = 0, \ldots, 9$). It is easy to verify that

$$M_n(\hat{y}_i) = \frac{N_n(\hat{y}_i)}{D_n(\hat{y}_i)}, \qquad i = 1, \ldots, n,$$

where

$$N_n(\hat{y}_i) = \sum_{j=0}^{i-1} \frac{1}{2j+1} + \sum_{j=0}^{n-i} \frac{1}{2j+1}, \tag{4.11a}$$

$$D_n(\hat{y}_i) = \sum_{j=0}^{i-1} \frac{(-1)^j}{2j+1} + \sum_{j=0}^{n-i} \frac{(-1)^j}{2j+1}. \tag{4.11b}$$

We write $n = 4k + (n \bmod 4)$. For $n \bmod 4 = 1$ and $n \bmod 4 = 2$ we have

$$N_n(\hat{y}_1) < N_n(\hat{y}_2) < \ldots < N_n(\hat{y}_{2k+1})$$

and

$$D_n(\hat{y}_{2i+1}) > D_n(\hat{y}_{2i+2}), \qquad i = 0, \ldots, k-1,$$

$$D_n(\hat{y}_{2i+1}) > D_n(\hat{y}_{2i+3}), \qquad i = 0, \ldots, k-1.$$

For $n \bmod 4 = 0$, the statements about N_n and D_n at $\hat{y}_{2k+1}$ are dropped, and for $n \bmod 4 = 3$, similar statements at $\hat{y}_{2k+2}$ are added. From the above we can deduce that for $n \bmod 4 = 1$ and $n \bmod 4 = 2$,

$$M_n(\hat{y}_{2i+1}) < M_n(\hat{y}_{2i+2}), \qquad i = 0, \ldots, k-1,$$

$$M_n(\hat{y}_{2i+1}) < M_n(\hat{y}_{2i+3}), \qquad i = 0, \ldots, k-1,$$

with a similar adjustment for $n \bmod 4 = 0$ and $n \bmod 4 = 3$ as before. Now we treat the cases n odd and n even separately.

When n is odd we find that for $n \bmod 4 = 1$,

$$M_n(\hat{y}_{2i}) \leq M_n(\hat{y}_{2k}), \qquad i = 1, \ldots, k-1 \tag{4.12}$$

by combining (4.9a) for $N = N_n(\hat{y}_{2i}), D = D_n(\hat{y}_{2i})$ with

$$\frac{N_n(\hat{y}_{2k}) - N_n(\hat{y}_{2i})}{D_n(\hat{y}_{2k}) - D_n(\hat{y}_{2i})} \leq \frac{N_n(\hat{y}_{2k}) - N_n(\hat{y}_{2i+2})}{D_n(\hat{y}_{2k}) - D_n(\hat{y}_{2i+2})}, \qquad i = 1, \ldots, k-2.$$

Analogously, for $n \bmod 4 = 3$, the statement (4.12) holds for $i = 1, \ldots, k$ with $\hat{y}_{2k}$ in the right hand side replaced by $\hat{y}_{2k+2}$.

The situation is more complicated when n is even though. But fortunately the following inequalities for the near-maxima at $\hat{y}_{2k}$, and the other local maxima near $\hat{y}_{2i}$, help us out. Using (4.9a) we find

$$M_{4k+1}(\hat{x}_{2k}) \leq M_{4k+3}(\hat{x}_{2k+2}).$$

From (4.9b) we obtain

$$M_{4k+2}(\hat{x}_{2i}) \leq M_{4k+1}(\hat{x}_{2i}), \qquad i = 1, \ldots, k.$$

And finally (4.9c) gives

$$M_{4k}(\hat{x}_{2i}) \leq M_{4k+2}(\hat{x}_{2i}), \qquad i = 1, \ldots, k.$$

The question that remains is now whether in case $n \bmod 4 = 1$ or $n \bmod 4 = 2$ the maximum is near $M_n(\hat{y}_{2k})$ or near $M_n(\hat{y}_{2k+1})$. Using (4.9c) we obtain

$$M_{4k+1}(\hat{x}_{2k+1}) \leq M_{4k+1}(\hat{x}_{2k})$$

and at last from (4.9d)

$$M_{4k+2}(\hat{x}_{2k+1}) \leq M_{4k+2}(\hat{x}_{2k}).$$

Since we know that when $n \bmod 4 = 3$, a maximum is exactly at $M_n(\hat{x}_{2k+2})$, we can use this value to compute an upper bound estimate for the Lebesgue constant $M_n^{(0)}$. Likewise a lower bound for $M_n^{(0)}$ can be obtained because $M_{4k}(\hat{x}_{2k}) \leq M_{4k}^{(0)} \leq M_n^{(0)}$ for general n. To conclude:

$$\max_n \max_{x \in [-1,1]} M_n(x) \approx M_{4k+3}(0)$$

and

$$\min_n \max_{x \in [-1,1]} M_n(x) \geq M_{4k}(1/n).$$

In other words, a sharp upper bound for $M_{4k+3}(0)$ is an accurate estimate for $M_n(x^*), n = 4k+i, 0 \leq i \leq 3$, and a lower bound for $M_{4k}(1/n)$ is a lower bound for $M_n(x^*), n = 4k+i, 0 \leq i \leq 3$.

To prove the actual bounds, we make use of the transformed variable y again. For the upper bound we have:

$$\begin{aligned}
M_{4k+3}(^1/_2) &= \frac{N_{4k+3}(^1/_2)}{D_{4k+3}(^1/_2)},\\
N_{4k+3}(^1/_2) &= 2\sum_{j=0}^{2k+1}\frac{1}{2j+1}\\
&\leq \ln(8k+8)+\gamma+\frac{1}{24(2k+2)^2}\\
&\leq \ln(2n+2)+\gamma+\frac{1}{24n}\\
D_{4k+3}(^1/_2) &= 2\left|\sum_{j=0}^{2k+1}\frac{(-1)^{j+2k+1}}{2j+1}\right|\\
&\geq \frac{\pi}{2}-\frac{2}{n+2}.
\end{aligned}$$

From these inequalities it follows that (stated in the variable x again)

$$\max_n \max_{x\in[-1,1]} M_n(x) \simeq \frac{2}{\pi-\frac{4}{n+2}}\left(\ln(n+1)+\ln 2+\gamma+\frac{1}{24n}\right).$$

For the lower bound, expressed in the transformed variable y, we use the fact that the numerator of $M_{4k}(^{n+1}/_{2n})$ can be expressed using the digamma function $\Psi(x)$ where for $x>0$ it holds that $\ln(x)-1/x\leq\Psi(x)$:

$$\begin{aligned}
M_{4k}(^{n+1}/_{2n}) &= \frac{N_{4k}(^{n+1}/_{2n})}{D_{4k}(^{n+1}/_{2n})},\\
N_{4k}(^{n+1}/_{2n}) &= \sum_{j=0}^{2k+2}\frac{1}{2j+1}+\sum_{j=0}^{2k+1}\frac{1}{2j+1}\\
&\geq \ln(8k+2)+\gamma = \ln(2n+2)+\gamma,\\
D_{4k}(^{n+1}/_{2n}) &\leq 2\left|\sum_{j=0}^{2k+2}\frac{(-1)^j}{2j+1}\right|\\
&\leq \frac{\pi}{2}+\frac{2}{n+3}
\end{aligned}$$

from which (4.4) follows.

CHAPTER 5

RADIAL ORTHOGONALITY AND LEBESGUE CONSTANTS ON THE DISK

In polynomial interpolation, the choice of a polynomial basis and the location of the interpolation points greatly influence the numerical conditioning of polynomial interpolation, and hence the quality of the computed interpolant. Moreover, some sets of interpolation points deliver near-best polynomial approximants, while others lead to divergence of the interpolation scheme. Fortunately, a univariate polynomial basis is always a Chebyshev system, thereby guaranteeing the existence and unicity of the polynomial interpolant for a set of distinct interpolation points. In the multivariate case, the situation is much more difficult. The location of the interpolation points also needs to be such that it guarantees unisolvence of the interpolation problem because no polynomial basis is a Chebyshev system. And because of the curse of dimensionality faced in polynomial interpolation, alternative techniques like radial basis interpolation have become very popular. But then the latter are prone to ill-conditioning. Here we propose an extremely well-conditioned alternative to radial basis interpolation on the disk (see Sections 5.1 and 5.2). At the same time we identify sets of interpolation points that guarantee a very small Lebesgue constant and consequently interpolants that are near-best polynomial approximants (see Sections 5.3 and 5.4). Both results follow from a detailed study of radial or spherical orthogonality on the disk.

5.1 Radial orthogonality

Let $\overline{B}_{d,p}(0;1)$ denote the closed unit ball centered at the origin in $\mathbb{R}^d$ equipped with the ℓ_p-norm. For each ℓ_p-norm this ball is a d-variate analogue of the closed interval $[-1,1]$.

For $p = \infty$ and $d = 2$ it is the unit square $[-1,1] \times [-1,1]$, for $p = 2$ and $d = 2$ the unit disk $\{(x,y); 0 \le x^2 + y^2 \le 1\}$ and for $p = 1$ and $d = 2$ it is the simplex $\{(x,y); 0 \le |x| + |y| \le 1\}$.

For the definition of our multivariate orthogonal polynomials we replace the cartesian coordinates $X = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by the new spherical coordinates $X = (x_1, \ldots, x_d) = (\lambda_1 z, \ldots, \lambda_d z)$ with $\lambda = (\lambda_1, \ldots, \lambda_d)$ belonging to the ℓ_p unit sphere $S_{d,p}(0;1) \subset \mathbb{R}^d$ and $z \in \mathbb{R}$. While λ contains the directional information of X, the radial variable z contains the signed distance information. A signed distance function is defined by

$$\text{sd}(X) = \text{sgn}(x_k) ||X||_p, \qquad k = \min\{j : x_j \neq 0\}. \tag{5.1}$$

Since λ is not unique, we choose it such that for given X we have $z = \text{sd}(X)$. We denote by $\mathbb{R}[\lambda] = \mathbb{R}[\lambda_1, \ldots, \lambda_d]$ the linear space of d-variate polynomials in the λ_k with real coefficients, by $\mathbb{R}(\lambda) = \mathbb{R}(\lambda_1, \ldots, \lambda_d)$ the commutative field of rational functions in the λ_k with real coefficients, by $\mathbb{R}[\lambda][z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}[\lambda]$ and by $\mathbb{R}(\lambda)[z]$ the linear space of polynomials in the variable z with coefficients from $\mathbb{R}(\lambda)$.

In the bivariate case we mostly use the notation $X = (x,y)$ instead of $X = (x_1, x_2)$ and $\lambda = (\alpha, \beta)$ instead of $\lambda = (\lambda_1, \lambda_2)$.

We introduce d-variate functions $V_m(X)$ that are polynomials in z with coefficients from $\mathbb{R}[\lambda]$:

$$V_m(X) = \mathcal{V}_m(\lambda; z) = \sum_{i=0}^{m} b_{m^2-i}(\lambda) z^i.$$

The $b_{m^2-i}(\lambda)$ are homogeneous polynomials in the λ_k of degree $m^2 - i$. Note that the functions $V_m(X)$ do not belong to $\mathbb{R}[X]$ but they belong to $\mathbb{R}[\lambda][z]$. Therefore the $V_m(X)$ can be viewed as spherical polynomials: for every $\lambda \in S_{d,p}(0;1)$ the function $V_m(X) = \mathcal{V}_m(\lambda; z)$ is a polynomial of degree m in the radial variable $z = \text{sd}(X)$. In addition, the functions

$$\mathcal{V}_m(\lambda; \lambda_1 x_1 + \ldots + \lambda_d x_d)$$

are polynomial in the x_k, they belong to $\mathbb{R}[\lambda][X]$ and play a crucial role in the sequel.

On the $V_m(X)$ we impose the orthogonality conditions

$$\int \cdots \int_{||X||_p \le 1} \left(\sum_{k=1}^{d} x_k \lambda_k\right)^i \mathcal{V}_m\left(\lambda; \sum_{k=1}^{d} x_k \lambda_k\right) w(z)\, dX = 0,\ i = 0, \ldots, m-1 \tag{5.2}$$

where $w(z)$ is a non-negative weight function with

$$\int \cdots \int_{||X||_p \le 1} w(z) dX < \infty.$$

The coefficients $b_{m^2-i}(\lambda)$ are obtained from the (symbolic/parameterized) linear system

$$\sum_{j=0}^{m} c_{i+j}(\lambda) b_{m^2-j}(\lambda) = 0, \qquad i = 0, \ldots, m-1 \tag{5.3}$$

where $c_i(\lambda)$ are the moments given by

$$c_i(\lambda) = \int \cdots \int_{||X||_p \le 1} \left(\sum_{k=1}^{d} x_k \lambda_k\right)^i w(z)\, dX, \qquad i = 0, \ldots, 2m-1. \tag{5.4}$$

The $c_i(\lambda)$ are homogeneous polynomials in the λ_k of degree i. This radial or spherical orthogonality was already introduced in [67] and [68] although it was not yet termed like that in the early references. An explanation why $b_{m^2-i}(\lambda)$ needs to be of degree $m^2 - i$ is also given there.

For a fixed directional vector $\lambda = \lambda^*$, the projected spherical polynomials $\mathcal{V}_m(\lambda^*; \lambda_1^* x_1 + \ldots + \lambda_d^* x_d)$ are univariate polynomials in the variable $z = \lambda_1 x_1 + \ldots + \lambda_d x_d$, orthogonal on the interval $[A, B] \subset \text{span}\{\lambda^*\}$ with

$$\begin{aligned} A &= \min_{(x_1,\ldots,x_d) \in \overline{B}_{d,p}(0;1)} (\lambda_1^* x_1 + \ldots + \lambda_d^* x_d), \\ B &= \max_{(x_1,\ldots,x_d) \in \overline{B}_{d,p}(0;1)} (\lambda_1^* x_1 + \ldots + \lambda_d^* x_d) \end{aligned} \tag{5.5}$$

and $\lambda^* \in S_{d,p}(0;1)$. Note that the weight function is multivariate instead of univariate. In addition, the weight at $X = (x_1, \ldots, x_d) \in \overline{B}_{d,p}(0;1)$ is $w(\text{sd}(X))$ with $z = \text{sd}(X)$ and $-1 \le z \le 1$ and not $w(\lambda_1 x_1 + \ldots + \lambda_d x_d)$ which has a different support.

We point out the similarity of the $\mathcal{V}_m(\lambda; \lambda_1 x_1 + \ldots + \lambda_d x_d)$ with radial basis functions. The variable of our multidimensional function is

$$\langle X, \lambda \rangle = \lambda_1 x_1 + \ldots + \lambda_d x_d \tag{5.6}$$

which is the projection of X onto a directional unit vector $\lambda \in S_{d,p}(0;1)$ and this for continuously varying λ. When $X \in \text{span}\{\lambda\}$, then $\langle X, \lambda \rangle = z||\lambda||_2^2$. In the case of radial basis functions the variable is a weighted $||X - C||_p$ for a set of distinct vectors C. So for the user the choice is between spherical orthogonality or radial functions forming a Chebyshev set. More on the comparison with radial basis functions is to be found in Section 5.5.

For symmetric weight functions $w(z)$, the zeroes $\zeta_{i,m}(\lambda), i = 1, \ldots, m$ of the spherical orthogonal polynomials $\mathscr{V}_m(\lambda; z)$ appear in symmetric pairs, with one zero describing a curve in the right halfplane because of (5.1) and the other zero tracking the same curve in the left halfplane but mirrored with respect to the origin. In Figure 5.1 we show the zeroes for the case $d = 2, p = \infty, m = 3$ and $w(z) = 1$: $\zeta_{1,3}(\lambda)$ lies in the left half plane, $\zeta_{2,3}(\lambda)$ equals zero, $\zeta_{3,3}(\lambda)$ lies in the right half plane. For $\lambda^* = (1,1)$ for instance, the zeroes $\zeta_{1,3}(\lambda^*), \zeta_{2,3}(\lambda^*), \zeta_{3,3}(\lambda^*)$ lie in the interval $[-2, 2]$ which is the support of orthogonality in $\text{span}\{(1,1)\}$.

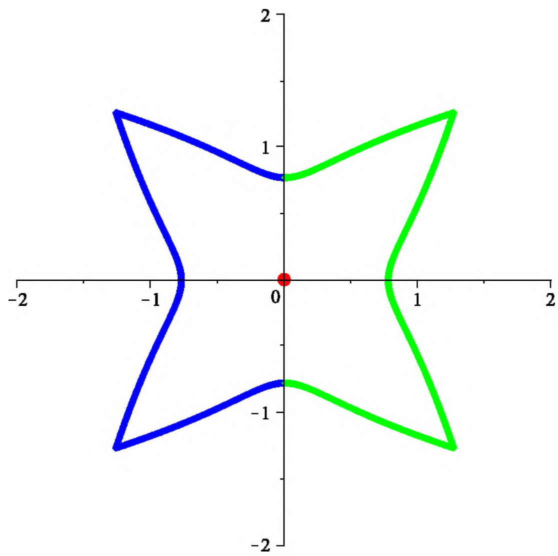

Figure 5.1 Zero curves of $\mathscr{V}_3(\lambda; z)$ orthogonal on $\overline{B}_{2,\infty}(0;1)$ for $w(z) = 1$.

In the case of Theorem 5.2 below, each curve $\zeta_{i,m}(\lambda)$ is a half circle. So a symmetric pair of zeroes describes a full circle. We then simply say that the radius of the circle equals the zero of the spherical orthogonal polynomial, and this is to be understood as a positive and a negative zero each describing half a circle. A use of this can be found in Section 5.3.

Note that for the moment the functions $V_m(X) = \mathscr{V}_m(\lambda;z)$ are unnormalized. We usually normalize them by requiring that $\gcd(b_{m^2-m}(\lambda),\ldots,b_{m^2}(\lambda)) = 1$, thus decreasing the degree of the $b_{m^2-i}(\lambda)$ with the same amount for each i. When the $b_{m^2-i}(\lambda)$ reduce to a constant, then the $V_m(X) = \mathscr{V}_m(\lambda;z)$ can be made monic. Examples of $V_m(X) = \mathscr{V}_m(\lambda;z)$ for different weight functions are given in [69].

5.2 Bivariate orthogonal cartesian basis

Now assume that $d = 2$, in other words we are considering the ℓ_p-ball $\overline{B}_{2,p}(0;1)$, and denote $\lambda_1 = \alpha$ and $\lambda_2 = \beta$. When substituting actual values for the λ_k then the function $\mathscr{V}_m(\lambda;\lambda_1 x_1 + \ldots + \lambda_d x_d)$ becomes a polynomial function in the x_k. The question that arises is whether these radially orthogonal functions can be used to construct a cartesian orthogonal basis for the linear space $\mathbb{R}[X]$. The answer is affirmative and the construction goes in three steps. Theorem 5.1 applies to general weight functions $w(z)$ and all ℓ_p-norms. Theorem 5.2 holds for specific weight functions and the closed Euclidean d-ball $\overline{B}_{d,2}(0;1)$. Theorem 5.3 is only valid for the specific weight function $w(z) = 1$ on the Euclidean disk $\overline{B}_{2,2}(0;1)$.

Theorem 5.1 *The set* $\{\mathscr{V}_m(\alpha_{m,k},\beta_{m,k};\alpha_{m,k}x+\beta_{m,k}y), 0 \le k \le m, m \in \mathbb{N}\}$ *is a basis for* $\mathbb{R}[x,y]$ *if*

$$\begin{vmatrix} \alpha_{m,0}^m & \alpha_{m,0}^{m-1}\beta_{m,0} & \cdots \alpha_{m,0}\beta_{m,0}^{m-1} & \beta_{m,0}^m \\ \alpha_{m,1}^m & \alpha_{m,1}^{m-1}\beta_{m,1} & \cdots \alpha_{m,1}\beta_{m,1}^{m-1} & \beta_{m,1}^m \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{m,m}^m & \alpha_{m,m}^{m-1}\beta_{m,m} & \cdots \alpha_{m,m}\beta_{m,m}^{m-1} & \beta_{m,m}^m \end{vmatrix} \neq 0, \qquad m \in \mathbb{N}.$$

proof 5.1 *It suffices to prove that for each m the* $(\alpha_{m,k}x+\beta_{m,k}y)^m$ *with* $k = 0,\ldots,m$ *are a basis for the homogeneous polynomials of degree m in x and y. So let us assume that a nontrivial vector* $(\gamma_0,\ldots,\gamma_m)$ *exists such that*

$$\sum_{k=0}^{m} \gamma_k(\alpha_{m,k}x+\beta_{m,k}y)^m = 0.$$

Then

$$\sum_{i=0}^{m}\binom{m}{i}x^{m-i}y^{i}\left(\sum_{k=0}^{m}\gamma_k\alpha_{m,k}^{m-i}\beta_{m,k}^{i}\right)=0$$

and hence

$$\sum_{k=0}^{m}\gamma_k\alpha_{m,k}^{m-i}\beta_{m,k}^{i}=0,\qquad i=0,\ldots,m.$$

But the latter is impossible because of the regularity of the coefficient matrix

$$A=(a_{i+1,k+1})_{0\le i,k\le m}=\left(\alpha_{m,k}^{m-i}\beta_{m,k}^{i}\right)_{0\le i,k\le m}.$$

A suitable selection for the values $\alpha_{m,k}$ and $\beta_{m,k}$ is for instance

$$\alpha_{m,k}=\frac{k}{||(k,m-k)||_p},\qquad \beta_{m,k}=\frac{m-k}{||(k,m-k)||_p}.$$

So with $\mathscr{V}_m(\lambda;z)$ computed for a general weight function $w(z)$, the functions

$$\begin{aligned}
&1\\
&\mathscr{V}_1((1,0);x)\\
&\mathscr{V}_1((0,1);y)\\
&\mathscr{V}_2((2,0)/||(2,0)||_p;x)\\
&\mathscr{V}_2((1,1)/||(1,1)||_p;(x+y)/||(1,1)||_p)\\
&\mathscr{V}_2((0,2)/||(0,2)||_p;y)\\
&\quad\vdots
\end{aligned}\tag{5.7}$$

provide a basis for the bivariate polynomials, but not yet an orthogonal basis. We now indicate how this can be achieved. In the sequel we focus on the Euclidean norm ($p=2$) and we consider weight functions of the form

$$w(z)=(1-z^2)^{\nu-1/2}\tag{5.8}$$

because this class is large enough for our purpose. The following result holds in d dimensions.

Theorem 5.2 *For $w(z)$ given by* (5.8) *the radial polynomials $\mathscr{V}_m(\lambda;z)$ are independent of*

λ and we have the additional orthogonality

$$\int \dots \int_{||X||_2 \le 1} \mathcal{V}_i\left(\cdot; \sum_{k=1}^{d} x_k \mu_k\right) \mathcal{V}_m\left(\cdot; \sum_{k=1}^{d} x_k \lambda_k\right) w(z)\, dX =$$

$$\begin{cases} 0, & i = 0, \dots, m-1, \\ \dfrac{\pi^{d/2}(2\nu + d - 1)\Gamma(\nu + 1/2)}{(2\nu + 2m + d - 1)\Gamma(\nu + (d+1)/2)} \mathcal{V}_m\left(\cdot; \sum_{k=1}^{d} \lambda_k \mu_k\right), & i = m. \end{cases}$$

proof 5.2 *For the polynomials $\mathcal{V}_m(\lambda; \lambda_1 x_1 + \ldots + \lambda_d x_d)$ satisfying the orthogonality conditions* (5.2) *with $w(z)$ given by* (5.8) *and a continuous function f defined on* $[-1,1]$ *the Funk-Hecke formula in [70] gives*

$$\int \dots \int_{||X||_2 \le 1} f(\mu_1 x_1 + \ldots + \mu_d x_d)\, \mathcal{V}_m(\lambda; \lambda_1 x_1 + \ldots + \lambda_d x_d)(1 - z^2)^{\nu - 1/2}\, dX$$
$$= \frac{\pi^{(d-1)/2}\Gamma(\nu + 1/2)}{C_m^{(\nu+(d-1)/2)}(1)\Gamma(\nu + d/2)}$$
$$\times \mathcal{V}_m(\lambda; \lambda_1 \mu_1 + \ldots + \lambda_d \mu_d) \int_{-1}^{1} f(t) C_m^{(\nu+(d-1)/2)}(t)(1 - t^2)^{\nu - 1 + d/2} dt$$

where $C_m^{(\nu)}(z)$ are the univariate Gegenbauer polynomials orthogonal with respect to the weight $(1 - z^2)^{\nu - 1/2}$ on $[-1,1]$. For the moments $c_i(\lambda)$ defined by (5.4) *we thus obtain with $f(t) = t^i, \mu = \lambda$ and $m = 0$ that*

$$c_i(\lambda) = \int \dots \int_{||X||_2 \le 1} (\lambda_1 x_1 + \ldots + \lambda_d x_d)^i (1 - z^2)^{\nu - 1/2}\, dX =$$
$$\frac{\pi^{(d-1)/2}\Gamma(\nu + 1/2)}{\Gamma(\nu + d/2)} \int_{-1}^{1} t^i (1 - t^2)^{\nu - 1 + d/2} dt.$$

Since these $c_i(\lambda)$ do not depend on λ, the coefficients solved from (5.3) *do not either and so we can write*

$$\mathcal{V}_m(\lambda; z) = \mathcal{V}_m(\cdot; z).$$

At the same time we see that the moments $c_0(\cdot), c_1(\cdot), c_2(\cdot), \ldots$ equal up to the factor

$$\frac{\Gamma(\nu + 1/2)\pi^{(d-1)/2}}{\Gamma(\nu + d/2)}$$

the moments of the univariate Gegenbauer polynomials $C_m^{(\nu+(d-1)/2)}(z)$. Hence we can *also write*

$$\mathscr{V}_m(\cdot;z) = \left(\frac{\Gamma(\nu+1/2)\pi^{(d-1)/2}}{\Gamma(\nu+d/2)}\right)^m C_m^{(\nu+(d-1)/2)}(z).$$

The expressions for the integral in the proposition follow from the Funk-Hecke formula in a similar way, now with

$$f(t) = C_i^{\nu+(d-1)/2}(t).$$

The above theorem guarantees the orthogonality of polynomials of different degree irrespective of the choice of λ, which may indeed be different when the degrees differ. In other words, with the $\mathscr{V}_m(\lambda;z)$ orthogonal with respect to the weight function $w(z) = (1-z^2)^{\nu-1/2}$ on the Euclidean disk, each of the $m+1$ functions $\mathscr{V}_m(\cdot;\alpha_{m,k}x+\beta_{m,k}y)$ of degree m is orthogonal to each of the $i+1$ functions $\mathscr{V}_i(\cdot;\alpha_{i,k}x+\beta_{i,k}y)$ of degree i. So the functions $\mathscr{V}_1(\cdot;x)$ and $\mathscr{V}_1(\cdot;y)$ are orthogonal to the functions $\mathscr{V}_2(\cdot;x), \mathscr{V}_2(\cdot;(x+y)/\sqrt{2}), \mathscr{V}_2(\cdot;y)$. Let us now deal with the remaining problem, being that of the mutual orthogonality of the $m+1$ polynomials of degree m in the basis.

Theorem 5.3 *The set* $\{\mathscr{V}_m(\alpha_{m,k},\beta_{m,k};\alpha_{m,k}x+\beta_{m,k}y), 0 \le k \le m, m \in \mathbb{N}\}$ *with* $\alpha_{m,k} = \cos(k\pi/(m+1))$ *and* $\beta_{m,k} = \sin((k\pi/(m+1))$ *is an orthogonal basis for* $\mathbb{R}[x,y]$ *with respect to the weight function* $w(z) = 1$.

proof 5.3 *The proclaimed result can be obtained from [70]. But a separate proof is immediate now and goes as follows. From Theorem 5.2 we know that functions of different degree are orthogonal because the weight function has the form* (5.8) *with* $\nu = 1/2$. *We also know that different functions of equal degree are only orthogonal if*

$$\mathscr{V}_m\left(\cdot;\sum_{k=1}^{d}\lambda_k\mu_k\right) = 0.$$

Since for $w(z) = 1$ *the* $\mathscr{V}_m(\cdot;z)$ *coincide up to a factor with the Gegenbauer polynomials* $C_m^{(1)}(z)$ *we need to have*

$$\lambda_1\mu_1 + \lambda_2\mu_2 = \alpha_{m,k}\alpha_{m,\ell} + \beta_{m,k}\beta_{m,\ell} = \cos(i\pi/(m+1))$$

for some $i = 1,\ldots,m$ *and whatever* $0 \le k,\ell \le m, k \ne \ell$ *which is satisfied for the above* $\alpha_{k,m}$ *and* $\beta_{k,m}$.

So with $\mathscr{V}_m(\cdot;z)$ orthogonal with respect to the weight function $w(z) = 1$ on the Euclidean disk, the polynomials

$$\begin{aligned}
&1 \\
&\mathscr{V}_1(\cdot;x) \\
&\mathscr{V}_1(\cdot;y) \\
&\mathscr{V}_2(\cdot;x) \\
&\mathscr{V}_2(\cdot;x\cos\pi/3 + y\sin\pi/3) \\
&\mathscr{V}_2(\cdot;x\cos 2\pi/3 + y\sin 2\pi/3) \\
&\quad\vdots
\end{aligned} \tag{5.9}$$

are a fully orthogonal basis on $\overline{B}_{2,2}(0;1)$ for $\mathbb{R}[x,y]$. In Section 5.5 we give an illustration of the use of this basis in least squares approximation.

5.3 Small Lebesgue constants on the disk

When moving to more variables, we face some immediate problems since

$$\operatorname{span}\{1, x, y, x^2, xy, y^2, \ldots\}$$

is not a Chebyshev system anymore. So an additional concern in polynomial interpolation is the unisolvence of the interpolation problem. Unless otherwise indicated, we consider polynomials of full homogeneous degree. In two variables this means that a polynomial of degree n has the form

$$p_n(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j$$

with $N+1 = (n+1)(n+2)/2$ coefficients. We consider the interpolation problem

$$p_n(x_k, y_k) = f(x_k, y_k), \qquad k = 0, \ldots, N, \qquad (x_k, y_k) \in \overline{B}_{2,p}(0;1).$$

Let $\{\phi_0, \ldots, \phi_N\} = \{x^i y^j; 0 \le i + j \le n\}$ and let $\{(x_k, y_k); 0 \le k \le N\}$ be such that the matrix

$$V_N = \left(\Phi_{\ell+1,k+1}\right)_{(N+1)\times(N+1)}, \qquad \Phi_{\ell+1,k+1} = \phi_k(x_\ell, y_\ell), \qquad 0 \le \ell, k \le N$$

is regular. The node sets that we consider in the sequel always guarantee this. Then the polynomial interpolant can be written as

$$p_n(x,y) = \sum_{i=0}^{N} f(x_i,y_i)\ell_i(x,y)$$

with

$$\ell_i(x,y) = \frac{\det V_{N,i}}{\det V_N},$$

where the matrix $V_{N,i}$ equals the matrix V_N except that the i-th row is replaced by $(\phi_0(x,y),\ldots,\phi_N(x,y))$. With the functions $\ell_i(x,y)$ we define the Lebesgue constant

$$\Lambda_n^{(2)} := \Lambda_n\left((x_0,y_0),\ldots,(x_N,y_N)\right) = \max_{(x,y)\in\overline{B}_{2,p}(0;1)} \sum_{i=0}^{N} |\ell_i(x,y)|.$$

The minimal growth of $\Lambda_n^{(2)}$ is different for different ℓ_p-balls. For instance, on the square the minimal order of growth is $O(\ln^2(n+1))$ and this order is achieved for the configurations of interpolation points given in [12] and [13].

On the disk the minimal order of growth is quite different, namely $O(\sqrt{n+1})$, as proved in [14]. No configurations of interpolation points obeying this order of growth are known. We analyze the Lebesgue constant on the disk for different unisolvent configurations and present the best that can be obtained so far.

On the simplex the minimal order of growth is not even known. Instead, in [15] some (non closed form) configurations of interpolation points are obtained from the solution of a minimization problem. There is clearly a lot of interest in the problem.

Several configurations of interpolation points on concentric circles guarantee unisolvence on the disk. Among others we mention [71, 72, 73]. We tried all configurations but report here only on the closed form set that gives the smaller Lebesgue constant $\Lambda_n^{(2)}$ on the disk. As can be expected, it is a configuration that increases the number of interpolation points towards the boundary.

Let us divide a total of $\lfloor n/2\rfloor+1$ concentric circles with center at the origin into k groups,

$$\nu_1+\ldots+\nu_k = \left\lfloor\frac{n}{2}\right\rfloor+1, \qquad \nu_i\in\mathbb{N}, \qquad i=1,\ldots,k,$$

with the j-th group containing ν_j circles with respective radii $r_1^{(j)}, \ldots, r_{\nu_j}^{(j)}$. On each circle in the j-th group we take the same number of $2n_j + 1$ equidistant interpolation points where

$$\begin{aligned} n_1 &= n - \nu_1 + 1, \\ n_2 &= n - 2\nu_1 - \nu_2 + 1, \\ &\vdots \\ n_k &= n - 2\nu_1 - \ldots - 2\nu_{k-1} - \nu_k + 1. \end{aligned}$$

Then it is easy to see that

$$\nu_1(2n_1 + 1) + \ldots + \nu_k(2n_k + 1) = N + 1$$

and that the Lebesgue constant $\Lambda_n^{(2)}$ decreases if k increases, for the simple reason that the points become more uniformly distributed over the circles as k approaches $\lfloor n/2 \rfloor + 1$ with $\nu_j = 1$ for $j = 1, \ldots, \lfloor n/2 \rfloor + 1$. In [71] it is proved that this configuration of points is unisolvent on the disk. For which of the larger k exactly the minimal value of $\Lambda_n^{(2)}$ is attained, depends on the interplay between the radii of the concentric circles and the distribution of the interpolation points over the disk. Smaller Lebesgue constants can be expected if the Dubiner distance between the interpolation points varies less [74]. We return to this issue in Section 5.4.

Remains the problem of how to choose the radii. In the Figures 5.2–5.5 we have taken the radii equal to the extended zeroes of the spherical Legendre polynomials, where this has to be interpreted as explained at the end of Section 5.1. In Figure 5.6 we illustrate that the growth rate of the Lebesgue constant $\Lambda_n^{(2)}$ is slowest for this choice: we compare the Lebesgue constants for the radii being the extended zeroes of the spherical Legendre, the extended zeroes of the spherical Chebyshev and the extended zeroes of the univariate Chebyshev polynomials, the latter being given by (2.20). Unless otherwise mentioned Chebyshev polynomials are of the first kind, in other words orthogonal with respect to the weight function $w(z) = 1/\sqrt{1 - z^2}$. For each degree n we have immediately taken k to be the maximal value $\lfloor n/2 \rfloor + 1$.

We illustrate this for $n = 6$ and $N + 1 = 28$. We point out that additional rotations with respect to each other of the concentric circles containing the interpolation points, have an effect on the Lebesgue constant under study, but never to the point that its order of magnitude for a certain configuration (meaning a certain value for k) is altered. A decrease of the Lebesgue constant due to such rotations is only marginal. In Figure 5.2 one finds the case $k = 1$, so $\nu_1 = 4$ with $n_1 = 3$, where the 28 interpolation points are distributed over 4 concentric circles each containing 7 equidistant points. The Lebesgue constant in this case is a whopping 6648. In Figure 5.3 the number k is increased to 2 and we take $\nu_1 = 2, \nu_2 = 2$, so 11 points on each of the 2 outer circles and 3 points on each of the 2 inner circles. This clearly improves the Lebesgue constant to about 51.17. In Figure 5.4 we take $k = 3$ with $\nu_1 = 1, \nu_2 = 1, \nu_3 = 2$, so 13 interpolation points on the outer circle, another circle with 7 points and 3 interpolation points on each of the 2 inner circles. The Lebesgue constant is further going down to approximately 10.58. Finally with $k = 4$ and all $\nu_j = 1$ for $j = 1, \ldots, 4$ the Lebesgue constant is smallest, namely 4.68. We have respectively 13, 9 and 5 interpolation points on 3 concentric circles and the last point at the origin. We repeat that the radii in the Figures 5.2–5.5 are taken as the extended zeroes of the spherical Legendre polynomials of respective degrees 8, 8, 8 and 7.

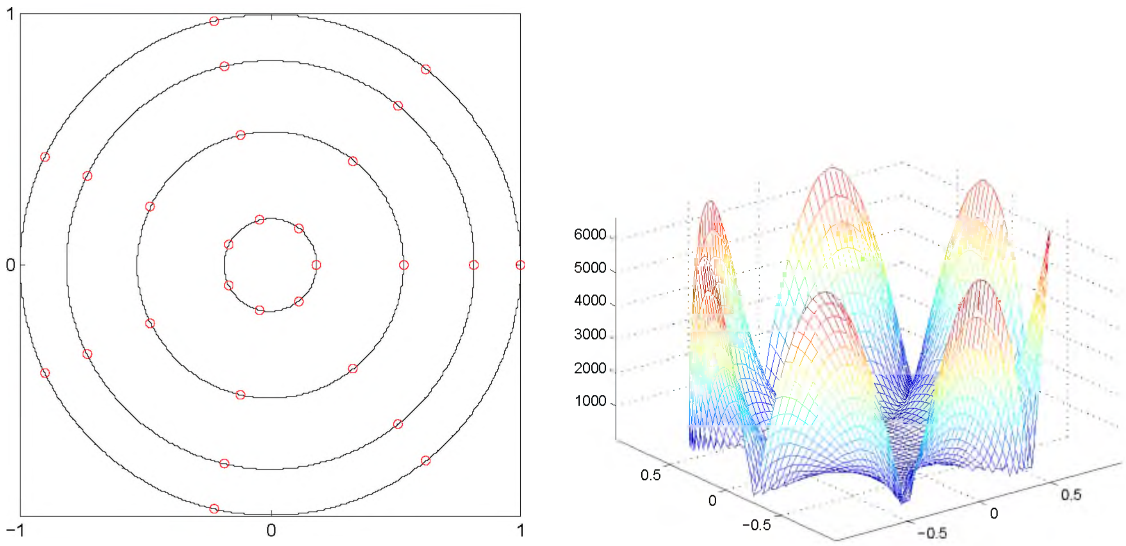

Figure 5.2 Case $k = 1$.

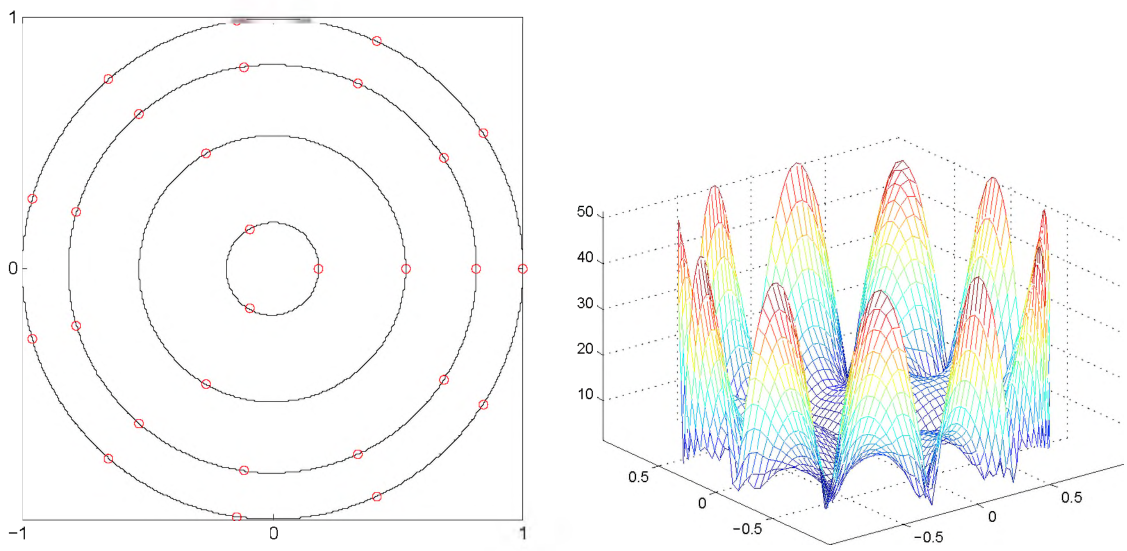

Figure 5.3 Case $k = 2$.

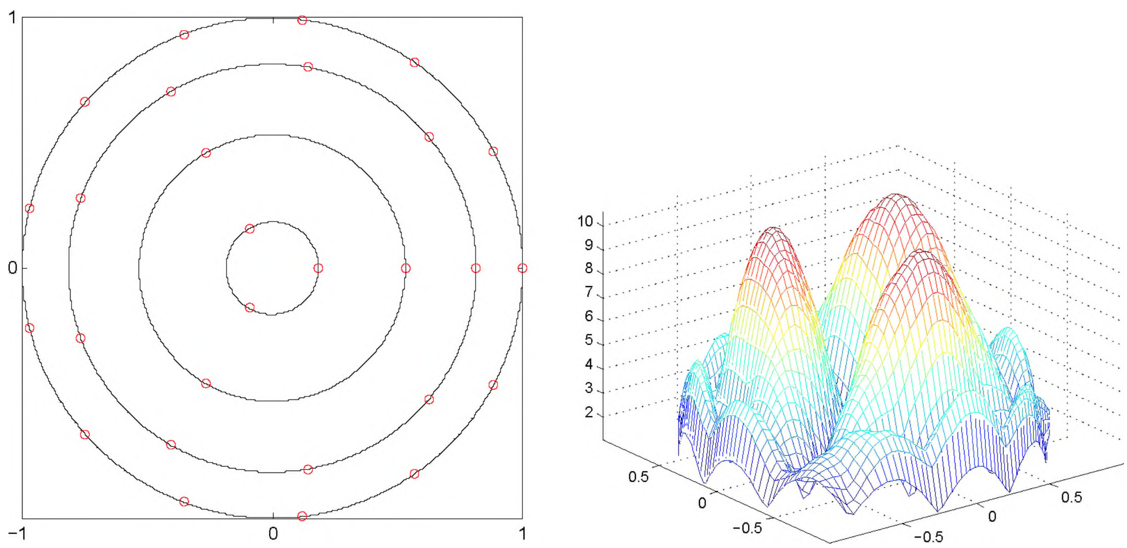

Figure 5.4 Case $k = 3$.

5.4 Exploring other configurations on the disk

The configurations leading to Lebesgue constants with minimal growth on the square $\overline{B}_{2,\infty}(0;1)$ are slightly different from the above. Let us analyze whether similar configurations to the ones on the square can be considered on the disk and whether they are any good. Our point of departure are the so-called Padua points [74].

A first observation is that the $N+1$ Padua interpolation points are distributed over the unit square on n concentric squares with increasing radius and with (from the center to

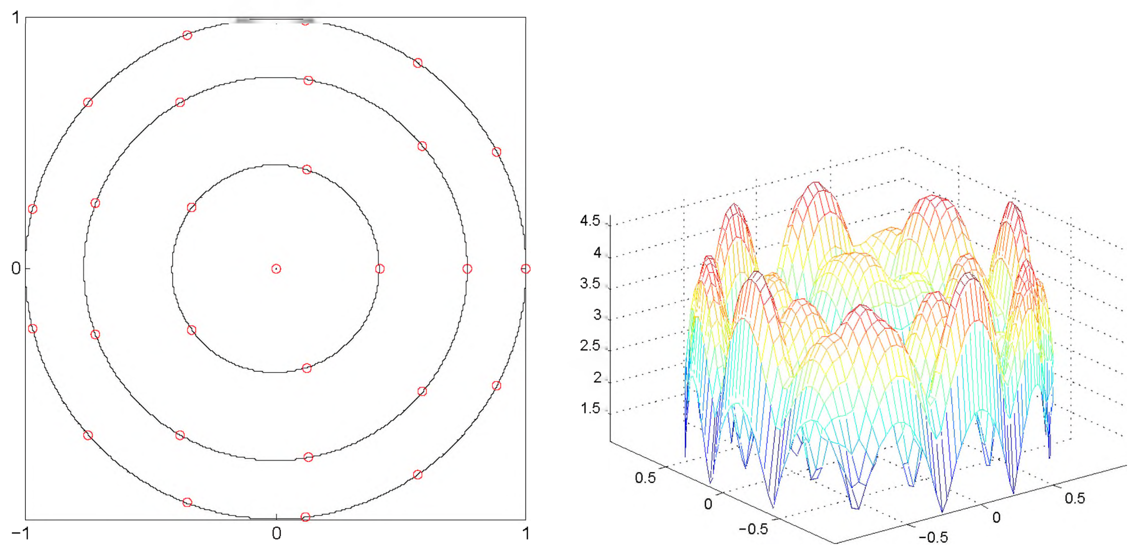

Figure 5.5 Case $k = 4$.

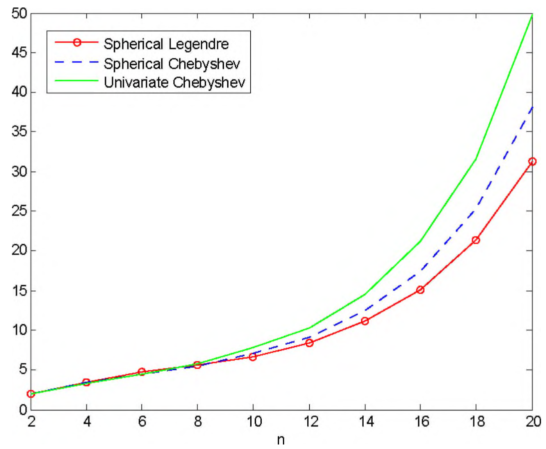

Figure 5.6 Growth of $\Lambda_n^{(2)}$ for $k = \lfloor n/2 \rfloor + 1$.

the boundary) i points on the i-th square for $i = 1, \ldots, n-1$ and $2n+1$ points on the n-th square, being the boundary of $\overline{B}_{2,\infty}(0;1)$. Also, we show that the radii of the inner $n-1$ concentric squares are the zeroes of the univariate Chebyshev polynomials of the second kind $U_n(z)$ and $U_{n-1}(z)$, excluding zero, where the symmetric zeroes are interpreted as in the Sections 2.3.3 and 5.2. To see this we organize the Padua interpolation points for degree n, explicited in [12] as

$$\left(x^{(j,k)} = (-1)^{j+k}\cos\left(\frac{j\pi}{n+1}\right), y^{(j,k)} = (-1)^{j+k}\cos\left(\frac{k\pi}{n}\right)\right),\ 0 \le j+k \le n$$

in the following way. First we note that the points in

$$S_n = \left\{ (-1)^{l+1} \left(\cos\left(\frac{l\pi}{n}\right), \cos\left(\frac{0\pi}{n+1}\right) \right) : l = 0, \ldots, n \right\} \cup \\ \left\{ (-1)^{l+1} \left(\cos\left(\frac{0\pi}{n}\right), \cos\left(\frac{l\pi}{n+1}\right) \right) : l = 1, \ldots, n \right\} \tag{5.10}$$

lie on the boundary of the unit square. Then we take the collection of points consisting of

$$S_{n-i} = \left\{ (-1)^{l+m+1} \left(\cos\left(\frac{l\pi}{n}\right), \cos\left(\frac{m\pi}{n+1}\right) \right) : m = \lceil i/2 \rceil, \\ l = m, \ldots, n-m \right\}, 1 \leq i \leq n-1, i \text{ odd}$$

$$S_{n-i} = \left\{ (-1)^{l+m+1} \left(\cos\left(\frac{m\pi}{n}\right), \cos\left(\frac{l\pi}{n+1}\right) \right) : m = \lceil i/2 \rceil, \\ l = m+1, \ldots, n-m \right\}, 1 \leq i \leq n-1, i \text{ even}$$

lie on the same square of radius

$$\left\| (-1)^{l+m+1} \left(\cos(l\pi/n), \cos(m\pi/(n+1)) \right) \right\|_\infty = \cos(m\pi/(n+1)), \quad i \text{ odd},$$
$$\left\| (-1)^{l+m+1} \left(\cos(m\pi/n), \cos(l\pi/(n+1)) \right) \right\|_\infty = \cos(m\pi/n), \quad i \text{ even}.$$

In Figure 5.7 this is illustrated for $n = 6$. These ℓ_∞ radii are the zeroes of

$$U_n(z) = \sin\left((n+1)\arccos(z)\right) \sin\left(\arccos(z)\right),$$
$$U_{n-1}(z) = \sin\left(n \arccos(z)\right) \sin\left(\arccos(z)\right).$$

This explains why the radii are two by two rather similar, except for the innermost square that contains only one point.

When carrying this configuration to the disk, replacing concentric squares by concentric circles, copying the distribution of the points and the values of the radii, then what remains to specify is the distribution of the points on the i-th circle for $i = 1, \ldots, n$. Here we can follow the simple rule that the points on the boundary of the unit disk are taken equidistantly and then (from the boundary to the center) the union of the points on each pair of concentric circles is also distributed equidistantly as if the points were lying on

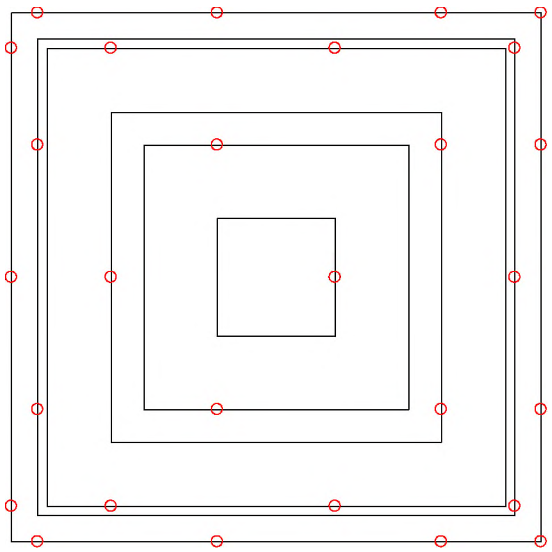

Figure 5.7 Padua points in $\overline{B}_{2,\infty}(0;1)$ for $n = 6$.

only one circle. Figure 5.8 for $n = 6$ illustrates this best. The accompanying Lebesgue constant $\Lambda_n^{(2)} = 7.76$. Another variation on this theme is to plainly take the set of the

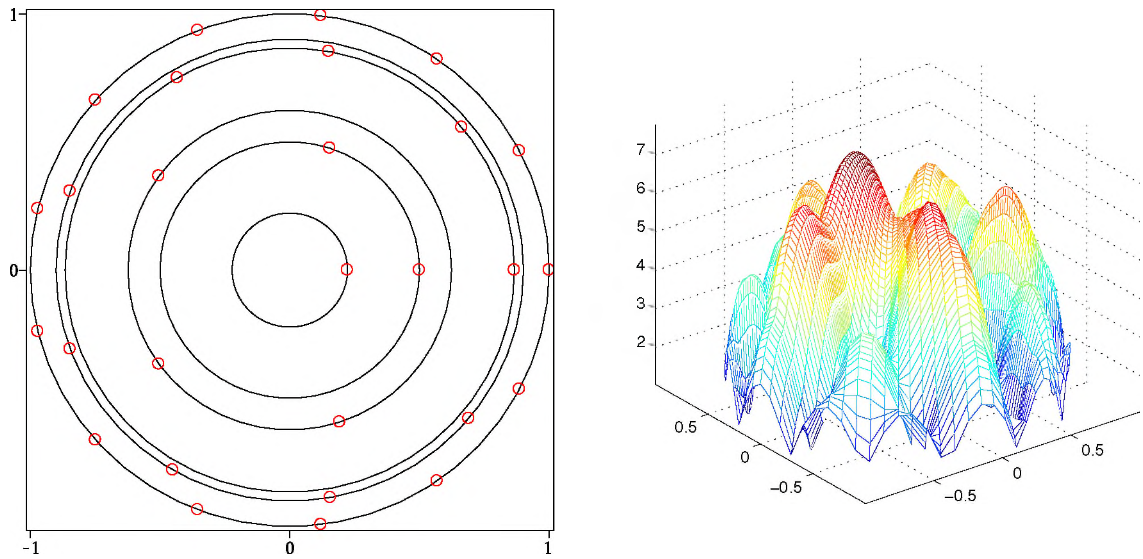

Figure 5.8 Padua-like configuration on the disk for $n = 6$.

Padua points and map the square on the disk using

$$t(x,y) = \left(x\frac{||(x,y)||_\infty}{||(x,y)||_2}, y\frac{||(x,y)||_\infty}{||(x,y)||_2} \right), \qquad (x,y) \in \overline{B}_{2,\infty}(0;1).$$

For $n = 6$ this leads to the configuration in Figure 5.9 with a matching Lebesgue constant of $\Lambda_6^{(2)} = 12.50$. Remember that smaller Lebesgue constants are to be expected from sets with a smaller variation in the Dubiner distance among the interpolation points [74]. The

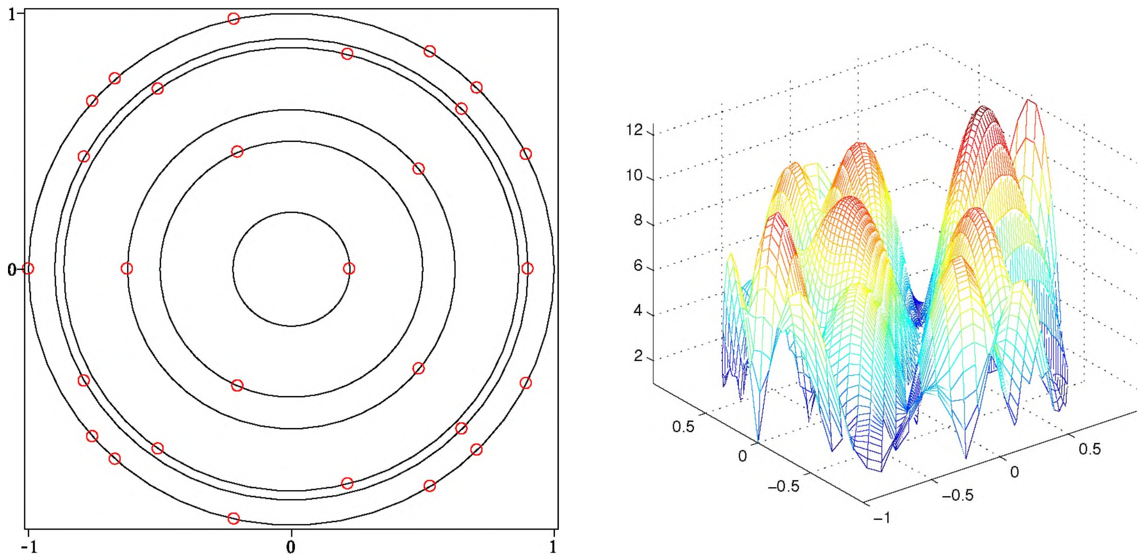

Figure 5.9 Padua points for $n = 6$ mapped to the disk.

zeroes of the Chebyshev polynomials $T_{n+1}(x)$, for instance, are equidistant with respect to the Dubiner distance. From this it is easy to conclude from Figure 5.10 for $n = 33$, that the leftmost configuration which is the one described in Section 5.3, gives a smaller Lebesgue constant than the configuration in the middle, which is similar to that in Figure 5.8, or the rightmost one, which is similar to that in Figure 5.9. In the rightmost configuration there are clearly accumulations of interpolation points, while in the configuration in the middle the interpolation points are a bit too much pushed out of the center region. Hence our conclusion that the sets of interpolation points leading to the better Lebesgue constants on the disk are for the moment the ones given in Section 5.3.

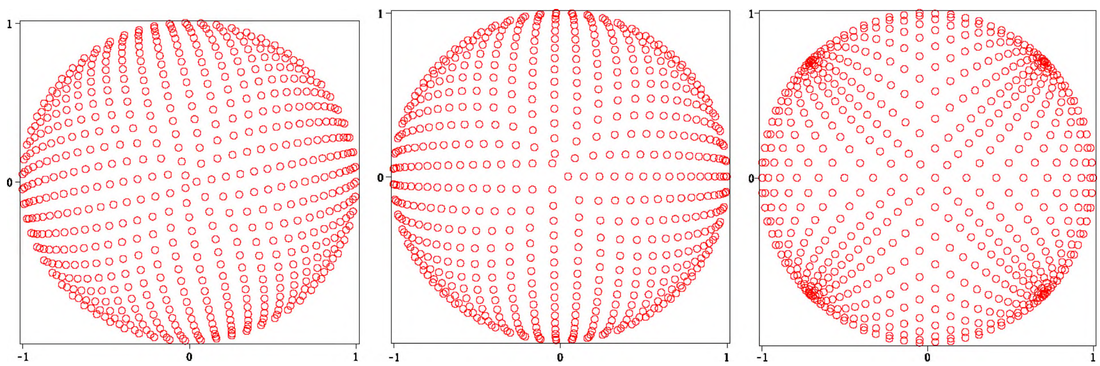

Figure 5.10 Point configurations as in Figures 5.5, 5.8 and 5.9 for $n = 33$.

5.5 Illustration

Let $f(x,y)$ be the matlab `peaks` function (Figure 5.11) on the Euclidean disk $\overline{B}_{2,2}(0;1)$,

$$\begin{aligned} f(x,y) = {} & 3(1-3x)^2 \exp\left(-9x^2-(3y+1)^2\right) \\ & - 10(3x/5-27x^3-243y^5)\exp(-9(x^2+y^2)) \\ & - (1/3)\exp\left(-(3x+1)^2-9y^2\right), \qquad (x,y) \in \overline{B}_{2,2}(0;1). \end{aligned}$$

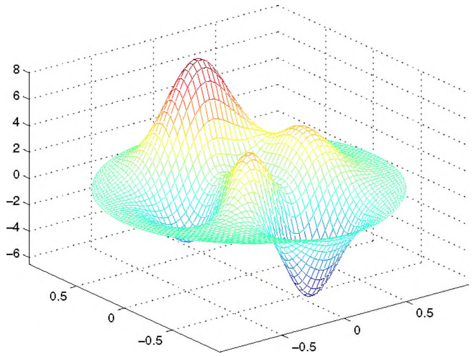

Figure 5.11 Graph of `peaks` function.

We illustrate the usefulness of the new orthogonal cartesian basis derived in Section 5.2 and the configuration of interpolation points described in Section 5.3 with $k = \lfloor n/2 \rfloor + 1$, by computing on the one hand the least squares approximant to $f(x,y)$

$$q_n(x,y) = \sum_{m=0}^{n} \sum_{k=0}^{m} v_{m,k} \mathscr{V}_m\left(\cdot; x\cos(k\pi/(m+1)) + y\sin(k\pi/(m+1))\right),$$

$$v_{m,k} = \frac{\int\int_{\overline{B}_{2,2}(0;1)} f(x,y) \mathscr{V}_m\left(\cdot; x\cos(k\pi/(m+1)) + y\sin(k\pi/(m+1))\right) dX}{||\mathscr{V}_m\left(\cdot; x\cos(k\pi/(m+1)) + y\sin(k\pi/(m+1))\right)||_2^2},$$

and on the other hand the polynomial interpolant of the same form with $v_{m,k}$ solved from the system of interpolation conditions

$$p_n(x_j,y_j) = f(x_j,y_j), \qquad j = 0,\ldots,N, \tag{5.11}$$

where the $\mathscr{V}_m$ are the spherical Legendre polynomials on the Euclidean unit disk and the interpolation points (x_j,y_j) are chosen as in Figure 5.5. From Theorem 5.2 we easily find that

$$||\mathscr{V}_m(\cdot;x\cos(k\pi/(m+1))+y\sin(k\pi/(m+1)))||_2^2 =$$

$$\int\int_{\overline{B}_{2,2}(0;1)} \mathscr{V}_m^2(\cdot;x\cos(k\pi/(m+1))+y\sin(k\pi/(m+1)))\,dX = \pi.$$

The polynomial $q_n(x,y)$ is the best ℓ_2 polynomial approximant for $f(x,y)$ on the disk $\overline{B}_{2,2}(0;1)$. Due to the mutual orthogonality of all basis functions $\mathscr{V}_m(\cdot;x\cos(k\pi/(m+1))+y\sin(k\pi/(m+1)))$ the coefficients $v_{m,k}$ do not have to be computed from a linear system. Instead, an explicit formula for the best polynomial approximant on the disk can now be written down.

Both p_n and q_n are also compared with the popular radial basis function interpolant [75]

$$r_n(x,y) = \sum_{k=0}^{N} \sigma_k \sqrt{1+||(x-x_k,y-y_k)||_2^2}$$

and the (better conditioned but slower converging) constrained radial basis function interpolant [75]

$$s_n(x,y) = \sum_{k=0}^{N} \tau_k ||(x-x_k,y-y_k)||_2^2 \ln(||(x-x_k,y-y_k)||_2)$$

where the constraint comes from adding a quadratic bivariate polynomial as described in [75]. In Table 5.1 we illustrate the errors $||f-q_n||_\infty, ||f-p_n||_\infty, ||f-r_n||_\infty$ and $||f-s_n||_\infty$ for different values of n. All errors were computed in higher precision (Maple) because of the ill-conditioning of the RBF interpolation problems.

Table 5.1 ℓ_∞ errors of approximant $q_n(x,y)$, interpolant $p_n(x,y)$, and radial basis interpolants $r_n(x,y)$ and $s_n(x,y)$.

n	$N+1$	$\|\|f-q_n\|\|_\infty$	$\|\|f-p_n\|\|_\infty$	$\|\|f-r_n\|\|_\infty$	$\|\|f-s_n\|\|_\infty$
10	66	1.160	1.747	1.412	1.961
12	91	0.596	0.909	0.648	1.091
14	120	0.329	0.332	0.225	1.117
16	153	0.202	0.202	0.043	0.559
18	190	0.051	0.050	0.006	0.509
20	231	0.030	0.018	0.001	0.255

In Table 5.2 we give $||f-q_n||_2$ and $||f-p_n||_2$ and in Figure 5.12 we show both the error curves $(q_{16}-f)(x,y)$ and $(p_{16}-f)(x,y)$.

Note that the interpolant computed for the interpolation points constructed in Section 5.3

Table 5.2 ℓ_2 errors of approximant $q_n(x,y)$ and interpolant $p_n(x,y)$.

n	$N+1$	$\|\|f-q_n\|\|_2$	$\|\|f-p_n\|\|_2$
10	66	0.494	0.717
12	91	0.251	0.377
14	120	0.134	0.182
16	153	0.058	0.081
18	190	0.014	0.025
20	231	0.007	0.009

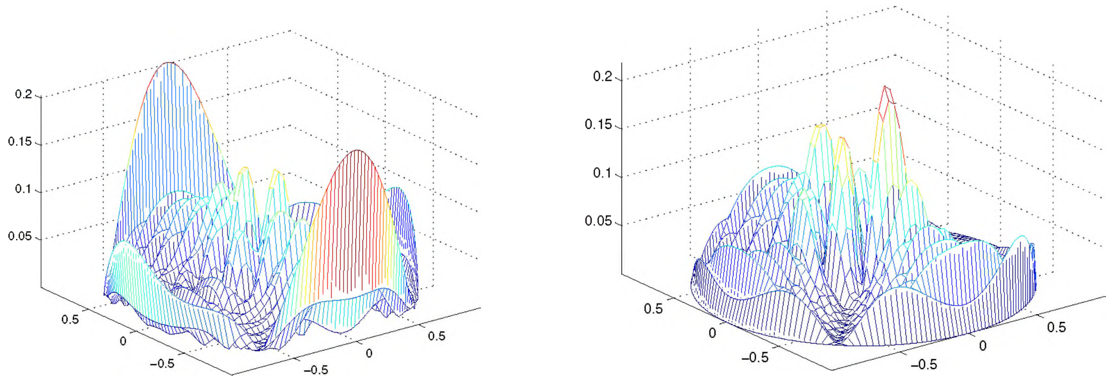

Figure 5.12 Error plots of polynomial approximant $q_{16}(x,y)$ (left) and polynomial interpolant $p_{16}(x,y)$ (right).

can indeed be called a near-best polynomial approximant, as one may expect from a set of good interpolation points.

Table 5.3 Condition number using mutually orthogonal basis versus tensor product basis and radial basis functions.

n	$N+1$	$\mathcal{V}_m(;\langle,\rangle)$	$T_i(x)T_j(y)$	$\sqrt{1+(\cdot)^2}$	$(\cdot)^2\ln(\cdot)$
10	66	$6.99e+00$	$4.67e+03$	$1.08e+08$	$3.09e+03$
12	91	$8.89e+00$	$2.88e+04$	$3.14e+09$	$7.59e+03$
14	120	$1.24e+01$	$1.76e+05$	$9.70e+10$	$1.67e+04$
16	153	$1.82e+01$	$1.07e+06$	$2.95e+12$	$3.34e+04$
18	190	$2.78e+01$	$6.41e+06$	$9.17e+13$	$6.26e+04$
20	231	$4.42e+01$	$3.83e+07$	$3.04e+15$	$1.10e+05$

In Table 5.3 one finds the condition numbers of the system of interpolation conditions (5.11) when written down using the new fully orthogonal basis compared to the use of:

- the classical tensor products $T_i(x)T_j(y)$ of Chebyshev polynomials,
- the radial basis functions $\sqrt{1+||(x-\cdot,y-\cdot)||_2^2}$,

- the constrained radial basis functions $||(x-\cdot,y-\cdot)||_2^2\ln(||(x-\cdot,y-\cdot)||_2$,

where the constraints come from adding a quadratic bivariate polynomial. The results speak for themselves: the new basis gives extremely well-conditioned systems of interpolation conditions: on the disk the mutually orthogonal polynomials in (5.9) lead to far better conditioning than the orthogonal polynomials in (5.7) of which the conditioning is comparable to that of $T_i(x)T_j(y)$! And it is just a matter of choosing the directions λ in (5.6) wisely.

CHAPTER 6

RESULTS

After the introduction in Chapter 1, our investigations began in Chapter 2, where we reviewed the known results on the conditioning of univariate polynomial interpolation. We now summarize the results presented in the Chapters 3 to 5.

In Chapter 3, we present the best Lebesgue constants in existence for rational interpolation with preassigned poles. The new results are based on a fairly unknown rational analogue of the Chebyshev orthogonal polynomials.

The (extended) zeros of the orthogonal rational function $\mathscr{T}_{n+1}(x)$ with predetermined denominator $q_m(x)$ of degree $m \leq n+1$ and constructed in Section 3.2, provide interpolation points for rational interpolation with poles prescribed by $q_m(x) = 0$, that are as good as the (extended) Chebyshev zeroes for polynomial interpolation. In the case of poles close to the interval of interpolation, they clearly outperform all other proposed sets of interpolation points. We compare with the results obtained in [63] and [10].

A rough analysis of the growth of the Lebesgue constant in the case of barycentric rational interpolation at equidistant interpolation points, is made in [10, 11], leading to the conclusion that it only grows logarithmically. In Chapter 4, we give a fine analysis, obtaining the precise growth formula

$$\frac{2}{\pi}\left(\ln(n+1) + \ln 2 + \gamma\right) + o(1)$$

for the Lebesgue constant under consideration, with γ being the Euler constant. The similarity between barycentric rational interpolation at equidistant points and polynomial interpolation at Chebyshev nodes (or the like) is remarkable.

After introducing the rational interpolation case in Section 4.1, tight lower and upper bound estimates are given in Section 4.2. The fine results are obtained from very high order numerical experiments in exact arithmetic. In Section 4.3 we indicate that the results can be extended to the rational interpolants introduced in[66]. The proof of the new bounds is detailed in Section 4.4.

If the choice of the polynomial basis and the location of the interpolation points play an important numerical role in univariate polynomial interpolation, they do so even more in the multivariate case discussed in Chapter 5. In Section 5.3 we explore the concept of spherical orthogonality for multivariate polynomials in more detail on the disk. We focus on two items:

- on the one hand, the construction in Section 5.4, of a fully orthogonal cartesian basis for the space of multivariate polynomials starting from this sequence of spherical orthogonal polynomials,
- and on the other hand, the connection described in Section 5.5, between these orthogonal polynomials and the Lebesgue constant in multivariate polynomial interpolation on the disk.

We point out the many links of the two topics under discussion with the existing literature and present a thorough discussion in Section 5.6. In Section 5.7 the new results are illustrated with an example of polynomial interpolation and approximation on the unit disk. The numerical example is also compared with the popular radial basis function interpolation.

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2010–2011	Universiteit Antwerpen	Guest Researcher

PUBLISHMENTS

Papers

1. Cuyt, A., Yaman, I., Ibrahimoglu, B.A., Benouahmane, B., (2012). "Radial orthogonality and Lebesgue constants on the disk", Numerical Algorithms,61(2), 291-313, 2012.

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3. Kurulay M, Ibrahimoglu B.A., Bayram M., (2010). "Solving a system of nonlinear fractional partial differential equations using three dimensional differential transform method", International Journal of Physical Sciences, 5(6), 906-912, 2010.

4. Ibrahimoglu B.A., Bayram M., (2008). "Türev değerlerini içeren rasyonel interpolasyon yöntemleri ve uygulamaları", Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 16, 2008.

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1. Ibrahimoglu, B.A., Cuyt, A., (2013). "Sharp bounds for Lebesgue constants of barycentric rational interpolation", Universiteit Antwerpen, Techical Report. 2013.