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ON δ-PRIMARY HIPERIDEALS AND FUZZY HIPERIDEALS EXPANSIONS

ASHRAF ABUMGHAISEEB

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ADVISER
PROF. DR. BAYRAM ALI ERSOY

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Thesis Adviser

Prof. Dr. Bayram Ali ERSOY

Yıldız Technical University

Approved By the Examining Committee

Prof. Dr. Bayram Ali ERSOY

Yıldız Technical University ______________________

Prof. Dr. A. Göksel AĞARGÜN, Member

Yıldız Technical University ______________________

Assoc. Prof. Dr. Fatih DEMİRKALE, Member

Yıldız Technical University ______________________

Assoc. Prof. Dr. Uğur ŞENGÜL, Member

Marmara University ______________________

Assoc. Prof. Dr. Esra Şengelen SEVIM, Member

Bilgi University ______________________

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CHAPTER 1

Krasner's Hyperring

1.1 Preliminaries

The first one who introduced the concept of hyperstructure was the French Mathematician F. Marty [1]. The theory of hyperstructure algebra is an extension of the theory of classical algebra. In 1934, Marty defined a hyperoperation $\circ$ as a relation from a nonempty set H to $P^*(H)$ which is the set of the nonempty subsets of H. It can be written as:

$$\circ: H \times H \longrightarrow P^*(H).$$

By the definition of a hyperoperation, Marty defined a hypergroup as follows:

If H is a nonempty set and $\circ$ is a hyperoperation defined on H, such that $\circ$ satisfies the following conditions,

1. $(x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ H = H \circ x$,

for all $x, y, z \in H$, then $(H, \circ)$ is a hypergroup where,

$$x \circ (y \circ z) = \bigcup_{r \in (y \circ z)} x \circ r \ , \ x \circ H = \bigcup_{h \in H} x \circ h.$$

In 1956, Krasner defined a type of hyperrings which is named Krasner Hyperring [1]. According to Krasner, a hyperring is a nonempty set with two operations, addition and multiplication where the addition is a hyperoperation but the multiplication is a binary operation as in the classical algebra.

Definition 1.1.1 [1] According to Krasner, the algebraic structure $(R, +, \cdot)$ where R is a nonempty set is a hyperring if:

1. $(R,+)$ is a canonical hypergroup. i.e.

   i. for all $x, y, z \in R,\ x + (y + z) = (x + y) + z$,

   ii. for all $x, y \in R,\ x + y = y + x$,

   iii. there exists a zero element $0 \in R$, such that $x + 0 = 0 + x = \{0\}$, for all $x \in R$,

   iv. for each $x \in R$, there exists a unique $x' \in R$ such that $0 \in x + x'$, x' is called the opposite of x, and we write $x' = -x$,

   v. if $z \in x + y$, then $y \in -x + z$ and $x \in -y + z$.

2. $(R,\ \cdot)$ is a semigroup with a 0 as bilaterally absorbing element, i.e. $0 \cdot x = x \cdot 0 = 0$, for all $x \in R$.

3. The multiplication $\cdot$ is distributive with respect to the hyperaddition +; that is, $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Corollary 1.1.1

i- $-(-x) = x$,

ii- $-(x + y) = -x - y$, where $-H = \{-h; h \in H\}$ for all $x, y \in R$.

Proof. Assume that $-x = z$, then $-(-x) = -z$ and $0 \in x + z$, $0 \in -(-x) + z$ but since the opposite is unique, then $-(-x) = x$. For the second part, suppose that $h \in -x - y$, then $-y \in h + x$ so that $x \in -y - h$ and $-h \in x + y$, then $h \in -(x + y)$. Similarly, if $h \in -(x + y)$ then $h \in -x - y$. Therefore, $-(x + y) = -x - y$.

Remark 1.1

i. A Krasner hyperring $(R, +,\ \cdot)$ is called commutative if $x \cdot y = y \cdot x$, for all $x, y \in R$.

ii. A Krasner hyperring has a unit element if there exists $1 \in R$, such that $1 \cdot x = x \cdot 1 = x$, for all $x \in R$.

Definition 1.1.2 [1] If R is a hyperring such that $(R \setminus \{0\},\ \cdot)$ is a hypergroup, then R is called a hyperfield.

Definition 1.1.3 [1] A Krasner hyperring R is called hyperdomain if R is a commutative hyperring with a unit element and $xy = 0$ implies that $x = 0$ or $y = 0$ for all $x, y \in R$.

In the second notion of the hyperrings, both addition and multiplication are hyperoperations. Our work exclusively focuses on hyperrings in sense of Krasner, so that, from now on, when we mention hyperrings, we mean a commutative Krasner hyperring with a unit unless otherwise specified.

After Krasner, many of scientists discussed the hyperrings and its applications. They published many papers. For example, in 2004, Davvaz extended the isomorphism theorems [2], and in 2006, another paper was published by Davvaz and Salasi [3] about the Chinese Reminder Theorem for hyperrings. Davvaz and Salasi developed the concepts of classical algebra and applied these concepts to the hyperrings. In 2010, [4] Muthusamy Velrajan and Arjunan Asokkumar extended the isomorphism theorems to hyperrings, where the addition and the multiplication are hyperoperations. Also they proved that, in order to define a quotient hyperring, the hyperideal concerned need not be normal. As application of hyperrings in coding theory Davvaz and Tusavi published the paper titled "Codes over Hyperrings" in 2016 [19].

In this thesis we concentrate on the hyperideals of the commutative Krasner hyperrings with units. In the next sections, we give some examples of hyperrings, and we use Krasner construction to construct hyperrings. In Chapter 2 we study the hyperideals, prime hyperideals and maximal hyperideals and give some examples. After that we discuss the external direct product of hyperideals. In the third chapter we study the hyperrings homomorphism, and the hyperring isomorphism theorems. In the fourth chapter, we concentrate on the primary hyperideals. We extend Zhao Dongsheng's definition about δ-primary ideal expansion of commutative rings in [15] to hyperideals of commutative hyperrings and commutative semihyperrings. We prove many theorems and give some examples related to this expansion.

1.2 Examples

In this section we provide some examples of hyperrings, illustrating the variety of algebraic properties of these hyperrings. The main challenge in each case of examples is verifying the associativity of hyperaddition.

Example 1.2.1[5] Let $H = \{0,1,2\}$, $A = \{1, 2\}, B = \{0, 2\}$, then for each of the following pair of tables we prove that $(H, +, \cdot)$ is a hyperring.

I.

$+$	0	1	2
0	{0}	{1}	{2}
1	{1}	A	H
2	{2}	H	A

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Proof. We verify the associativity for hyperaddition.

$$0 + (1 + 2) = 0 + H = H, \qquad (0 + 1) + 2 = 1 + 2 = H$$

$$0 + (2 + 1) = 0 + H = H, \qquad (0 + 2) + 1 = 2 + 1 = H$$

$$1 + (0 + 2) = 1 + 2 = H, \qquad (1 + 0) + 2 = 1 + 2 = H$$

$$1 + (2 + 0) = 1 + 2 = H, \qquad (1 + 2) + 0 = H + 0 = H$$

$$2 + (0 + 1) = 2 + 1 = H, \qquad (2 + 0) + 1 = 2 + 1 = H$$

$$(2 + 1) + 0 = H + 0 = H, \qquad 2 + (1 + 0) = 2 + 1 = H$$

$$(2 + 2) + 1 = A + 1 = \{1,2\} + 1 = A \cup H = H, \qquad 2 + (2 + 1) = 2 + H = H$$

II.

$+$	0	1	2
0	{0}	{1}	{2}
1	{1}	B	{1}
2	{2}	{1}	{0}

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Proof. As in "I" we prove the associativity,

$$0+(1+2)=0+1=1, \qquad (0+1)+2=1+2=1$$

$$0+(2+1)=0+1=1, \qquad (0+2)+1=2+1=1$$

$$1+(0+2)=1+2=1, \qquad (1+0)+2=1+2=1$$

$$1+(2+0)=1+2=1, \qquad (1+2)+0=1+0=1$$

$$2+(0+1)=2+1=1, \qquad (2+0)+1=2+1=1$$

$$(2+1)+0=1+0=1, \qquad 2+(1+0)=2+1=1$$

Example 1.2.2 [5] Let $=\{0,1,2,\ 3\}$, $A=\{1,2,3\}$, $B=\{0,2,3\}, C=\{0,1,3\}$, $D=\{0,1,2\}$. Then for each of the following pairs of tables with hyperaddition and multiplication, $(H,+,\ \cdot)$ forms a hyperfield.

I.

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	H	A	A
2	$\{2\}$	A	H	A
3	$\{3\}$	A	A	H

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

II.

$+$	0	1	2	3
0	$\{0\}$	1	$\{2\}$	$\{3\}$
1	$\{1\}$	B	A	A
2	$\{2\}$	A	C	A
3	$\{3\}$	A	A	D

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

1.3 Krasner's Construction and a Counter Example

Theorem 1.3.1 [6] Let $(R,+,\ \cdot)$ be a commutative ring and G be a subset of R such that $(G,\ \cdot)$ is a group, define an equivalence relation $\sim$ on R as follows:

$$x \sim y \text{ if and only if } xG = yG.$$

Let $\bar{x} = xG$, and $R/G = \{xG : x \in R\}$.

Define " + " on R/G by: $xG + yG = \{zG | z = xg_1 + yg_2 , g_1, g_2 \in G\}$ and define a multiplication on R/G by $xG \cdot yG = xyG$. Then $(R/G ,+,\cdot)$ form a hyperring. If we choose R to be a field, then $(R/G ,+,\ \cdot)$ will be a hyperfield.

Proof. It is clear that $\sim$ is an equivalence relation. Define the hyperaddition on the set of equivalence classes R/G by $\bar{x} + \bar{y} = \{\bar{z} : z = xg_1 + yg_2 , g_1, g_2 \in G\}$ and since R is commutative under addition we have $\bar{x} + \bar{y} = \bar{y} + \bar{x}$.

Define the sum of the subsets of R as $\bar{x} \dotplus \bar{y} = \{z : z = x_1 + y_1;\ x_1 \in \bar{x}, y_1 \in \bar{y}\}$.

Let $\bar{x}, \bar{y}, \bar{z} \in R/G$. Then

$$(\bar{x} + \bar{y}) + \bar{z} = \bigcup_{\bar{t} \subseteq \bar{x}+\bar{y}} (\bar{t} + \bar{z}) = \bigcup_{\bar{t} \subseteq \bar{x} \dotplus \bar{y}} \{\bar{v} \in \bar{R} : \bar{v} \subseteq \bar{t} \dotplus \bar{z}\} = \{\bar{v} \in \bar{R} : \bar{v} \subseteq \bigcup_{\bar{t} \subseteq \bar{x} \dotplus \bar{y}} \bar{t} \dotplus \bar{z}\}$$

$= \{\bar{v} \in \bar{R} : \bar{v} \subseteq (\bar{x} \dotplus \bar{y}) \dotplus \bar{z}\}$. Similarly, $\bar{x} + (\bar{y} + \bar{z}) = \{\bar{v} \in \bar{R} : \bar{v} \subseteq \bar{x} \dotplus (\bar{y} \dotplus \bar{z})\}$. But the addition "$\dotplus$ " of subsets of R is associative then $(\bar{x} \dotplus \bar{y}) \dotplus \bar{z} = \bar{x} \dotplus (\bar{y} \dotplus \bar{z})$, which implies that $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$. The zero element of R/G is $0G = \bar{0}$ and $\bar{0} + \bar{x} = \{\bar{t} : t = 0g_1 + xg_2 = xg_2 + 0g_1 : g_1, g_2 \in G\} = \bar{x} + \bar{0} = \bar{x}$.

For the inverse, let $\bar{x} = xG \in R$, then there exists $-x \in R$ and $-xG = -\bar{x} \in \bar{R}$ so $0 \in \bar{x} \dotplus (-\bar{x})$ implying that $\bar{0} \in \bar{x} + (-\bar{x})$. To prove that $-\bar{x}$ is unique, suppose that $\bar{0} \in \bar{x} + \bar{y}$ then $0 \in \bar{x} \dotplus \bar{y}$ and there exist $x \in \bar{x}, y \in \bar{y}$ such that $0 = x + y$, then $y = -x$ and $y \in -\bar{x}$, this means that $\bar{y} = -\bar{x}$.

Finally, let $\bar{z} \in \bar{x} + \bar{y}$ then $\bar{z} \subseteq \bar{x} \dotplus \bar{y}$. Then, there exist $x' \in \bar{x}, y' \in \bar{y}$ such that $z = x' + y'$ and $y' = z + (-x')$, then $\bar{y} = \overline{y'} = \overline{z + (-x')} \subseteq \bar{z} \dotplus (\overline{-x'}) = \bar{z} \dotplus (\overline{-x})$ therefore, $\bar{y} \in \bar{z} - \bar{x}$.

Since $(\bar{x} + \bar{y}) = \{\bar{t} : t = xg_1 + yg_2 , g_1, g_2 \in G\}$ then $\bar{z}(\bar{x} + \bar{y}) = \{\bar{z}\bar{t} : t = xg_1 + yg_2 , g_1, g_2 \in G\} = \{zG(xg_1 + yg_2)G, g_1, g_2 \in G\} = zxG + zyG = \overline{zx} + \overline{zy} = \bar{z}\bar{x} + \bar{z}\bar{y}$ because $\bar{z}\bar{t} = zGtG = ztG = \overline{zt}$. By the same argument we can prove that $(\bar{x} + \bar{y})\bar{z} = \bar{x}\bar{z} + \bar{y}\bar{z}$. ■

Example 1.3.1 Let $R = (\mathbb{Z}_{15}, +, \cdot)$ and $G = \{1, 4, 7, 13\}$ be a subset of R, then $(G, \cdot)$ is a group. Define $P(x) = \bar{x} = xG$, then $P(1) = \{1,4,7,13\}$, $P(2) = \{2,8,14,11\}$, $P(3) = \{3, 12, 6, 9\}$, $P(5) = \{5\}$, $P(0) = \{0\}$, $P(10) = \{10\}$.

$R/G = \{P(0), P(1), P(2), P(3), P(5), P(10)\}$. So $P(x) + P(y) = \{P(t): t = xg_1 + yg_2,\ g_1, g_2 \in G\} = \{P(t): t \in P(x) \dot{+} P(y)\}$. For instance we compute $P(2) + P(3)$ by the following table:

+	3	12	6	9
2	5	14	8	11
8	11	5	14	2
14	2	11	5	8
11	14	8	2	5

Therefore, $P(2) + P(3) = \{P(5), P(2)\}$. The hyperaddition and the multiplication on R/G can be written as follows:

$+$	$P(0)$	$P(1)$	$P(2)$	$P(3)$	$P(5)$	$P(10)$
$P(0)$	$\{P(0)\}$	$\{P(1)\}$	$\{P(2)\}$	$\{P(3)\}$	$\{P(5)\}$	$\{P(10)\}$
$P(1)$	$\{P(1)\}$	$\{P(2), P(5)\}$	$\{P(3), P(0)\}$	$\{P(1), P(10)\}$	$\{P(3)\}$	$\{P(2)\}$
$P(2)$	$\{P(2)\}$	$\{P(3), P(0)\}$	$\{P(1), P(10)\}$	$\{P(5), P(2)\}$	$\{P(1)\}$	$\{P(3)\}$
$P(3)$	$\{P(3)\}$	$\{P(1), P(10)\}$	$\{P(5), P(2)\}$	$\{P(0), P(3)\}$	$\{P(2)\}$	$\{P(1)\}$
$P(5)$	$\{P(5)\}$	$\{P(3)\}$	$\{P(1)\}$	$\{P(2)\}$	$\{P(10)\}$	$\{P(0)\}$
$P(10)$	$\{P(10)\}$	$\{P(2)\}$	$\{P(3)\}$	$\{P(1)\}$	$\{P(0)\}$	$\{P(5)\}$

$\cdot$	$P(0)$	$P(1)$	$P(2)$	$P(3)$	$P(5)$	$P(10)$
$P(0)$	$P(0)$	$P(0)$	$P(0)$	$P(0)$	$P(0)$	$P(0)$
$P(1)$	$P(0)$	$P(1)$	$P(2)$	$P(3)$	$P(5)$	$P(10)$
$P(2)$	$P(0)$	$P(2)$	$P(1)$	$P(3)$	$P(10)$	$P(5)$
$P(3)$	$P(0)$	$P(3)$	$P(3)$	$P(3)$	$P(0)$	$P(0)$
$P(5)$	$P(0)$	$P(5)$	$P(10)$	$P(0)$	$P(10)$	$P(5)$
$P(10)$	$P(0)$	$P(10)$	$P(5)$	$P(0)$	$P(5)$	$P(10)$

Example 1.3.2 Let $R = \mathbb{Z}_3[i] = \{0, 1, 2, i, 1+i,\ 2+i, 2i, 1+2i, 2+2i\}$ be a field with nine elements, and let $G = \{1,2\}$. Then,

$$P(1) = \{1, 2\}, P(i) = \{i, 2i\}, \qquad P(1+i) = \{1+i, 2+2i\},$$

$$P(2+i) = \{2+i, 1+2i\},\ P(0) = \{0\}.$$

As a result, $\bar{R} = R/G = \{\, P(1),\ P(i),\ P(1+i),\ P(2+i),\ P(0)\}$ is a hyperfield. The hyperaddition and the multiplication can be written by the following tables:

$+$	$P(0)$	$P(1)$	$P(i)$	$P(1+i)$	$P(2+i)$
$P(0)$	$\{P(0)\}$	$\{P(1)\}$	$\{P(i)\}$	$\{P(1+i)\}$	$\{P(2+i)\}$
$P(1)$	$\{P(1)\}$	$\{P(1), P(0)\}$	$\{P(1+i), P(1+2i)\}$	$\{P(2+i), P(i)\}$	$\{P(i), P(1+i)\}$
$P(i)$	$\{P(i)\}$	$\{P(1+i), P(1+2i)\}$	$\{P(0), P(i)\}$	$\{P(1), P(2+i)\}$	$\{P(1), P(1+i)\}$
$P(1+i)$	$\{P(1+i)\}$	$\{P(2+i), P(i)\}$	$\{P(1), P(2+i)\}$	$\{P(0), P(1+i)\}$	$\{P(1), P(i)\}$
$P(2+i)$	$\{P(2+i)\}$	$\{P(1+i), p(2i)\}$	$\{P(1), P(1+i)\}$	$\{P(1), P(i)\}$	$\{P(0), P(2+i)\}$

$\cdot$	$P(0)$	$P(1)$	$P(i)$	$P(1+i)$	$P(2+i)$
$P(0)$	$P(0)$	$P(0)$	$P(0)$	$P(0)$	$P(0)$
$P(1)$	$P(0)$	$P(1)$	$P(i)$	$P(1+i)$	$P(2+i)$
$P(i)$	$P(0)$	$P(i)$	$P(1)$	$P(2+i)$	$P(1+i)$
$P(1+i)$	$P(0)$	$P(1+i)$	$P(2+i)$	$P(i)$	$P(1)$
$P(2+i)$	$P(0)$	$P(2+i)$	$P(1+i)$	$P(1)$	$P(i)$

Therefore, $(R/G, +, \ \cdot)$ is a hyperfield where $P(1)$ is the multiplicative identity.

Definition 1.3.1 [7] Let R be a commutative hyperring and a be nonzero element of R, then a called a zero divisor if there exists a nonzero element $b \in R$, such that $ab = 0$.

The following definitions of a hyperdomain and a hyperfield are extensions to the definitions of an integral domain and a field in classical algebra.

Definition1.3.2 [1] A hyperring R that has no zero divisor is called a hyperdomain.

In Example 1.3.2, $P(3), P(5), P(10)$ are zero divisors because $P(3).P(5) = P(0)$ and $P(3).P(10) = P(0)$. Therefore, $\bar{R}$ in Example 1.3.2 is a hyperring but not a hyperdomain.

Definition 1.3.3 [1] A subset H of a hyperring R is called a subhyperring if itself is a hyperring under the hyperaddition and multiplication which are defined on R.

Theorem 1.3.2 [8] A nonempty subset H of a hyperring R is a subhyperring if it closed under hyperaddition and the multiplication which are defined on R, such that if for any $a, b \in H$ we have $a + (-b) \subseteq H$.

Proof. Firstly, H is closed under hyperaddition and multiplication. Since the hyperaddition and the multiplication are associative on R, the same is true for H. If $a \in H$ then $0 \in (a + (-a)) \subseteq H$, so $0 \in H$. To prove $-x \in H$ whenever $x \in H$, choose $a = 0, b = x$, then $0 + (-x) = \{-x\} \subseteq H$, implies $-x \in H$. Again by the closure of H, we have the multiplication is distributive to hyperaddition and also we have for all $a, b, c \in H$, if $c \in (a+b)$ then $a \in (c-b)$. Hence H is a subhyperring of R.■

Theorem 1.3.3 [1] Let R be commutative ring with identity, define $\bar{R}$ as follows, $\bar{R} = \{\bar{x} = \{x, -x\} | x \in R\}$, and define the hyperaddition by $\bar{x} \oplus \bar{y} = \{\overline{x+y}, \overline{x-y}\}$ and the multiplication by $\bar{x} \otimes \bar{y} = \overline{xy}$. Then $(\bar{R}, \oplus, \otimes)$ is a hyperring.

Proof. Let $\bar{a}, \bar{b}, \bar{c} \in \bar{R}$, then $\bar{a} \oplus \bar{b} = \{\overline{a+b}, \overline{a-b}\} \subseteq \bar{R}$, and

$\bar{a} \oplus (\bar{b} \oplus \bar{c}) = \bar{a} \oplus \{\overline{b+c}, \overline{b-c}\} = \bar{a} \oplus \overline{b+c} \cup \bar{a} \oplus \overline{b-c} = \{\overline{a+b+c},$
$\overline{a-(b+c)}, \overline{a+(b-c)}, \overline{a-(b-c)}\} = \{\overline{(a+b)+c}, \overline{(a+b)-c}\} \cup$
$\{\overline{(a-b)-c}, \overline{(a-b)+c}\} = \overline{a+b} \oplus \bar{c} \cup \overline{a-b} \oplus \bar{c} = \{\overline{a+b}, \overline{a-b}\} \oplus \bar{c}$

$= (\bar{a} \oplus \bar{b}) \oplus \bar{c}$.

$\bar{a} \oplus \bar{b} = \{\overline{a+b}, \overline{a-b}\} = \{\{a+b, -(a+b)\}, \{a-b, -(a-b)\}\}$

$= \{\{b+a, -(b+a)\}, \{b-a, -(b-a)\}\} = \{\overline{b+a}, \overline{b-a}\} = \bar{b} \oplus \bar{a}$,

$\bar{a} = \overline{-a} = \{-a, -(-a)\} = \{-a, a\}$.

The zero element is $\bar{0} = \{0\}$ because $\bar{0} \oplus \bar{a} = \{\overline{0+a}, \overline{0-a}\} = \{\bar{a}, \overline{-a}\} = \{\bar{a}\}$, $\bar{a} \oplus \bar{0} = \{\overline{a+0}, \overline{a-0}\} = \{\bar{a}\}$.

Let $\bar{0} \in \bar{a} \oplus \bar{x} = \{\overline{a+x}, \overline{a-x}\}$, then $\bar{0} = \overline{a+x}$ or $\bar{0} = \overline{a-x}$, if $\bar{0} = \overline{a+x} = \{a+x, -(a+x)\} \Longrightarrow 0 = a+x \Longrightarrow x = -a \Longrightarrow \bar{x} = \overline{-a}$, similarly, if $\bar{0} = \overline{a-x}$ we get $x = a \Longrightarrow \bar{x} = \bar{a} = \overline{-a}$, therefore for all $\bar{a} \in \bar{R}$, there exists $-\bar{a} = \bar{a} \in R$ such that $\bar{0} \in \bar{a} \oplus \bar{x}$.

Assume that $\bar{c} \in \bar{a} \oplus \bar{b} = \{\overline{a+b}, \overline{a-b}\}$ then $c = a+b$ or $c = a-b$ implies $a = c + (-b)$ or $a = c+b$ so that $\bar{a} = \overline{c+(-b)}$ or $\bar{a} = \overline{c+b}$ then $a \in \{(c+(-b))^{-}, (c+b)^{-}\} = c^{-} \oplus -b^{-}$.

For multiplication, if $\bar{a}, \bar{b}, \bar{c} \in \bar{R}$ then $\bar{a} \otimes \bar{b} = \overline{ab} \in \bar{R}$ and $(\bar{a} \otimes \bar{b}) \otimes \bar{c} = (\overline{ab}) \otimes \bar{c} = \overline{abc} = \overline{a(bc)} = \bar{a} \otimes (\bar{b} \otimes \bar{c})$.

Since R has an identity element 1, then $\bar{1} = \{1, -1\} \in \bar{R}$ is the identity element of $\bar{R}$ because $\bar{1} \otimes \bar{x} = \overline{1 \cdot x} = \overline{x \cdot 1} = \bar{x}$, for all $\bar{x} \in \bar{R}$.

For all $\bar{x} \in \bar{R}$, $\bar{x} \otimes \bar{0} = \overline{x \cdot 0} = \bar{0} = \overline{0 \cdot x} = \bar{0} \otimes \bar{x}$.

$\bar{a} \otimes (\bar{b} \oplus \bar{c}) = \bar{a} \otimes \{\overline{b+c}, \overline{b-c}\} = \bar{a} \otimes (\overline{b+c}) \cup \bar{a} \otimes (\overline{b-c}) = \overline{a(b+c)} \cup \overline{a(b-c)} = \overline{ab+ac} \cup \overline{ab+a(-c)} = \{ab+ac, -(ab+ac)\} \cup \{ab+a(-c), -(ab+a(-c))\} = \{ab+ac, -(ab+ac), ab-ac, -(ab-ac)\} = \{\overline{ab+ac}, \overline{ab-ac}\} = ab \oplus ac = \bar{a} \otimes \bar{b} \oplus \bar{a} \otimes \bar{c}$. Therefore, $(\bar{R}, \oplus, \otimes)$ is a hyperring. ■

Example 1.3.3 Let $R = \mathbb{Z}_8$. Define $\bar{R}$ as in the previous theorem, and $\bar{R} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. The hyperaddition and the multiplication can be written as in the following tables,

$\oplus$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\{\bar{0}\}$	$\{\bar{1}\}$	$\{\bar{2}\}$	$\{\bar{3}\}$	$\{\bar{4}\}$
$\bar{1}$	$\{\bar{1}\}$	$\{\bar{2}, \bar{0}\}$	$\{\bar{3}, \bar{1}\}$	$\{\bar{4}, \bar{2}\}$	$\{\bar{3}\}$
$\bar{2}$	$\{\bar{2}\}$	$\{\bar{3}, \bar{1}\}$	$\{\bar{4}, \bar{0}\}$	$\{\bar{3}, \bar{1}\}$	$\{\bar{2}\}$
$\bar{3}$	$\{\bar{3}\}$	$\{\bar{4}, \bar{2}\}$	$\{\bar{3}, \bar{1}\}$	$\{\bar{2}, \bar{0}\}$	$\{\bar{1}\}$
$\bar{4}$	$\{\bar{4}\}$	$\{\bar{3}\}$	$\{\bar{2}\}$	$\{\bar{1}\}$	$\{\bar{0}\}$

$\otimes$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{0}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{4}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$

Theorem 1.3.4 [1] Let $(H, \cdot)$ be a group and let $G = H \cup \{0, u, v\}$ where u, v are orthogonal idempotent elements and $u \neq v$ i.e. $uv = vu = 0$ and $u^2 = u, v^2 = v$. Define the hyperaddition on G by

$$g + 0 = 0 + g = \{g\} \text{ for all } g \in G,$$

$$g + g = \{g, 0\} \text{ for all } \in G,$$

$$\text{if } g_1 \neq g_2 ,\ g_1 + g_2 = G \backslash \{g_1 , g_2, 0\}, g_1, g_2 \in G \backslash \{0\}.$$

The multiplication can be defined as;

$$g \cdot 0 = 0 \cdot g = 0, \qquad \text{for all } g \in G.$$

$h \cdot u = u \cdot h = u$, $h \cdot v = v \cdot h = v$, for all $h \in H$ and $uv = vu = 0$. Then $(G, +, \cdot)$ is a hyperring.

Example 1.3.4 Consider the set $\mathbb{Z}_{10}$, let $H = \mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$ and the orthogonal idempotent elements of $\mathbb{Z}_{10}$ are 5 , 6 because $5 \cdot 6 = 0$, $5^2 = 5, 6^2 = 6$.

Let $G = H \cup \{0, 5, 6\}$, according to Theorem 1.3.4, $(G, +, \cdot)$ is a hyperring. The hyperaddition and the multiplication as in the following table.

+	0	1	3	7	9	5	6
0	{0}	{1}	{3}	{7}	{9}	{5}	{6}
1	{1}	{1, 0}	{7,9,5,6}	{3,9,5,6}	{3,7,5,6}	{3,7,9,6}	{3,7,9,5}
3	{3}	{7,9,5,6}	{3, 0}	{1,9,5,6}	{1,7,5,6}	{1,7,9,6}	{1,7,9,5}
7	{7}	{3,9,5,6}	{1,9,5,6}	{7, 0}	{1,3,5,6}	{1,3,9,6}	{1,3,9,5}
9	{9}	{3,7,5,6}	{1,7,5,6}	{1,3,5,6}	{9, 0}	{1,3,7,6}	{1,3,7,5}
5	{5}	{3,7,9,6}	{1,7,9,6}	{1,3,9,6}	{1,3,7,6}	{5, 0}	{1,3,7,9}
6	{6}	{3,7,9,5}	{1,7,9,5}	{1,3,9,5}	{1,3,7,5}	{1,3,7,9}	{6, 0}

$\cdot$	0	1	3	7	9	5	6
0	0	0	0	0	0	0	0
1	0	1	3	7	9	5	6
3	0	3	9	1	7	5	6
7	0	7	1	9	3	5	6
9	0	9	7	3	1	5	6
5	0	5	5	5	5	5	0
6	0	6	6	6	6	0	6

Theorem 1.3.5 [1] Let $(H,\ \cdot)$ be a group and $G = H \cup \{0\}$, define the hyperaddition and the multiplication on G as follows;

$$a + 0 = 0 + a = \{a\}\ ,\ a + a = \{a, 0\}, \text{for all}\ \ a \in G,$$

$$a + b = H\backslash\{a, b\}\ , \text{for all}\ \ a, b \in H, a \neq b.$$

The multiplication can be defined by $a \odot b = a \cdot b$ for all $a, b \in H$,

$a \odot 0 = 0 \odot a = 0$ for all $a \in G$. Then $(G, +, \odot)$ is a hyperring.

Example 1.3.5 Let $H = \mathbb{Z}_{10}^{*} = \{1, 3, 7, 9\}$ and $G_1 = \mathbb{Z}_{10}^{*} \cup \{0\}$, according to Theorem 1.3.5, $(G, +, \odot)$ is a hyperring. For G in Example 1.3.4, $G_1 \subset G$. It is easy to show that G_1 is a subhyperring of G under the same hyperaddition and multiplication of G.

CHAPTER 2

Hyperideals

In this chapter we provide definitions of a hyperideal, prime hyperideals and primary hyperideal. We give an example for each one of them. For all cases we talk about hyperideals of hyperring in the sense of Krasner.

2.1 Hyperideals and Examples

Definition 2.1.1 [1] Let $(R,+,\cdot)$ be a hyperring. A nonempty subset I of a hyperring R is called left (right) hyperideal if for all $a,b \in I$ we have $(a-b) \subseteq I$ and $r \cdot a \in I$ $(a \cdot r \in I)$ for all $r \in R$. It is called a hyperideal if it is left and right hyperideal.

Example 2.1.1 Let $(R,+,\ \cdot)$ be a commutative hyperring, then the subset

$$I = < a_1, a_2, \dots, a_m > = \{r_1 a_1 + r_2 a_2 + \ \dots + r_m a_m : r_i \in R\}$$

is a hyperideal which generated by $a_1, a_2, \dots\ , a_m$.

In Example 1.3.1, $< P(3) > = \{P(0), P(3)\}$, $< P(5) > = \{P(0), P(5), P(10)\}$ are hyperideals where $< P(5),\ P(10) > = R$.

Definition 2.1.2 [1] Let A, B be nonempty subsets of the hyperring $(R,+,\ \cdot\ )$ then :

$$A + B = \{x : x \in a + b\ , a \in A, b \in B\}$$

$$A \cdot B = \{x : x \in \textstyle\sum_{i=1}^{n} a_i b_i\ \ , a_i \in A, b_i \in B, n \in \mathbb{N}\}.$$

Theorem 2.1.1 [1] If I, J are two hyperideals of the hyperring $(R,+,\ \cdot\ )$. Then $I + J$ and $I \cdot J$ are hyperideals of $(R,+,\ \cdot)$.

Proof. Let $x, y\ \in I + J$ then $x \in a_1 + b_1$ and $y \in a_2 + b_2$ for some $a_1, a_2 \in I,\ b_1, b_2 \in J$. Then $x - y \in\ a_1 + b_1 - (a_2 + b_2) \Longrightarrow x - y \in (a_1 - a_2) + (b_1 - b_2) \subseteq I + J$ and for $r \in R$ we have $rx \in r(a_1 + b_1) = ra_1 + rb_1 \subseteq I + J$. So $I + j$ is a hyperideal.

For the product, let $, y \in I \cdot J$, then $x \in \sum_{i=1}^{n} a_i b_i,\ a_i \in I, b_i \in J$,

$y \in \sum_{j=1}^{m} c_j d_j\ ,\ c_j \in I, d_j \in J,$ then $x - y \in \sum_{i=1}^{n} a_i b_i - \sum_{j=1}^{m} c_j d_j = \sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{m} (-c_j) d_j \subseteq I \cdot J$. If $\in R$, then $r \cdot x \in r \cdot \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (r a_i) b_i \subseteq I \cdot J$. Hence $I \cdot J$ is a hyperideal of R.■

Corollary 2.1.1 If $I_1, I_2, \cdots, I_n$ are hyperideals of a hyperring R, then the sum $\sum_{i=1}^{i=n} I_i$ and the product $I_1 I_2 \cdots I_n$ are hyperideals of the hyperring R.

Proof. By Using Theorem 2.1.1 and induction.

Corollary 2.1.2 If $I_1, I_2, \cdots, I_n$ are hyperideals of a hyperring R, then the intersection $\bigcap_{i=1}^{n} I_i$ is a hyperideal of R.

Proof. Since $0 \in I_i$ for all i, then $0 \in \bigcap_{i=1}^{n} I_i$, let $a, b \in \bigcap_{i=1}^{n} I_i$, then $a, b \in I_i$ for all i, so $a - b \subseteq I_i$ for all i implying that $a - b \subseteq \bigcap_{i=1}^{n} I_i$. Now, let $r \in R, a \in \bigcap_{i=1}^{n} I_i$, then $a \in I_i$ and so $ra \in I_i$, for all i. Therefore $ra \in \bigcap_{i=1}^{n} I_i$ and hence $\bigcap_{i=1}^{n} I_i$ is a hyperideal of R.

Definition 2.1.3 [9] Let I be a hyperideal of a hyperring $(R, +, \cdot)$, then I is said to be prime hyperideal of R if $a \cdot b \in I$ implies that $a \in I$ or $b \in I$.

Definition 2.1.4 [9] Let I be a hyperideal of the hyperring $(R, +, \cdot)$, then I is said to be a primary hyperideal of R if $I \neq R$, $a \cdot b \in I, a \notin I$ implies $b^n \in I$ for some $n \in \mathbb{Z}^+$.

Definition 2.1.5 [9] A proper hyperideal I of the hyperring $(G, +, \cdot)$ is said to be maximal, if for any proper hyperideal B such that $I \subseteq B \subseteq R$, then $I = B$ or $B = R$.

In Example 1.3.4, $\{0, 6\}, \{0, 5\}$ are maximal hyperideals of R, but in Example 1.3.3 $I = \{\bar{0}, \bar{2}, \bar{4}\}$, $J = \{\bar{0}, \bar{4}\}$ are proper hyperideals where I is maximal hyperideal but J is not. In Example 1.3.1 both $< P(3) > = \{P(0), P(3)\}$ and $< P(5) > = \{P(0), P(5), P(10)\}$ are maximal hyperideals.

Theorem 2.1.3 [9] Let R be a hyperring with at least one proper hyperideal, then R has at least one maximal hyperideal.

Proof. Let $\mathcal{H}$ be the set of all chains of proper hyperideals of R which is ordered by inclusion, pick any chain $\{I_\alpha : \alpha \in \Delta\}$ of hyperideals from $\mathcal{H}$, and let $I = \bigcup_{\alpha \in \Delta} I_\alpha$. We show that I is a maximal hyperideal of R. Firstly let $a, b \in I$ then $a \in I_\alpha$, $b \in I_\beta$ for some $\alpha, \beta \in \Delta$, without loss of generality assume that $I_\alpha \subseteq I_\beta$, then $a, b \in I_\beta$ implying that

$a - b \subseteq I_\beta$ and so $a - b \in I$, and for all $r \in R$ we have $ra \in I_\beta$ implying that $ra \in I$. Therefore I is a hyperideal of R. If $I = R$ then $1 \in I \Longrightarrow 1 \in I_\beta$ for some β implying that $I_\beta = R$, which is a contradiction because I_β is proper hyperideal of R. Therefore $I \neq R$. Consequently, I is a proper hyperideal of R which is an upper bound of the chain $\{I_\alpha : \alpha \in \Delta\}$, then by Zorn's lemma I is a maximal hyperideal of R. ■

Definition 2.1.6 [10] A hyperring that has only one maximal hyperideal is called local hyperring.

Theorem 2.1.3 [10] A hyperring R is a local hyperring if and only if the nonunit elements of R form a hyperideal.

Proof. Assume that R is a local hyperring, M is the maximal hyperideal of R and x is any element in $R - M$. If x is not unit, then we have a hyperideal $< x >$ generated by x, but the hyperideal $< x >$ is either maximal or contained in a maximal hyperideal other than M, and in both cases we have a contradiction because M is the unique maximal hyperideal of R, so that x must be a unit.

Conversely, suppose that R is a hyperring for which the nonunit elements form a hyperideal I, to show that I is a unique maximal hyperideal, suppose that J is another hyperideal such that $J \nsubseteq I$, pick an element $x \in J - I$, then x is a unit, so $x^{-1}x = 1 \in J$, therefore, $J = R$. Hence I is the unique maximal hyperideal in R and so R is a local hyperring. ■

As in the ordinary algebra, we can construct a hyperring from another hyperring, consider the following theorem and its proof.

Theorem 2.1.4 [8] Let $(R, +, \cdot)$ be a commutative hyperring and A be a subhyperring of R. Then the set of cosets $R/A = \{r + A : r \in R\}$ is a commutative hyperring under the hyperaddition $x + A + y + A = x + y + A$ and multiplication $(s + A)(t + A) = st + A$ if and only if A is a hyperideal of R. This hyperring is called a quotient hyperring.

Proof. Firstly, let us prove that $(R/A, +)$ is a commutative hypergroup. If $a + A, b + A \in R/A$ then $a + A + b + A = a + b + A \subseteq R/A$. Since the hyperaddition is associative on R then $(a + A + b + A) + c + A = (a + b + A) + c + A = (a + b) + c + A = a + (b + c) + A = a + A + (b + A + c + A)$.

$a + A + b + A = a + b + A = b + a + A = b + A + a + A$. And the zero element is $0 + A = A$, also for $a + A$ in R/A the additive inverse is $-a + R$ because $0 + A \subseteq a +$

$(-a)+A=a+A+(-a)+A$. Finally, let $a+A, b+A, c+A \in R/A$ such that $c+A \in a+A+b+A=a+b+A$ then $c \in a+b$ and since R is a hypergroup then $a \in c+(-b)$, therefore $a+A \in c+A+(-b)+A$, and $(R/A,+)$ is a hypergroup. To complete the proof, it is easy to check that the multiplication is associative and distributive over the hyperaddition. So the multiplication is well defined if and only if A is a hyperideal of R. Assume that $s_1+A=s_2+A$, $t_1+A=t_2+A$ then we must show that $s_1t_1+A=s_2t_2+A$. By the definition $s_1 \in s_2+a$, $t_1 \in t_2+b$ where $a,b \in A$ then $s_1t_1 \in (s_2+a)(t_2+b)=s_2t_2+s_2b+at_2+ab$, so $s_1t_1+A \subseteq s_2t_2+s_2b+at_2+ab+A=s_2t_2+A$ because A absorbs s_2b+at_2+ab, and by the same method $s_2 \in s_1+a_1$, $t_2 \in t_1+b_1$, $a_1,b_1 \in A$. We have $s_2t_2+A \subseteq s_1t_1+A$. Therefore $s_1t_1+A=s_2t_2+A$ and hence the multiplication is well defined.

Conversely, assume that A is a subhyperring but not a hyperideal, then there exists $a \in A, r \in R$, but $ra \notin A$ or $ar \notin A$, say $ra \notin A$. Now $a+A=0+A$ but $(a+A)(r+A)=ar+A$, $(0+A)(r+A)=0+A=A$, $ar+A \neq A$. Therefore the multiplication is not well defined and hence A must be a hyperideal.■

Example 2.1.2 In Example 1.3.1 $< P(3) > = \{P(0),P(3)\}$ is a hyperideal and

$$R/< P(3) > = \{< P(3) >, P(1)+< P(3) >, P(2)+< P(3) >\}$$

$$= \{\{P(0),P(3)\},\{P(1),P(10)\},\{P(2),P(5)\}\}.$$

The hyperaddition and multiplication of $(R/< P(3) > ,+, \cdot)$ as in the following tables.

+	$\{P(0),P(3)\}$	$\{P(1),P(10)\}$	$\{P(2),P(5)\}$
$\{P(0),P(3)\}$	$\{P(0),P(3)\}$	$\{P(1),P(10)\}$	$\{P(2),P(5)\}$
$\{P(1),P(10)\}$	$\{P(1),P(10)\}$	$\{P(2),P(5)\}$	$\{P(0),P(3)\}$
$\{P(2),P(5)\}$	$\{P(2),P(5)\}$	$\{P(0),P(3)\}$	$\{P(1),P(10)\}$

$\cdot$	$< P(3) >$	$P(1)+< P(3) >$	$P(2)+< P(3) >$
$< P(3) >$	$< P(3) >$	$< P(3) >$	$< P(3) >$
$P(1)+< P(3) >$	$< P(3) >$	$P(1)+< P(3) >$	$P(2)+< P(3) >$
$P(2)+< P(3) >$	$< P(3) >$	$P(2)+< P(3) >$	$P(1)+< P(3) >$

Example 2.1.3 In Example 1.3.3, $\bar{R} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, $I =< \bar{4} > = \{\bar{0}, \bar{4}\}$ is a hyperideal of $\bar{R}$, then $\bar{R}/_I = \left\{\{\bar{0}, \bar{4}\}, \{\bar{1}, \bar{3}\}, \{\bar{2}\}\right\}$ is a hyperring the hyperaddition and multiplication is as in the following tables.

$+$	$\{\bar{0}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$
$\{\bar{0}, \bar{4}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$
$\{\bar{1}, \bar{3}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{0}, \bar{2}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$
$\{\bar{2}\}$	$\{\bar{2}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{0}, \bar{4}\}$

$\cdot$	$\{\bar{0}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$
$\{\bar{0}, \bar{4}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{0}, \bar{4}\}$
$\{\bar{1}, \bar{3}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$
$\{\bar{2}\}$	$\{\bar{0}, \bar{4}\}$	$\{\bar{2}\}$	$\{\bar{4}\}$

Theorem 2.1.5 [1] Let R be a commutative hyperring with unity, and I be a hyperideal of R. Then R/I is a hyperdomain if and only if I is prime.

Proof. Assume that R / I is a hyperdomain, let $ab \in I$, then $(a + I)(b + I) = ab + I = I$ where I is the zero element. So that $a + I = I$ or $b + I = I$, then $a \in I$ or $b \in I$ hence I is prime hyperideal. To prove that R / I is a hyperdomain if I is a prime hyperideal, let

$(a+I)(b+I)=ab+I=I,$ then $ab \in I$ implying that $a \in I$or $b \in I$, so that $a+I=I$ or $b+I=I$.

Theorem 2.1.6 [1] Let R be a commutative hyperring with unity and I be a hyperideal of R, then R/I is a hyperfield if and only if I is maximal.

Proof. Assume that $R\,/\,I$ is a hyperfield, let S be a hyperideal in R such that $I \subset S$ but $I \neq S,$ let $a \in S-I$ and since $R\,/\,I$ is a hyperfield there exists $r \in R$ such that $(a+S)(r+S)=ar+S=1+S$. Then, there exist $s_1\,,s_2 \in S$ such that $ar+s_1=1+s_2$, so $1 \in ar+s$ for some $s \in S$, we get $1 \in S$ which implies that $S=R$, therefore I is maximal hyperideal of R. Conversely, assume that I is maximal hyperideal of R, let $a+I \in R/I,$ but $a \notin I$, then $<a,I>=R$, and $1 \in <a,I>$, so $1=ra+i$ for some $i \in I,$ then $1+I \in ra+i+I$ for some $i \in I$. Then $1+I \in ra+i+I=ra+I$ implying that $1+I=ra+I=(r+I)(a+I)$. Therefore, $a+I$ is a unit, and hence $R\,/\,I$ is a hyperfield. ■

Corollary 2.1.4 Let R be a commutative hyperring with unity and I be a maximal hyperideal of R. Then I is prime hyperideal of R.

Proof. Suppose that I is a maximal hyperideal of R. Then by Theorem 2.1.6 we conclude that R/I is a hyperfield and so R/I is a hyperdomain. Then by Theorem 2.1.5 I is a prime hyperideal of R.

Example 2.1.3 In Example 1.3.3, $I=<\bar{4}>=\{\bar{0},\bar{4}\}$, I is not a prime hyperideal, because $\bar{2}\cdot\bar{2}=\bar{4} \in I$, but $\bar{2} \notin I$ in the other hand it easy to check that I is primary hyperideal, since $\bar{2}\cdot\bar{2}=\bar{4} \in I, \bar{2} \in I$ but $\bar{2}^2=\bar{4} \in I$.

Example 2.1.4 In Example 1.3.1, $I=\{P(0),P(5),P(10)\}$ is maximal hyperideal , so that $\bar{R}=R/I=\{I,P(1)+I\}$ is a hyperfield, the hyperaddition and multiplication for this hyperfield as in the following tables:

$+$	I	$P(1)+I$
I	I	$P(1)+I$
$P(1)+I$	$P(1)+I$	$\{P(2),P(5)\}+I$

$\cdot$	I	$P(1)+I$
I	I	$P(1)+I$
$P(1)+I$	$P(1)+I$	$P(1)+I$

Example 2.1.5 In Example 1.3.3 $I = \{\bar{0}, \bar{2}, \bar{4}\}$ is a maximal hyperideal, so that $F = R/I = \{I, 1 + I\}$ is a hyperfield.

Example 2.1.6 In Example 1.3.4 $I = \{0\ ,5\}$, $J = \{0\ ,6\}$ are maximal hyperideals , so $F_1 = G\ /\ I$, $F_2 = G\ /\ J$ are hyperfields.

Definition 2.1.7 [1] The subhyperring H of a hyperring R is called normal if and only if $x + H - x \subseteq H$ for all $x \in R$, and it can be written as $H \lhd R$.

Corollary 2.1.5 [4] H is a normal hyperideal of a hyperring R, if and only if $x - x \subseteq H$, for all $x \in R$.

Proof. Let H be a normal hyperideal of a hyperring R, then $x + H - x \subseteq H$, for all $x \in R$, but $x + 0 - x = x - x \subseteq x + H - x \subseteq H$. This means that $x - x \subseteq H$, for all $x \in R$. Conversely if $-x \subseteq H$, then $H + x - x \subseteq H + H$, which implies that $x + H - x \subseteq H$.

Definition 2.1.8 [1] Let H be a hyperideal of the hyperring R, the normalizer of H which is denoted by $N(H)$ is $N(H) = \{x: x + H - x \subseteq H, x \in R\}$.

Theorem 2.1.7 [1] If I is a hyperideal of a hyperring R, then $N(I)$ is a hyperideal of R, and $N(I)$ is the largest hyperideal in R that contains I.

Proof. Let $x, y \in N(I)$, then $(x - x) \subseteq I, (y - y) \subseteq I$, then $(x - x) - (y - y) \subseteq I$ and so $(x - y) - (x - y) \subseteq I$, now if $t \in (x - y)$, then $t - t \subseteq (x - y) - (x - y) \subseteq I$. Then $t \in N(I)$, so that $(x - y) \subseteq N(I)$.

Let $r \in R, x \in N(I)$, then $(x - x) \subseteq I$, so $r(x - x) = (rx - rx) \subseteq I$, we have $rx \in N(I)$. Therefore $N(I)$ is a hyperideal of the hyperring R. If H is a hyperideal of R such that $I \lhd H$, then for all $x \in H$ we have $x - x \subseteq I$. This means that $x \in N(I)$ so that $H \subseteq N(I)$. $N(I)$is the largest hyperideal in R that contains I. ■

Corollary 2.1.6 [12] If I is a hyperideal of a hyperring R, then I is normal in R if and only if $N(I) = R$.

Corollary 2.1.7 [12] If M is a maximal hyperideal of a hyperring R, then either $N(M) = M$ or $N(M) = R$.

Theorem 2.1.8 [12] If I, J are normal hyperideals of a hyperring R, then $I \cap J$ is a normal hyperideal of R.

Theorem 2.1.9 If I, J are hyperideals of a hyperring R, such that $I \subseteq J$, then $N(I) \subseteq N(J)$, if I is normal in R, then J is normal in R.

Proof. Let $x \in N(I)$, then $(x - x) \subseteq I \subseteq J$, which implies that $x \in N(J)$, so that $N(I) \subseteq N(J)$. On the other hand, if I is normal in R then by Corollary 2.1.7 $N(I) = R$. But $N(I) \subseteq N(J)$, then $N(J) = R$, so that again by Corollary 2.1.7 J is normal in R.■

Theorem 2.1.10 [12] If I, J are hyperideals of the hyperring R, then

i. $N(I) + N(J) \subseteq N(I + J)$,
ii. $N(I) \cap N(J) = N(I \cap J)$.

Proof. By the definition of addition we know that $I \subseteq I + J$, $J \subseteq I + J$ and by Theorem 2.1.9 we have $N(I) \subseteq N(I + J), N(J) \subseteq N(I + J)$. Then $N(I) + N(J) \subseteq N(I + J)$, For the second part, $I \cap J \subseteq I$, $I \cap J \subseteq J$, $N(I \cap J) \subseteq N(I)$, $N(I \cap J) \subseteq N(I)$ then $N(I \cap J) \subseteq N(I) \cap N(J)$. Conversely, let $x \in N(I) \cap N(J)$. Then $x - x \in I$ and $x - x \in J$, so that $x - x \in I \cap J$, we have $x \in N(I \cap J)$. Therefore $N(I) \cap N(J) = N(I \cap J)$.■

Theorem 2.1.11 [8] If I is normal hyperideal of a hyperring R and J is any hyperideal of R, then $I + J$ is normal hyperideal of R and $N(I \cap J) = N(J)$.

Proof. Since $I \subseteq I + J$ and $I \lhd R$, then $x - x \subseteq I \subseteq I + J$,for all $x \in R$. Then by Corollary 2.1.5 $I + J \lhd R$. To prove the second part, we know that $I \cap J \subseteq J$, then $N(I \cap J) \subseteq N(J)$. To prove the equality, let $x \in N(J)$, then $x - x \subseteq J$, but I is normal, then $x - x \subseteq I$, and $x - x \subseteq I \cap J$, so that $x \in N(I \cap J)$. Hence $N(I \cap J) = N(J)$. ■

Definition 2.1.9 [11] Let I be a hyperideal of the hyperring R then the radical of I which is denoted by $\rho(I)$ is defined by $\rho(I) = \{x : x^n \in I , n \in \mathbb{N} , x \in R\}$.

The following theorems are analogous to those in classical algebra.

Theorem 2.1.12 [11]The radical of a hyperideal I of a hyperring R is a hyperideal and if $I \lhd R$, then $\rho(I) \lhd R$.

Proof. Firstly, let us prove that $\rho(I)$ is a hyperideal of R. Since $I \subseteq \rho(I)$, then $\rho(I) \neq \emptyset$. Let $x, y \in \rho(I)$, then there exists $n, m \in \mathbb{N}$ such that $x^n, y^m \in I$. We claim that $(x - y)^{n+m} \in I$, by binomial expansion

$$(x - y)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} x^{n+m-i} y^i$$

and since $0 \leq i \leq n+m$, then for all values of i we have, either $x^{n+m-i} \in I$ or $y^i \in I$, so that $x^{n+m-i}y^i \in I$, implying that $(x-y)^{n+m} \in I$, which means that $x-y \in \rho(I)$. If $r \in R$, then $r^n \in R$ and $r^n x^n = (rx)^n \in I$, so $rx \in \rho(I)$. Hence $\rho(I)$ is a hyperideal of R. If I is normal in R, then $x - x \subseteq I \subseteq \rho(I)$ for all $x \in R$ so that $\rho(I)$ is a normal hyperideal of R. ■

The following theorem is an extension to one in classical algebra in [11].

Theorem 2.1.13 [11] Let I, J be hyperideals of a hyperring R. Then the following properties are hold;

1) $\rho(I) \supseteq I$,
2) $\rho(\rho(I)) = \rho(I)$,
3) $\rho(IJ) = \rho(I \cap J) = \rho(I) \cap \rho(J)$,
4) $\rho(I) = R \Leftrightarrow I = R$,
5) $\rho(I+J) = \rho(\rho(I) + \rho(J))$,
6) If P is a prime hyperideal of a hyperring R, then for all $n \in \mathbb{Z}^+$, $\rho(P^n) = P$.

Proof. The part (1) is clear from the definition of $\rho(I)$. To prove the second part, we know from (1) that $\rho(I) \subseteq \rho(\rho(I)$. Now let $x \in \rho(\rho(I))$. Then $x^n \in \rho(I)$ for some $x \in \mathbb{Z}^+$, which means that $(x^n)^m = x^{nm} \in I , m \in \mathbb{Z}^+$. Then $x \in \rho(I)$ and hence $\rho(\rho(I)) = \rho(I)$. For part (3), since $IJ \subseteq I,\ IJ \subseteq J$ then $IJ \subseteq I \cap J$. So that $\rho(IJ) \subseteq \rho(I \cap J)$. Now let $x \in \rho(I \cap J)$, then there exists $n \in \mathbb{Z}^+$ such that $x^n \in I \cap J$, and $x^n x^n = x^{2n} \in IJ$, so $x \in \rho(IJ)$ and $\rho(IJ) = \rho(I \cap J)$. On the other hand $I \cap J \subseteq I, I \cap J \subseteq J$, implying that $\rho(I \cap J) \subseteq \rho(I), \rho(I \cap J) \subseteq \rho(J)$, so that $\rho(I \cap J) \subseteq \rho(I) \cap \rho(J)$. Now let $x \in \rho(I) \cap \rho(J)$, then there exist $n, m \in \mathbb{Z}^+$such that $x^n \in I , x^m \in J$, so that $x^{n+m} \in I , x^{n+m} \in J$. Then $x^{n+m} \in I \cap J$, which means that $x \in \rho(I \cap J)$. Thus $\rho(I \cap J) = \rho(I) \cap \rho(J)$.

To prove (4) it is clear that if $I = R$, then $\rho(I) = R$. Conversely, if $\rho(I) = R$ this means that $1 \in I$, so $I = R$.

For part (5), we know that, $I \subseteq \rho(I), J \subseteq \rho(J)$ so that $I + J \subseteq \rho(I) + \rho(J)$ implying that $\rho(I+J) \subseteq \rho(\rho(I) + \rho(J))$, furthermore $I \subseteq I + J , J \subseteq (I+J)$ which implies that $\rho(I) \subseteq \rho(I+J), \rho(J) \subseteq \rho(I+J)$ then $\rho(I) + \rho(J) \subseteq \rho(I+J)$, and by part (2) we have $\rho(\rho(I) + \rho(J)) \subseteq \rho(\rho(I+J)) = \rho(I+J)$, hence $\rho(I+J) = \rho(\rho(I) + \rho(J))$.

For part (6) if $x \in P$, then $x^n \in P^n$, $x \in (P^n)$, and $P \subseteq \rho(P^n)$. Conversely, let $x \in \rho(P^n)$. Then there exists $m \in \mathbb{N}$, such that $x^m \in P^n$. Then either $x^m \in P$ or $x^k \in P$ for some $k < m$, and since P is a prime hyperideal then in the two cases we have $x \in P$. Hence for all $n \in \mathbb{N}$, $\rho(P^n) = P$. ■

Definition 2.1.10 [11] If I, J are two hyperideals of a hyperring R, then I, J are called coprime if $I + J = R$.

Theorem 2.1.14 [11] If I, J are two hyperideals of a hyperring R, such that $\rho(I), \rho(J)$ are coprime, then I, J are coprime.

Proof. By part(5) of Theorem 2.1.13, we know that $\rho(I + J) = \rho(\rho(I) + \rho(J))$, and since $\rho(I), \rho(J)$ are coprime. Then $\rho(I) + \rho(J) = R$, implying that $\rho(I + J) = \rho(R) = R$. By Part (4) of Theorem 2.1.13 $\rho(I + J) = R$ if and only if $I + J = R$, thereby I, J are coprime hyperideals of R■.

Theorem 2.1.15 [11] Let I, J, K be hyperideals of the hyperring R. Then the following properties hold.

1) $I(J + K) = IJ + IK$,
2) if $I + J = R$, then $IJ = I \cap J$.

Proof. (1) Let $a \in I(J + K)$ then $a \in \sum_{i=1}^{n} b_i c_i$, where $b_i \in I, c_i \in (J + K)$, then $c_i \in (d_i + e_i)$, $d_i \in J$, $e_i \in K$, so that

$$a \in \sum_{i=1}^{n} b_i(d_i + e_i) = \sum_{i=1}^{n} b_i d_i + b_i e_i \subseteq IJ + IK.$$

We get $I(J + K) \subseteq IJ + IK$.

Conversely, let $z \in IJ + IK$, then $z \in a + b$, where $a \in IJ, b \in IK$ and $a = \sum_{i=1}^{n} x_i y_i$, $b = \sum_{i=1}^{m} c_i d_i$, where $x_i, c_i \in I, y_i \in J, d_i \in K$, also since $J \subseteq J + K$, $K \subseteq J + K$, we have

$$\sum_{i=1}^{n} x_i y_i \subseteq I(J + K), \sum_{i=1}^{m} c_i d_i \subseteq I(J + K).$$

Then

$$z \in \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{m} c_i d_i \subseteq I(J + K).$$

Thus $IJ + IK \subseteq I(J + K)$, and hence $I(J + K) = IJ + IK$.

To prove (2), we know that $IJ \subseteq I, IJ \subseteq J$ so that $IJ \subseteq I \cap J$. Now, let $x \in I \cap J$. Since $I + J = R$, then $1 \in a + b$, where $a \in I, b \in J$, we get

$$x = x \cdot 1 \in x(a + b) = xa + xb \subseteq IJ.$$

Then $I \cap J \subseteq IJ$, and hence If $I + J = R$, thus $IJ = I \cap J$.■

2.2 External Direct Product

In this section we will construct a hyperring by finite number of hyperrings. This method is called the external direct product of hyperrings. It is an analogue to the direct product in [8, 12].

Definition 2.2.1[8] Let $R_1, R_2, \cdots, R_n$ be a finite collection of hyperrings. Then the external direct product of $R_1, R_2, \cdots, R_n$ which can be written as

$$R_1 \otimes R_2 \otimes \cdots \otimes R_n = \prod_{i=1}^{n} R_i$$

is the set of all *n*-tuples $x = (x_1, x_2, \cdots, x_n)$ for which $x_i \in R_i$. The hyperaddition and the multiplication operations are componentwise. The identity element is $(1, 1, \cdots, 1)$.

Theorem 2.2.1 The direct product $R = \prod_{i=1}^{n} R_i$ is a hyperring under the operations defined above.

Proof.

Let $a = (a_1, a_2, \cdots, a_n), b = (b_1, b_2, \cdots, b_n), c = (c_1, c_2, \cdots, c_n) \in R$. Then

1) $a + b = \{(x_1, x_2, \cdots, x_n) | x_i \in a_i + b_i \subseteq R_i\} \subseteq R$.
2) $a + (b + c) = (a_1, a_2, \cdots, a_n) + ((b_1, b_2, \cdots, b_n) + (c_1, c_2, \cdots, c_n))$

$$= (a_1, a_2, \cdots, a_n) + (b_1 + c_1, b_2 + c_2, \cdots, b_n + c_n)$$

$$= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \cdots, a_n + (b_n + c_n))$$

$$= ((a_1 + b_1) + c_1 , (a_2 + b_2) + c_2 , \cdots, (a_n + b_n) + c_n)$$

$$= (a + b) + c.$$

3) $a + b = (a_1 + b_1 , a_2 + b_2 , \cdots, a_n + b_n) = (b_1 + a_1, b_2 + a_2, \cdots, b_n + a_n) = b + a$.

4) The identity of hyperaddition is $0 = (0, 0, \cdots, 0)$, and for all $a \in R$, we have $0 + a = (0 + a_1, 0 + a_2, \cdots, 0 + a_n) = (a_1 + 0, a_2 + 0, \cdots, a_n + 0) = a + 0 = a$.

5) The additive inverse of $a = -a$ exits, because $-a = (-a_1, -a_2, \cdots, -a_n) \in R$, and $0 = (0, 0, \cdots, 0) \in a + b = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$ if and only if $0 \in a_i + b_i$, if and only if $b_i = -a_i$, if and only if $b = -a$.

6) If $a \in b + c$, then $a_i \in b_i + c_i$ for $i = 1, 2, \cdots, n$. So that $b_i \in a_i + -c_i$. Hence $b \in a + -c$.

7) $ab = (a_1b_1, a_2b_2, \cdots, a_nb_n) \in R$.

8) $a \cdot 1 = (a_1 \cdot 1, a_2 \cdot 1, \cdots, a_n \cdot 1) = (a_1, a_2, \cdots, a_n) = a \cdot 1 = a$.

9) $a \cdot 0 = (a_1 \cdot 0, a_2 \cdot 0, \cdots, a_n \cdot 0) = (0, 0, \cdots, 0) = 0$.

10) $a \cdot (b + c) = (a_1, a_2, \cdots, a_n) \cdot ((b_1, b_2, \cdots, b_n) + (c_1, c_2, \cdots, c_n))$

$$= (a_1, a_2, \cdots, a_n) \cdot (b_1 + c_1, b_2 + c_2, \cdots, b_n + c_n)$$

$$= (a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2), \cdots, a_n \cdot (b_n + c_n))$$

$$= (a_1 \cdot b_1 + a_1 \cdot c_1, a_2 \cdot b_2 + a_2 \cdot c_2, \cdots, a_n \cdot b_n + a_n \cdot c_n)$$

$$= a \cdot b + a \cdot c.$$

Similarly, $(b + c) \cdot a = b \cdot a + c \cdot a$.

Hence, the direct product $R = \prod_{i=1}^{n} R_i$ is a hyperring. ■

CHAPTER 3

Hyperring Homomorphism

In this chapter, we discuss the hyperring homomorphism, all definitions and theorems are extended to those in classical algebra in [11].

3.1 Chinese Remainder Theorem for Hyperrings

Definition 3.1.1[1] A hyperring homomorphism is a function f from a hyperring R_1 to a hyperring R_2 such that for all $a, b \in R_1$

i) $f(a+b) = f(a) + f(b)$.

ii) $f(a \cdot b) = f(a) \cdot f(b)$.

iii) $f(1) = 1$.

Definition 3.1.2 [8] The kernel of the homomorphism f from a hyperring R_1 to a hyperring R_2 which is denoted by $\ker(f)$ is defined by

$$\ker(f) = \{x : f(x) = 0 \, , x \in R_1\}.$$

Theorem 3.1.1 [1] Let f be a hyperring homomorphism from $R_1$to R_2, then $\ker(f) = \{0\}$ if and only if f is one to one.

Proof. Assume that $\ker(f) = \{0\}$. Let $f(x) = f(y)$ for some $x, y \in R_1$. Then

$$0 = f(0) \in f(x) - f(x) = f(y) - f(x) = f(y - x).$$

So that, there exists $t \in (y - x)$ such that $f(t) = f(0) = 0$, and since $\ker(f) = \{0\}$, then $t = 0$, which implies that $y - x = 0$ and $y = x$. Hence f is one to one.

Conversely, assume that f is one-to-one, $x \in \ker(f)$. Then $f(x) = f(0) = 0$, and $x = 0$. Therefore $\ker(f) = \{0\}$. ■

Theorem 3.1.2 Let R be a hyperring and $I_1, I_2, \cdots, I_n$ be hyperideals of R. Let f be a function from R to the hyperring $\prod_{i=1}^{n} R/I_i$ defined by

$f(x) = (x + I_1, x + I_2, \cdots, x + I_n)$, then f is a hyperring homomorphism.

Proof. Let $x, y \in R$, then by Theorem 2.1.2 we show that:

1) $f(x + y) = (x + y + I_1, x + y + I_2, \cdots, x + y + I_n)$

$$= (x + I_1 + y + I_1, x + I_2 + y + I_2, \cdots, x + I_n + y + I_n)$$

$$= (x + I_1, x + I_2, \cdots, x + I_n) + (y + I_1, y + I_2, \cdots, y + y_n)$$

$$= f(x) + f(y).$$

2) $f(xy) = (xy + I_1, xy + I_2, \cdots, xy + I_n)$

$$= ((x + I_1)(y + I_1), (x + I_2)(y + I_2), \cdots, (x + I_n)(y + I_n))$$

$$=(x + I_1, x + I_2, \cdots, x + I_n) \cdot (y + I_1, y + I_2, \cdots, y + y_n)$$

$$=f(x) \cdot f(y).$$

3) $f(1) = (1 + I_1, 1 + I_2, \cdots, 1 + I_n)$, where $(1 + I_1, 1 + I_2, \cdots, 1 + I_n)$ is the identity element of $\prod_{i=1}^{n} R/I_i$.■

The following theorem is an extension to Chinese Remainder Theorem for ring in [8].

Theorem 3.1.3 [13] (Chinese Remainder Theorem for Hyperrings)

Let R be a hyperring and $I_1, I_2, \cdots, I_n$ be hyperideals of R. Let the homomorphism f from R to $\prod_{i=1}^{n} R/I_i$ be defined as in the previous theorem. Then:

1) For $i \neq j$ if I_i, I_j are coprime , then $\prod_{i=1}^{n} I_i = \bigcap_{i=1}^{n} I_i$,

2) Whenever $i \neq j, I_i, I_j$ are coprime if and only if the homomorphism f is onto,

3) $\bigcap_{i=1}^{n} I_i = \{0\}$ if and only if the homomorphism f is one-to-one.

Proof.

1) The first statement will be proved by induction, for $n = 2$, it is true by Theorem 3.1.5. Assume that it is true for $m = n - 1$, this means that

$$I = \prod_{i=1}^{n-1} I_i = \bigcap_{i=1}^{n-1} I_i .$$

Now assume that $I_i + I_n = R$, for $i = 1, 2, \cdots, n-1$, so that for each i we have $x_i \in I_i,\ y_i \in I_n$ such that $1 \in x_i + y_i$. Then $x_i \in 1 + (-y_i) \in 1 + I_n$, implying that

$$\prod_{i=1}^{n-1} I_i \in 1 + I_n.$$

So that

$$1 \in I_n + \prod_{i=1}^{n-1} I_i = I_n + I \Rightarrow R = I_n + I.$$

Therefore, again by Theorem 3.1.5 we have

$$\prod_{i=1}^{n} I_i = II_n = I \cap I_n = \bigcap_{i=1}^{n} I_i.$$

2) Assume that f is onto, without loss of generality, it is enough to prove that I_1, I_2 are co-prime. Since f is onto, then there exists $x \in R$ such that $f(x) = (1 + I_1, 0 + I_2, \cdots, 0 + I_n)$. But $f(x) = (x + I_1, x + I_2, \cdots, x + I_n)$, so that $x + I_1 = 1 + I_1$, $x + I_2 = I_2$, this means that $x \in I_2$ and $x + i \in 1 + I_1$ for some $i \in I_1$, so that $x \in 1 + I_1 + i = 1 + I_1$, then $1 \in x + I_1$ but since $x \in I_2$, we have that $1 \in x + I_1 + I_2 = I_1 + I_2$. Therefore, $I_1 + I_2 = R$ and hence I_1, I_2 are co-prime.

Conversely, assume that I_i, I_j are co-prime for $i \neq j$. We will prove that f is onto, so it is enough to prove that for each $y \in R$, there exists $x \in R$ such that $f(x) = (y + I_1, I_2, \cdots, I_n)$. Since I_1, I_i are co-prime then $I_1 + I_i = R$ for $i = 2, 3, \cdots, n$, let $y \in u_2 + v_2$, $1 \in u_i + v_i$ for $i = 3, 4, \cdots, n$, where $u_i \in I_1, v_i \in I_i$ Let $x = \prod_{i=2}^{n} v_i$, then $x \in (y - u_2) \prod_{i=3}^{n} (1 - u_i)$. So that $x \in y + I_1$, which means that $x + I_1 = y + I_1$, also $x = \prod_{i=2}^{n} v_i \in I_i$.

Therefore $f(x) = (x + I_1, x + I_2, \cdots, x + I_n) = (y + I_1, I_2, \cdots, I_n)$, and f is onto.

3) By Theorem 3.1.1 we prove that $\ker(f) = \{0\}$ if and only if f is one-to-one, for the homomorphism f we have $\ker(f) = \{x \in R : x + I_i = I_i, i = 1,2, \cdots, n\} = \{x \in R : x \in I_i, i = 1, 2, \cdots, n\} = \bigcap_{i=1}^{n} I_i$. Hence f is one-to-one if and only if $\ker(f) = \{0\} = \bigcap_{i=1}^{n} I_i$. ■

Theorem 3.1.4 Let $I_1, I_2, \cdots, I_n$ be prime hyperideals of a hyperring R, and I be a hyperideal of R such that $I \subseteq \bigcup_{i=1}^{n} I_i$. Then $I \subseteq I_i$ for some i.

Proof. The proof will be done by contrapositive, so we prove that if $I \nsubseteq I_i$ for each i then $I \nsubseteq \bigcup_{i=1}^{n} I_i$, and we do it by induction, so for $n = 1$, it is trivial case, assume that the statement is true for $n - 1$ prime hyperideals. Now suppose that $I_1, I_2, \cdots, I_n$ are prime hyperideals of a hyperring R such that $I \nsubseteq I_j$ for $j = 1, 2, \cdots, n$. Then for each $i, (1 \leq i \leq n)$ we have $I_i \nsubseteq I_j$, for $j = 1, 2, \cdots, i-1, i+1, \cdots, n$. So by the hypothesis of induction, for each i with $1 \leq i \leq n$ we can find $x_i \in I$ but $x_i \notin \bigcup_{j \neq i} I_j$. Now if $x_i \notin I_i$ for some i, we have done, otherwise we have $x_i \in I_i$ for all i . Then take

$$y = \sum_{i=1}^{n} x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$$

which is an element of I , but not in any I_i. So $y \notin \bigcup_{i=1}^{n} I_i$, which completes the proof.■

Theorem 3.1.5 Let $I_1, I_2, \cdots, I_n$ be hyperideals of a hyperring R, and I be a prime hyperideal such that $\bigcap_{i=1}^{n} I_i \subseteq I$, then $I_i \subseteq I$ for some i, particularly if $I = \bigcap_{i=1}^{n} I_i$ then $I = I_i$ for some i.

Proof. We will prove the first part by contradiction, so assume that the claim is false. This means that for each i there exist $x_i \in I_i$ but $x_i \notin I$. Then

$$x_1 x_2 \cdots x_n \in I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n \subseteq I$$

which is a contradiction as I is a prime hyperideal, so $I_i \subseteq I$ for some i and the first part is true. Now if $= \bigcap_i I_i$, then $I \subseteq I_i$ for each i, and by the first part of this theorem we have $I_i \subseteq I$ for some i. Therefore $I_i = I$ for some i.■

Definition 3.1.3 [7] If I, J are hyperideals in a hyperring R, then their hyperideal quotient which is denoted by $(I : J)$ is the set $\{x \in R : xJ \subseteq I\}$. In particular $(0 : I)$ is called the annihilator of I and is denoted by $Ann(I)$ which is the set of all $x \in R$, such that $xI = 0$. The set of all zero divisors of the hyperring R can be defined as $D = \bigcup_{x \neq 0} Ann(< x >)$.

Theorem 3.1.6 [7] If I, J are hyperideals of a hyperring R, then the hyperideal quotient $(I : J)$ is also a hyperideal of R.

Proof. $0j = 0 \in I$, so that $0 \in (I : J)$ and $(I : J) \neq \emptyset$. Now let $a, b \in (I : J)$. Then $aJ \subseteq I, bJ \subseteq I$. So that $(a - b)J \subseteq aJ - bJ \subseteq I$, therefore $(a - b) \subseteq I$. If $c \in R$, $d \in (I : J)$, then $c(dJ) = cdJ \subseteq I$, and $ca \in (I : J)$. Hence $(I : J)$ is a hyperideal of R■.

Theorem 3.1.7: Let $A, B, C, \{A_i\}$, and $\{B_i\}$ be hyperideals of a hyperring R, then the followings are satisfied.

(i) $A \subseteq (A : B)$,

(ii) $(A : B)B \subseteq A$,

(iii) $((A : B) : C) = (A : BC) = ((A : C) : B)$,

(iv) $(\bigcap_i A_i : B) = \bigcap_i (A_i : B)$,

(v) $(A : \sum_i B_i) = \bigcap_i (A : B_i)$.

Proof.

(i) By the definition of hyperideal quotients for any $x \in A$, we have $xB \subseteq A$, so that $x \in (A : B)$. Therefore $A \subseteq (A : B)$.

(ii) Let $x \in (A : B)B$. Then $x \in \sum_{i=1}^{n} x_i y_i$, such that $x_i \in (A : B), y_i \in B$, and $x_i B \subseteq A$ for $i = 1, 2, \cdots, n$, and $x_i y_i B \subseteq A$. Thus $(\sum_{i=1}^{n} x_i y_i) B \subseteq A$. Therefore $x \in A$, and $(A : B)B \subseteq B$.

(iii) Let $x \in ((A : B) : C)$. Then $xC \subseteq (A : B)$ and $xBC \subseteq A$ so that $x \in (A : BC)$, which means that $((A : B) : C) \subseteq (A : BC)$. To prove the equality, let $x \in (A : BC)$, then $xBC \subseteq A$. Now take $y \in xC$, then $y = xc$ for some $c \in C$, then $yB = xcB \subseteq A$. So that $y \in (A : B)$ which implies that $xC \subseteq (A : B)$, and $x \in ((A : B) : C)$. Hence $((A : B) : C) = (A : BC)$. Since the hyperring is commutative, then we have $((A : B) : C) = (A : BC) = ((A : C) : B)$.

(iv) By the definition of the hyperideal quotient, let $x \in (\bigcap_i A_i : B)$, then $xB \subseteq \bigcap_i A_i$, so that $xB \subseteq A_i$, for all i hence $x \in (A_i : B)$, for all i, which implies that $x \in \bigcap_i (A_i : B)$. Similarly, if $x \in \bigcap_i (A_i : B)$ we have $x \in (\bigcap_i A_i : B)$. Therefore, $(\bigcap_i A_i : B) = \bigcap_i (A_i : B)$.

(v) Let $x \in (A : \sum_i B_i)$, then $x \sum_i B_i \subseteq A$, and since $B_i \subseteq \sum_i B_i$, we have $xB_i \subseteq x \sum_i B_i \subseteq A$. Then $x \in (A : B_i)$ for all i, and $x \in \bigcap_i (A : B_i)$. Hereby $(A : \sum_i B_i) \subseteq \bigcap_i (A : B_i)$. On the other hand, let $x \in \bigcap_i (A : B_i)$, then $x \in (A : B_i)$ for all i, so that $xB_i \subseteq A$ and $x \sum_i B_i \subseteq A$ for all i which implies that $x \in (A : \sum_i B_i)$. ■

3.2 Hyperring Isomorphism Theorems

Theorem 3.2.1 [4] (First Isomorphism Theorem) Let R,S be hyperrings and $f: R \to S$ be a hyperring homomorphism, then $\ker(f)$ is a hyperideal of R and $f(R)$ is a subhyperring of S such that $R/\ker(f)$ is isomorphic to $f(R)$.

Proof. We prove that $\ker(f)$ is a hyperideal of R. Let $x,y \in \ker(f), r \in R$. Then $f(x-y) = f\big(x+(-y)\big) = f(x)+f(-y) = f(x)-f(y) = 0, \quad f(rx) = rf(x) = 0$. So that $\ker(f)$ is a hyperideal of R. Since f is homomorphism, it is easy to show that $f(R)$ is a subhyperring of the hyperring S. Define $g: R/\ker(f) \to f(R)$ by $g(\bar{x}) = f(x)$ where $\bar{x} = x + \ker(f)$. Firstly we have to prove that g is well defined. Suppose that $\bar{x} = \bar{y}$, where $x,y \in R$, then $x \in \bar{y}$. This means that $x \in y+k$ for some $k \in Ker(f)$, then $f(x) \in f(y+k) = f(y)+f(k) = f(y)+0 = f(y)$. Thus $f(x) = f(y)$ and $g(\bar{x}) = g(\bar{y})$, so that g is well defined. Let $x,y \in R$, then

$$
\begin{aligned}
g(\bar{x}+\bar{y}) &= g(\{\bar{z}: z \in x+y\}) \\
&= \{g(\bar{z}): z \in x+y\} \\
&= \{f(z): z \in x+y\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
g(\bar{x})+g(\bar{y}) &= f(x)+f(y) \\
&= f(x+y) \\
&= \{f(z): z \in x+y\}.
\end{aligned}
$$

Therefore, $g(\bar{x}+\bar{y}) = g(\bar{x})+g(\bar{y})$.

For the multiplication we have,

$$g(\bar{x}\cdot\bar{y}) = g(\overline{xy}) = f(xy) = f(x)f(y) = g(\bar{x})\cdot g(\bar{y}).$$

Therefore, g is a homomorphism and $g(\bar{0}) = f(0) = 0$, so by Theorem 3.1.1, g is one-to-one. To prove that g is onto, let $y \in f(R)$, then we have $x \in R$ such that $f(x) = y$, so that $g(\bar{x}) = f(x) = y$ so g is onto. Hence $R/\ker(f)$ is isomorphic to $f(R)$. ■

Corollary 3.2.2 Let R,S be hyperrings, and f be a homomorphism from R onto S. Then $R/\ker(f)$ isomorphic to S.

Theorem 3.2.3 [4] (Second isomorphism Theorem) Let R be a hyperring, and A be a subhyperring of R and B a hyperideal of R, then $A \cap B$ is a hyperideal of A and

$$A/A \cap B \cong A+B/B.$$

Proof. To prove that $A \cap B$ is a hyperideal of A, let $x, y \in A \cap B$. Then $x-y \subseteq A$ because A is a subhyperring. Also $x-y \subseteq B$ since B is a hyperideal, so that $x-y \subseteq A \cap B$. If $r \in A, x \in A \cap B$, then $rx \in A \cap B$. Therefore $A \cap B$ is a hyperideal of A. Now define a function f from A to $(A+B)/B$ by $f(a)=a+B$ for each $a \in A$. Then,

$$\begin{aligned} f(a+b) &= f(\{x : x \in a+b\}) \\ &= \{f(x) : x \in a+b\} \\ &= \{x+B : x \in a+b\} \\ &= a+b+B \\ &= a+B+b+B = f(a)+f(b). \end{aligned}$$

For multiplication we have,

$$f(ab)=ab+B=(a+B)(b+B)=f(a)f(b).$$

For the zero elements, $f(0)=0+B=B$. Therefore f is a homomorphism. To prove that f is onto, let $x+B \in (A+B)/B$. Then $x \in y+B$ for some $y \in A+B$ so that $y \in a+b$ for some $a \in A, b \in B$, then $y \in a+B$, which implies that $y+B=a+B$. Thus $f(a)=a+B=y+B=x+B$. Hence f is onto.

Let $b \in A$, then $b \in \ker(f)$ if and only if $f(b)=b+B=B$, if and only if $b \in B$. Then $b \in \ker(f)$ if and only if $b \in A \cap B$, so that $\ker(f)=A \cap B$. Hence by the first isomorphism theorem and Corollary 3.2.2 we have $A/A \cap B \cong (A+B)/B$. ■

Theorem 3.2.4 [4] (Third Isomorphism Theorem) Let R be a hyperring and I, J be hyperideals of R such that $J \subseteq I$, then I/J is a hyperideal of R/J and

$$R/J \big/ I/J \cong R/I \,.$$

Proof. Since I, J are hyperideals of R, then I, J are nonempty and so $I/J=\{a+J : a \in I\}$ is also nonempty. Let $a, b \in I, r \in R$, then by the definition of hyperaddition and multiplication defined on the quotient hyperring we have

$$(a+J)+(b+J)=a+b+J \text{ and } (r+J)(a+J)=ra+J.$$

Since I is a hyperideal, then $a+b, ra$ are contained in I, so that I/J is a hyperideal of R/J. Now, consider the function $f: R/J \longrightarrow R/I$ defined by $f(r+J)=(r+I)$. We want to prove that f is well defined. Assume that $r_1 + J = r_2 + J$, then $r_1 \in r_2 + J$, and so $r_1 \in r_2 + r_3$, for some $r_3 \in J \subseteq I$. Therefore $r_1 \in r_2 + I$. Hence $r_1 + I = r_2 + I$ and so f is well defined. To show that f is onto, let $r + I \in R/I$, then $f(r + J) = r + I$, thus f is onto. Finally, $r + J \in \ker(f)$ if and only if $r + I = 0 + I$, which means that $r \in I$. Thus $\ker(f) = I/J$. Hence by the first isomorphism theorem we have that ${R/J}\big/{I/J} \cong R/I$. ■

The following theorem is called the fourth isomorphism theorem and sometimes called lattice isomorphism theorem.

Theorem 3.2.5 [12] (Fourth isomorphism theorem) Let R be a hyperring, and J be a hyperideal of R. Then the correspondence $I \longrightarrow I/J$ is an inclusion preserving bijection between the set of hyperideals I of R that contains J and the set of hyperideals $\hat{I}$ of R/J.

Proof. Let $f: R \longrightarrow R/J$ defined by $f(r) = r + J$. We show that f is a homomorphism from R onto R/J. Let $a, b \in R$, then $f(a + b) = a + b + J = a + J + b + J = f(a) + f(b)$, also $f(ab) = ab + J = (a + J)(b + J)$ and $f(1) = 1 + J$. Therefore f is a homomorphism. To show that f is onto, let $r + J \in R/J$, then $f(r) = r + J$, so that f is onto. Now let K be the set of all hyperideals I of R, such that $J \subseteq I$, then we will show that for each hyperideal I in the set K, $f(I)$ is a hyperideal of R/J and this is directly come from the definition of the homomorphism, let $a_1 + J, a_2 + J$ be elements of $f(I)$, then $(a_1 + J) - (a_2 + J) = a_1 - a_2 + J = f(a_1 - a_2)$, where $a_1, a_2 \in I$. So that $(a_1 + J) - (a_2 + J) \subseteq f(I)$. For $r + J \in R/J$, we have $f(ra_1) = ra_1 + J = (r + J)(a_1 + J)$. Then $(r + J)(a_1 + J) \in f(I)$. Therefore, $f(I)$ is a hyperideal of R/J.

Now, we show that for every hyperideal $\hat{I}$ of R/J, there is a unique hyperideal $I \in K$, such that $f(I) = \hat{I}$. Let $\hat{I}$ be a hyperideal of R/J and $I = f^{-1}(\hat{I}\,)$. Firstly, we prove that $I = f^{-1}(\hat{I}\,)$ is a hyperideal of R. Since $\hat{I}$ is hyperideal of R/J, then the zero element J of R/J is an element of $\hat{I}$, such that $f^{-1}(J) = J \subseteq I$. Let $a, b \in I = f^{-1}(\hat{I}\,)$, then $f(a), f(b) \in \hat{I}$, so $f(a) - f(b) = f(a - b) \subseteq \hat{I}$, which implies that $a - b \subseteq f^{-1}(\hat{I}) = I$. If $r \in R$, then $f(r) \in R/J$, so that $f(r)f(a) = f(ra) \in \hat{I}$, therefore $ra \in f^{-1}(\hat{I}) = I$. Hence I is a hyperideal of R and $J \subseteq I$. It is easy to show that the function f is inclusion preserving between the set of hyperideals I of R that contains J and the set of hyperideals $\hat{I}$ of R/J. To complete the proof we have to show that f is one-to-one from the set K to the set of hyperideals in R/J. Assume that $f(I_1) = f(I_2)$ for some $I_1, I_2 \in K$, since f is inclusion preserving, we have $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$, and hence $I_1 = I_2$. ■

3.3 Extension and Contraction

Let $f: A \longrightarrow B$ be a hyperring homomorphism, if I is a hyperideal of A. Then $f(I)$ need not to be a hyperideal of B. For example from classical algebra, let $f: \mathbb{Z} \longrightarrow \mathbb{Q}$ defined by $f(x) = x$. Then $f(\mathbb{Z}) = \mathbb{Z}$, but $\mathbb{Z}$ is not a hyperideal in $\mathbb{Q}$. In this section we study the extension and contraction of the hyperideals and we generate a hyperideal from $f(I)$.

Definition 3.3.1 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism and let I be a hyperideal of the hyperring A, then the extension of $f(I)$ is which denoted by I^e is defined by

$$I^e = < f(I) > = \left\{ b \in B : b \in \sum_i b_i f(x_i) : b_i \in B, x_i \in I \right\}.$$

Definition 3.3.2 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism and J be a hyperideal of B. Then the contraction of J, which is denoted by J^c is defined by

$$J^c = f^{-1}(J) = \{a \in A : f(a) \in J\}.$$

The following theorems are analogues to those in classical algebra that mentioned in [11].

Theorem 3.3.1 Let $f: A \longrightarrow B$ be a hyperring homomorphism and I be a hyperideal of A, J be a hyperideal of B. Then;

1. I^e is a hyperideal of B,
2. J^c is a hyperideal of A, and if J is prime hyperideal of B, then J^c is prime hyperideal of A,
3. $I \subseteq I^{ec}$ and $J^{ce} \subseteq J$,
4. $J^c = J^{cec}$ and $I^e = I^{ece}$,
5. Let C be the set all hyperideals that contracted in A and E be the set of all hyperideals that extended in B. Then $C = \{I : I^{ec} = I\}$, $E = \{J : J^{ce} = J\}$ and $I \longmapsto I^e$ is a bijection map from C onto E , which inverse $J \longmapsto J^c$.

Proof.

1. It is straightforward from the definition of I^e that I^e is a hyperideal of B.
2. Let $a, b \in J^c = f^{-1}(J)$, then $f(a), f(b) \in J$, so that $f(a) - f(b) = f(a - b) \subseteq J$. Therefore $a - b \subseteq f^{-1}(J) = J^c$. If $r \in A$, then $f(r) \in B$, and $f(r)f(a) = f(ra) \in J$, so that $ra \in f^{-1}(J) = J^c$. Hence J^c is a hyperideal of A. Assume that J is a prime hyperideal, let $ab \in J^c = f^{-1}(J)$. Then $f(ab) = f(a)f(b) \in J$ and

since J is prime, then either $f(a) \in J$ or $f(b) \in J$. Therefore either $a \in f^{-1}(J)$ or $b \in f^{-1}(J)$, so that J^c is a prime hyperideal whenever J is prime.

Note: If I is a prime hyperideal, then I^e need not to be prime, example from classical algebra, if $f: \mathbb{Z} \longrightarrow \mathbb{Q}$ and if $I \neq 0$, then $I^e = \mathbb{Q}$, which is not prime.

3. Let $x \in I$, then $f(x) \in I^e$ which means that $x \in f^{-1}(I^e) = I^{ec}$, so that $I \subseteq I^{ec}$. For the second part of (3) let $a \in J^{ce}$, then $a \in \sum_i f(b_i)c_i$, for some $b_i \in J^c$, $c_i \in B$, and since $b_i \in J^c$ then $b_i = f^{-1}(d_i)$ for some $d_i \in J$. So that $a \in \sum_i d_i c_i \subseteq J$. Hence, $J^{ce} \subseteq J$.
4. By (3) we know that $J^{ce} \subseteq J$, then $J^{cec} \subseteq J^c$, again from the first part of (3), since J^c is a hyperideal of A, then $J^c \subseteq J^{cec}$, therefore $J^c = J^{cec}$. For the second part, from (3), $I \subseteq I^{ec}$ so that $I^e \subseteq I^{ece}$. Since I^e is a hyperideal of B then, by the second part of (3) we have $I^e \supseteq I^{ece}$. Therefore $I^{ece} = I^e$.
5. Let I be a hyperideal in set C, then $I = J^c$ for some hyperideal J in B, so that from (4), $J^c = J^{cec}$. Therefore, $I = J^c = J^{ece} = I^{ec}$. Now, let J be a hyperideal in the set E, then $J = I^e$ for some $I \in A$, then by (4) we have $I^{ece} = I^e$. Therefore, $J = I^e = I^{ece} = J^{ce}$. Now we show the map $I \longmapsto I^e$ is one-to-one and to do this, let $I_1^e = I_2^e$, but $I_1 = J_1^c$, $I_2 = J_2^c$ for some $J_1, J_2 \in E$. So that $I_1^e = J_1^{ce} = I_2^e = J_2^{ce}$ which implies that $J_1^{cec} = J_2^{cec}$, and by (4) we have that $J_1^c = J_2^c$, hence $I_1 = I_2$ and the map is one-to-one. To show that the map $I \longmapsto I^e$ is onto, let J be a hyperideal in E, then $J^c \in C$ and $J^c \longmapsto J^{ce} = J$. Therefore, the map is onto. So for a hyperideal I in C, we have $I \longrightarrow I^e$ is bijection and by the definition C and E, $I^e = J$ for some $J \in E$, and $J \longmapsto J^c = I^{ec} = I$. Then $J \longmapsto J^c$ is the inverse of $I \longmapsto I^e$.■

Theorem 3.3.2 Let $f: A \longrightarrow B$ be a hyperring homomorphism, and let I_1, I_2 be hyperideals of A and J_1, J_2 be hyperideals of B. Then

1. $(I_1 + I_2)^e = I_1^e + I_2^e$.
2. $(J_1 + J_2)^c \supseteq J_1^c + J_2^c$.
3. $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$.
4. $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$.
5. $(I_1 I_2)^e = I_1^e I_2^e$.
6. $(J_1 J_2)^c \supseteq J_1^c J_2^c$.
7. $(I_1 : I_2)^e \subseteq (I_1^e : I_2^e)$.

8. $(J_1{:}J_2)^c \subseteq (J_1^c{:}J_2^c)$.

Proof.

1. $(I_1+I_2)^e = \{x \in B : x \in \sum_i f(x_i)b_i\,,\ x_i \in I_1+I_2\,, b_i \in B\}$
$= \{x \in B : x \in \sum_i f(a_{1i}+a_{2i})b_i\,, a_{1i} \in I_1, a_{2i} \in I_2\,, x_i \in a_{1i}+a_{2i}\}$
$= \{x \in B\,:\, x \in \sum_i f(a_{1i})b_i + \sum_i f(a_{2i})b_i\}$
$= \{a+b\,:\, a \in \sum_i f(a_{1i})b_i\,,\ b \in \sum_i f(b_{2i})b_{i\,,}\ x \in a+b\}$
$= \{a\,:\, a \in \sum_i f(a_{1i})b_{i\,,}a_{1i} \in I_1\} + \{b : b \in \sum_i f(a_{2i})b_{i\,,}a_{2i} \in I_2\}$
$= I_1^e + I_2^e$.
2. Let $y \in J_1^c + J_2^2$. Then $y \in a+b$, for some $a \in J_1^c, b \in J_2^c$. Thus, $f(a) \in J_1$, $f(b) \in J_2$. So that $f(a+b) = f(a)+f(b) \in J_1+J_2$. Therefore, $y \in a+b \subseteq (J_1+J_2)^c$.
3. Let $y \in (I_1 \cap I_2)^e$, then $y \in \{\sum_i f(b_i)c_i, b_i \in I_1 \cap I_2\,, c_i \in B\}$, therefore $y \in I_1^e \cap I_2^e$.
4. $y \in (J_1 \cap J_2)^c \Leftrightarrow f(y) \in J_1 \cap J_2 \Leftrightarrow y \in J_1^c$ and $y \in J_2^c \Leftrightarrow y \in J_1^c \cap J_2^c$.
5. Since $1 \in B$, then $BB = B$ and since f is homomorphism and the multiplication of hyperideals is commutative, then $(I_1I_2)^e = Bf(I_1I_2) = BBf(I_1)f(I_2) = Bf(I_1)Bf(I_2) = I_1^eI_2^e$.
6. Let $y \in J_1^cJ_2^c$. Then $y = ab$, where $a \in J_1^c, b \in J_2^c$, then $f(a) \in J_1, f(b) \in J_2$. Therefore, $f(y) = f(ab) = f(a)f(b) \in J_1J_2$. Hence $y \in f^{-1}(J_1J_2) = (J_1J_2)^c$.
7. By Theorem 3.1.7(ii) we have, $(I_1{:}I_2)I_2 \subseteq I_1$, so that by (5) $(I_1{:}I_2)^eI_2^e = ((I_1{:}I_2)I_1)^e \subseteq I_1^e$. Now if $x \in (I_1:I_2)^e$, then $xI_2^e \subseteq I_1^e$, so that $x \in (I_1^e : I_2^e)$ which means that $(I_1:I_2)^e \subseteq (I_1^e : I_2^e)$.
8. By Theorem 3.1.7(ii) we have, $(J_1{:}J_2)J_2 \subseteq J_1$ and by (6) we have $(J_1{:}J_2)^cJ_2^c \subseteq ((J_1{:}J_2)J_2)^c \subseteq J_1^c$. If $x \in (J_1:J_2)^c$. Then $xJ_2^c \subseteq J_1^c$, so that $x \in (J_1^c : J_2^c)$ which means that $(J_1 : J_2)^c \subseteq (J_1^c{:}J_2^c)$. ■

Theorem 3.3.3 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism from a hyperring A onto a hyperring B. Let I be a prime hyperideal of A such that $\ker(f) \subseteq I$. Then $f(I)$ is a prime hyperideal in B.

Proof. First of all, we show that $f(I)$ is a hyperideal of B. Let $y_1, y_2 \in f(I)$, $s \in B$ and since f is onto, then there exists $x_1, x_2 \in I, r \in A$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and $f(r) = s$. Since I is a hyperideal of I, then $x_1 - x_2 \subseteq I, rx_1 \in I$, and $f(x_1 - x_2) \subseteq f(I)$,

$f(rx_1) \in f(I)$. So that $y_1 - y_2 = f(x_1) - f(x_2) = f(x_1 - x_2) \subseteq f(I)$ and $sy_1 = f(r)f(x_1) = f(rx_1) \in f(I)$. So that $f(I)$ is a hyperideal of B. To prove that $f(I)$ is a prime hyperideal, let $b_1, b_2 \in B$ such that $b_1 b_2 \in f(I)$, then there exists $a_1, a_2 \in A, c \in I$ where $f(a_1) = b_1$, $f(a_2) = b_2$ and $f(a_1 a_2) = f(a_1)f(a_2) = b_1 b_2 = f(c)$. Therefore, $0 \in f(a_1 a_2) - f(c) = f(a_1 a_2 - c) = \{f(z): z \in a_1 a_2 - c\}$, then for some $z \in a_1 a_2 - c$, $f(z) = 0$, which means that $z \in \ker(f)$. So that $a_1 a_2 \in z + c \subseteq \ker(f) + I \subseteq I$ and since I is a prime hyperideal, then $a_1 \in I$ or $a_2 \in I$, therefore, $b_1 = f(a_1) \in f(I)$ or $b_2 = f(a_2) \in I$. Hence $f(I)$ is a prime hyperideal of B. ■

Theorem 3.3.4: Let $f: A \longrightarrow B$ be a hyperring homomorphism from a hyperring A to a hyperring B. If J is a prime hyperideal of B, then $f^{-1}(J)$ is a prime hyperideal of A.

Proof. The result follows directly from Theorem 3.3.1■

Theorem 3.3.5 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism from a hyperring Aonto a hyperring B. Let I be a prime hyperideal of A such that, $\ker(f) \subseteq I$, then I^e is a prime hyperideal of B.

Proof. In Theorem 3.3.1, we show that I^e is a hyperideal of B, to show that I^e is prime hyperideal, let $ab \in I^e$ for some $a, b \in B$, since f is onto, then there exist $x, y \in A$ such that $f(x) = a, f(y) = b$. So that, $ab = f(x)f(y) = f(xy) \in \sum_i s_i f(c_i)$, $s_i \in B, c_i \in I$. Since f is onto, then $s_i = f(r_i)$, for some $r_i \in A$. Therefore,

$$ab = f(xy) = \sum_i f(r_i)f(x_i) = \sum_i f(r_i x_i) = f\left(\sum_i r_i x_i\right) \subseteq f\left(\sum_i r_i x_i\right) + 0.$$

So that,

$$0 \in f\left(\sum_i r_i x_i\right) - f(xy) = f\left(\left(\sum_i r_i x_i\right) - xy\right).$$

Then there exists $t \in (\sum_i r_i x_i) - xy$ such that $f(t) = 0$, which means that $t \in \ker(f)$. Therefore $xy \in \sum_i r_i x_i + t \subseteq I + \ker(f) \subseteq I$. Since I is prime hyperideal, then $x \in I$, or $y \in I$. So that $a = f(x) \in f(I) \subseteq I^e$, or $b = f(y) \in f(I) \subseteq I^e$. Hence, I^e is a prime hyperideal of B.■

Theorem 3.3.6 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism from a hyperring A onto a hyperring B, then there is one-to-one correspondence between the prime hyperideal in B and the prime hyperideal in A that contains $\ker(f)$.

Proof. Let H be the set of all prime hyperideals in a hyperring A that contains $\ker(f)$, let K be the set of all prime hyperideals in B. Let the function, $\psi: H \longrightarrow K$ be defined such that $I \longrightarrow I^e$, which has an inverse $J \longrightarrow J^c$. We show that ψ is one-to-one and onto with respect to elements of H and K. So it is enough to show that $I^{ec} = I$. Let $x \in I$, then $f(x) \in f(I) \subseteq I^e$, so that $x \in f^{-1}(I^e) = I^{ec}$. Hence $I \subseteq I^{ec}$, to prove the converse, let $x \in I^{ec}$, then $f(x) \in I^e$, so that $f(x) \in \sum_i c_i f(x_i)$, $c_i \in B$, $x_i \in I$. Since f is onto then there exists $r_i \in A$ such that $c_i = f(r_i)$. Therefore

$$f(x) \in \sum_i f(r_i) f(x_i) = \sum_i f(r_i x_i) + 0.$$

So that

$$0 \in \sum_i f(r_i x_i) - f(x) = f\left(\left(\sum_i r_i x_i\right) - x\right).$$

Then there exists $c \in (\sum_i r_i x_i) - x$ such that $f(c) = 0$, then $c \in \ker(f)$. So that $x \in (\sum_i r_i x_i) + c \subseteq I + \ker(f) \subseteq I$, $I^{ec} \subseteq I$. Hence $I = I^{ec}$ and so ψ is one-to-one and onto.■

CHAPTER 4

On Primary Hyperideals

In this chapter, we discuss more theorems on the primary hyperideals. Recall that the definition of the primary hyperideal, we say that a hyperideal Q of a commutative hyperring R is a primary hyperideal if $Q \neq R$ and for any $ab \in Q$, either $a \in Q$ or $b^n \in Q$, for some $n \in \mathbb{N}$.

4.1 Primary Decomposition

Definition 4.1.1 [1] An element x in a hyperring R which has the property that $x^n = 0$, for some integer $n > 0$ is called nilpotent. The set of all nilpotent elements in R is called nilradical of R and denoted by $nil(R)$.

We need to prove the following proposition before Theorem 4.1.1.

Proposition 4.1.1 Let R be a hyperring and $x, y \in R$. Then $(x+y)^n \subseteq \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$.

Proof. We will prove this proposition by induction and by using the rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

For $n = 1$ it is clear that $(x+y)^1 = x + y$, assume that it is true for $n-1$, which means that

$$(x+y)^{n-1} \subseteq \sum_{r=0}^{n-1}\binom{n-1}{r}x^{n-1-r}y^r.$$

Now consider;

$$(x+y)^n = (x+y)(x+y)^{n-1} \subseteq (x+y)\sum_{r=0}^{n-1}\binom{n-1}{r}x^{n-1-r}y^r$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} x^{n-r} y^r + \sum_{r=0}^{n-1} \binom{n-1}{r} x^{n-1-r} y^{r+1}$$

$$= \sum_{r=0}^{n-1} \binom{n-1}{r} x^{n-r} y^r + \sum_{r=0}^{n-1} \binom{n-1}{r+1-1} x^{n-(r+1)} y^{r+1}$$

$$= \sum_{r=0}^{n} \binom{n-1}{r} x^{n-r} y^r - \binom{n-1}{n} y^n + \sum_{r=0}^{n} \binom{n-1}{r-1} x^{n-r} y^r - \binom{n-1}{-1} x^n$$

$$= \sum_{r=0}^{n} \left[\binom{n-1}{r}\binom{n-1}{r-1}\right] x^{n-r} y^r = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r . \blacksquare$$

Theorem 4.1.1 [11] If N is the nilradical of a hyperring R, then N is a hyperideal of R and R/N has no nonzero nilpotent.

Proof. Let $x, y \in N, a \in R$. Then $x^n = y^m = 0$, for some $m, n \in \mathbb{N}$. We have $(ax)^n = a^n x^n = 0$ and $(-y)^m = (-1)^m y^m = 0$, so that $ax, -y \in N$. Also,

$$(x - y)^{m+n-1} \subseteq \sum_{i=0}^{m+n-1} \binom{m+n-1}{i} x^i (-y)^{m+n-1-i}.$$

Where $\binom{m+n-1}{i}$ is the binomial coefficient. If $i \geq n$, then $x^i = 0$, and if $i < n$, then $m + n - 1 - i = m + (n - 1 - i) \geq m$, so that $(-y)^{m+n-1-i} = 0$, therefore $x + y \in N$. Hence N is a hyperideal of R. Let $x + N \in R/N$ be a nilpotent element, then there exists $n \in \mathbb{N}$, such that $(x + N)^n = x^n + N = N$, so that $x^n \in N$, this means that $(x^n)^m = x^{nm} = 0$, then $x \in N$ and so $x + N = N$. Hence every nonzero element of R/N is not nilpotent.■

Theorem 4.1.2 [11] Q is a primary hyperideal of a hyperring R if and only if R/Q is not trivial and every zero divisor in R/Q is nilpotent.

Proof. Assume that Q is a primary hyperideal of a hyperring R. Then certainly R/Q is not trivial. Let $x + Q \in R/Q$ be a zero divisor, then there exists $y \notin Q$ such that $(x + Q)(y + Q) = xy + Q = Q$. So that $xy \in Q$, but $y \notin Q$, then $x^n \in Q$ for some $n \in \mathbb{N}$. So $(x + Q)^n = x^n + Q = Q$, that is $x + Q$ is nilpotent. Conversely, suppose that R/Q is not trivial and every zero divisor in R/Q is nilpotent. Certainly $Q \neq R$, let $x, y \in R$ such that $xy \in Q$, then either $x \in Q$ or $x \notin Q$. If $x \in Q$, then we have done. So if $x \notin Q$, we have $(x + Q)(y + Q) = xy + Q = Q$. So that $y + Q$ is a zero divisor and by assumption $(y^n + Q) = (y + Q)^n = Q$ for some $n \in \mathbb{N}$. Then $y^n \in Q$. Hence Q is primary.■

Proposition 4.1.2 [11] Let I be the nilradical of a hyperring R, then I is the intersection of all prime hyperideals of R.

Proof. Let P be a prime hyperideal of R, x be a nilpotent element of R, then $x^m = 0 \in P$ for some integer m. Let $n \geq 1$ be the smallest integer such that $x^n \in P$, then $x\, x^{n-1} \in P$, but P is a prime hyperideal, this means that $x \in P$ or $x^{n-1} \in P$ and this can be done only if $n = 1$, therefore $x \in P$ and so

$$x \in \bigcap_{P\ prime} P \Longrightarrow I \subseteq \bigcap_{P\ prime} P$$

To prove the converse, we have to show that $\bigcap_{P\ prime} P \subseteq I$. We show that if $x \notin I$ then $x \notin P$ for some prime hyperideal P. Let $x \notin I$, then $x^n \neq 0$ for all $n \in \mathbb{N}$. Let S be the set of all hyperideals J_α such that $a^n \notin J_\alpha$ for all $n \in \mathbb{N}$. If we order S by inclusion, we have $J_1 \subseteq J_2 \subseteq \cdots J_m$ is a chain, then $J = \bigcup J_i$ is an upper bound hyperideal belong to S. By Zorn's lemma, S has a maximal element, say M. We show that M is a prime hyperideal. By contrary assume that $ab \in M$ but $a \notin M$ and $b \notin M$. Then the hyperideals $M + \langle a \rangle$ and $M + \langle b \rangle$ are not in S, then $x^r \in M + \langle a \rangle$ and $x^t \in M + \langle b \rangle$ for some $r, t \in \mathbb{N}$, then $x^{r+t} \in M + \langle ab \rangle = M$. Hence M is not in S which a contradiction. Thus either a or $b \in M$ and so M is prime hyperideal such that $x \notin M$. So

$$x \notin \bigcap_{P\ prime} P\,.$$ ■

Theorem 4.1.3 [7] Let $f: A \longrightarrow B$ be a hyperring homomorphism from a hyperring A to a hyperring B. Let J be a primary hyperideal in B, then J^c is a primary hyperideal in A.

Proof. Let $x, y \in A$, such that $xy \in J^c$. Then $f(xy) = f(x)f(y) \in J$. So that $f(x) \in J$ or $[f(y)]^n = f(y^n) \in J$. Therefore, $x \in f^{-1}(J) = J^c$ or $y^n \in f^{-1}(J) = J^c$. Hence J^c is a primary hyperideal of A.■

Theorem 4.1.4 [11] Let Q be a primary hyperideal in the hyperring R, then $\rho(Q)$ which is the radical of a hyperideal Q, is the smallest prime hyperideal that contains Q.

Proof : Note that $\rho(Q) = \{x : x^n \in Q$, for some $n \in \mathbb{N}\}$, by Theorem 3.1.2, $\rho(Q)$ is a hyperideal of R and $Q \subseteq \rho(Q)$. Let $ab \in \rho(Q)$, then $(ab)^m \in Q$ for some $m \in \mathbb{N}$. Then $a^m b^m \in Q$ which is a primary. So that either $a^m \in Q$ or $b^{mn} \in Q$. Therefore either $a \in$

$\rho(Q)$ or $b \in \rho(Q)$. Hence $\rho(Q)$ is a prime hyperideal. To show that $\rho(Q)$ is the smallest prime hyperideal that contains Q. Let Q_1 be another prime hyperideal of R, such that $Q \subseteq Q_1$, $x \in \rho(Q)$, then $x^n \in Q$ for some $n \in \mathbb{N}$. Then $x^n \in Q_1$ and since Q_1 is prime, then $x \in Q_1$. Hence $\rho(Q) \subseteq Q_1$. ■

Definition 4.1.2 [11] If Q is a primary hyperideal in a hyperring R where $P = \rho(Q)$ is the smallest prime hyperideal that contains Q, then we say that Q is a P-primary hyperideal.

In the next example we show that the primary hyperideals need not to be powers of prime hyperideals.

Example 4.1.1 Let $F[x,y]$ be a field of polynomials of two variables over the set of real numbers and $G = \{1,-1\}$ be a subset of $F[x,y]$, since $(G, \cdot)$ is a group, then by Krasner's construction in Theorem 1.3.3 $R = F[x,y]/G$ is a hyperring and $J = <x,y^2>/G = \{fG | f \in <x,y^2>\}$ is a hyperideal of R, so $J = R(xG) + R(y^2G)$. Define $\phi: R \longrightarrow F[y]G/<y^2>G$ by $p(x,y)G \mapsto p(0,y)G + <y^2>G$, so that it is easy to prove that ϕ is an onto hyperring homomorphism and

$$\ker(\phi) = \{p(x,y)G \in R | p(0,y)G \in <y^2>G\}.$$

Certainly, $xG, y^2G \in \ker(\phi)$, so that $J \subseteq \ker(\phi)$. Now if $p(x,y)G \in \ker(\phi)$, then $p(x,y)G = p_1(y)G + xp_2(x,y)G$ for some $p_1(y)G \in F[y]G$ and $p_2(x,y)G \in R$. So that $p_1(y)G = p(0,y)G \in <y^2>G$. Thus $p(x,y)G \in <x,y^2>G = J$ and hence $\ker(\phi) = J$, then by the first isomorphism theorem $R/J \cong F[y]G/<y^2>G$, and $<y^2>G = y^2F[y]G = (yF[y])^2G$ is a primary hyperideal of $F[y]G$. So all zero divisors of $F[y]G/<y^2>G$ and hence all zero divisors of R/J are nilpotent. Thus J is a primary hyperideal of R. Let $\rho(J) = <x,y>G = R(xG) + R(yG)$ and so

$$[\rho(J)]^2 = [R(xG)]^2 + [R(yG)]^2 + [R(xG)][R(yG)]$$

$$= R(xG)^2 + R(yG)^2 + R(xyG).$$

So $[\rho(J)]^2 \subseteq J \subseteq \rho(J)$. Thus J is not a power of its radical. Now we want to prove that J is not a power of any prime hyperideal of R. Now, assume that $J = P^n$ for some prime hyperideal P and $n \geq 1$. Then $[\rho(J)]^2 \subset J \subseteq P$ and $P^n = J \subset \rho(J)$. Now we check that $P = \rho(J)$. Let $\alpha \in P$, then $\alpha^n \in P^n \subseteq \rho(J)$, so that $\alpha \in \rho(\rho(J)) = \rho(J)$. If $\beta \in \rho(J)$, then $\beta^2 \in [\rho(J)]^2 \subseteq P$, so $\beta \in P$ since P is prime. Hence $P = \rho(J)$. So that $P^2 \subset P^n \subset P$, which is impossible. Thus J is not a power of a prime hyperideal.

Theorem 4.1.5 [9] Let I be a hyperideal of a hyperring R. Then the radical hyperideal $\rho(I)$ is the intersection of all prime hyperideals that contains I.

Proof. Let I be a hyperideal of a hyperring R and let $N_{R/I}$ be the nilradical of the hyperring R/I. Then by Theorem 4.1.2, $N_{R/I}$ is the intersection of all prime hyperideals of R/I. Recall that the natural homomorphism map $f: R \longrightarrow R/I$ defined by $f(x) = x + I$ in Theorem 3.2.4. We showed that $f^{-1}(P/I)$ is a hyperideal of R for any hyperideal $f(P) = P/I$. Now we show that if P/I is a prime hyperideal, then P is a prime hyperideal of R. Assume that P/I is a prime hyperideal, let $xy \in P$, then $f(xy) = xy + I = (x + I)(y + I) \in P/I$, then either $x + I \in P/I$ or $y + I \in P/I$, then either $x \in P$ or $y \in P$, so that P is a prime hyperideal of R whenever P/I is a prime hyperideal of R/I. Therefore $N_{R/I}$ is the intersection of all prime hyperideals P/I in R/I. Also $x + I \in N_{R/I}$ if and only if $(x + I)^n = x^n + I = I$ so that $x^n \in I$ and $x \in \rho(I)$. Therefore $N_{R/I} = r(I)/I$. Hence $\rho(I)$ is the intersection of all prime hyperideals that contains I.■

Theorem 4.1.6 [11] The set of all zero divisor of a hyperring R is equal to its own radical, that is $D = \rho(D) = \bigcup_{x \neq 0} \rho(Ann(< x >))$ where $D = \bigcup_{x \neq 0} Ann(< x >)$ is the set of all zero divisor of R.

Proof. Clearly, $D \subseteq \rho(D)$. Suppose that $a \in \rho(D)$, so that $a^n \in D$ for some $n \in \mathbb{N}$. Then $xa^n = 0$, for some $x \in R, x \neq 0$. If $n = 1$, then $a \in D$, else if $x > 1$, then $(xa^{n-1})a = 0$. So by induction, either $xa^{n-1} = 0$ when $a \in D$ or $xa^{n-1} = 0$ when $a \in D$. Thus $D = \rho(D) = \rho(\bigcup_{x \neq 0} Ann < x >) = \bigcup_{x \neq 0} \rho(Ann < x >)$.■

Definition 4.1.3 [10] The Jacobson radical of a hyperring R is the intersection of all maximal hyperideals of R.

Note: In Corollary 2.1.1 we concluded that all maximal hyperideals are prime hyperideals, so that immediately we get nilradical is a subset of Jacobson radical.

Theorem 4.1.7 [10] Let J be the Jacobson radical of a hyperring R. Then $x \in J$ if and only if any element $c \in 1 - xy$, c is a unit for all $y \in R$.

Proof. Let $x \in J$, assume that $c \in 1 - xy$, but c is not unit. Then by Zorn's lemma $< c > \subseteq M$ for some maximal hyperideal M of R. So we have $c \in M$, $x \in J \subseteq M$ and $1 \in c + xy \subseteq M$. Therefore $M = R$, which is a contradiction because M is a maximal hyperideal of R. Hence x is a unit. Conversely, assume that every element in the set $1 - xy$ is unit, and let $x \notin J$. Then $x \notin M$ for some maximal hyperideal M in R because J is the intersection of maximal hyperideals. So that by maximality of M, we have $R =< M \cup \{x\} > = \{m + xy: m \in M , y \in R\}$. Then there exist $m \in M , y \in R$ such that $1 = m + xy$, then $m \in 1 - xy$. If every element in $1 - xy$ is a unit, then m is a unit, so that $M = R$, which is a contradiction. Hence $x \in M$, for all maximal hyperideal M so that $x \in J$. ■

Theorem 4.1.8 [11] Let Q be a hyperideal of a hyperring R such that $M = \rho(Q)$ is a maximal hyperideal of R, then Q is a primary hyperideal. Particularly, all powers of a maximal hyperideal M are M-primary hyperideal.

Proof. Assume that $\rho(Q) = M$. Then by the proof Theorem 4.1.6, M/Q is the nilradical of R/Q. So that by Theorem 4.1.2 M/Q is the intersection of all prime hyperideals of the hyperring R/Q. Since M is maximal in R, then M/Q is maximal in R/Q, so by the maximality of M/Q we conclude that M/Q is the only prime hyperideal in R/Q. Therefore, every element in R/Q is either a unit or nilpotent, but the zero divisors are not unit. Then all zero divisors of R/Q must be in $M/Q$and so every zero divisor in R/Q is nilpotent. Hence by Theorem 4.1.2, Q is a primary hyperideal. Now if M is any maximal hyperideal of R, then we have $\rho(M^n) = M$ for all $n \in \mathbb{N}$. So by what we just proved M^n is primary and so M-primary. ■

Theorem 4.1.9:[11] Let P be a prime hyperideal of a hyperring R, and let $Q_1, \cdots, Q_n$ be P-primary hyperideals then $Q = \bigcap_{i=1}^{n} Q_i$ is also P-primary.

Proof. By Theorem 3.1.3, we have $\rho(Q) = \bigcap_{i=1}^{n} \rho(Q_i) = \bigcap_{i=1}^{n} P = P$. So it enough to show that Q is P-primary. Let $x, y \in R$ and $xy \in Q$. Then $xy \in Q_i$ for all i. If $x \in Q$, then we have done, if $x \notin Q$, then $x \notin Q$, we have $x \notin Q_j$ for some j. So that $y^m \in Q_j$ for some $m \in \mathbb{N}$, and since Q_j is primary then $y \in \rho(Q_j) = P$. Therefore $y^r \in Q$, for some $r \in \mathbb{N}$. Hence Q is P-primary. ■

Theorem 4.1.10 [11] Let P be a prime hyperideal of a hyperring R, and let Q be P-primary and $x \in R$. Then

(i) $x \in Q \Longrightarrow (Q: x) = R$.

(ii) $x \notin Q \Longrightarrow (Q: x)$ is P-primary, this means that $r(Q: x) = P$.

(iii) $x \notin P \Longrightarrow (Q: x) = Q$.

Proof.

(i) If $x \in Q$, then for all $y \in R$, we have $xy \in Q$. Therefore $(Q: x) = R$.

(ii) Suppose that $x \notin Q$. If $y \in (Q: x)$, then $xy \in Q$, so that $y^r \in Q$ for some $r \in \mathbb{N}$. Therefore $y \in \rho(Q) = P$. Thus $Q \subseteq (Q: x) \subseteq P$, which implies that $P = \rho(Q) \subseteq \rho(Q: x) \subseteq \rho(P) = P$. Hence $\rho(Q: x) = P$. Now we will show that $(Q: x)$ is primary. Surely, $1 \notin (Q: x)$ so that $(Q: x) \neq R$. Let $a, b \in R$ such that $ab \in (Q: x)$, then we want to show $a \in (Q: x)$ or $b^r \in (Q: x)$ for some $r \in \mathbb{N}$. Suppose that $b^r \notin (Q: x)$, for all $r \in \mathbb{N}$, then $b \notin \rho(Q: x) = P$, but $axb \in Q$, so that $ax \in Q$ or $b^s \in Q$, for some $s \in \mathbb{N}$. If $b^s \in Q$, then $b \in \rho(Q) = P$ which contradicts the assumption that $b \notin P$. Hence $ax \in Q$, and so $a \in (Q: x)$. Therefore $(Q: x)$ is primary.

(iii) Suppose that $x \notin P$, we know that $Q \subseteq (Q: x)$. To prove the equality, let $y \notin Q$. If $xy \in Q$, then $x^r \in Q$, for some $r \in \mathbb{N}$ and this means that $r \in \rho(Q) = P$, which contradicts the assumption $x \notin P$. Therefor $xy \notin Q$ and we conclude that $y \notin (Q: x)$. Then $(Q: x) \subseteq Q$ and so the equality holds.■

Definition 4.1.4 [14] A primary decomposition of a hyperideal I in a hyperring R is an expression of I as a finite intersection of primary hyperideals. That is

$$I = \bigcap_{i=1}^{n} Q_i \tag{4.1}$$

where $\{Q_i\}$ are primary hyperideals of R.

Note: In general the decomposition (4.1) may not exist.

We call the decomposition (4.1) is minimal if

(i) $\rho(Q_1), \rho(Q_2), \cdots, \rho(Q_n)$ are distinct

(ii) $\bigcap_{j \neq i} Q_j \nsubseteq Q_i$, (for all $i = 1, 2, \cdots, n$)

Claim. Any primary decomposition of a hyperideal I can be replaced by minimal decomposition.

Proof. Consider the primary decomposition $I = \bigcap_{i=1}^{n} Q_i$. If $\rho(Q_{j1}) = \cdots = \rho(Q_{jk}) = P$, $k < n$, then by Theorem 4.1.10, $Q = \bigcap_{i=1}^{k} Q_{ji}$, is also P-primary. So we can replace $Q_{j1}, \cdots, Q_{jk}$ by Q in the decomposition, and continue this process until Condition (i) holds for the minimality. If the Condition (ii) is violated, so we can omit hyperideals until Condition (ii) is satisfied.

We shall call a hyperideal Q to be decomposable, if it has a primary decomposition.

Theorem 4.1.11 [14] (The First Uniqueness Theorem) Let J be a decomposable hyperideal of the hyperring R and let $J = \bigcap_{i=1}^{n} Q_i$, be a minimal primary decomposition. Put $P_i = \rho(Q_i)$ for $i = 1, \cdots, n$. Then the set $\{P_1, \cdots, P_n\}$ contains prime hyperideals which occur in the set of hyperideals $\{\rho(J: x), x \in R\}$, so that the set $\{P_1, \cdots, P_n\}$ is independent of the particular decomposition of J.

Proof. Let $x \in R$, then by Theorem 3.1.3 and Theorem 4.1.11(ii) we have

$$\rho(J: x) = \rho\left(\bigcap_{i=1}^{n} Q_i: x\right) = \rho\left(\bigcap_{i=1}^{n}(Q_i: x)\right) = \bigcap_{i=1}^{n} \rho(Q_i: x) = \bigcap_{i, x \notin Q_i} P_i.$$

Now, if $\rho(J: x)$ is prime then $\rho(J: x) \in \{P_i | \; x \notin Q_i\}$ and this proves that $\rho(J: x) = P_i$ for some $P_i \in \{P_1, \cdots, P_n\}$. Conversely, since the primary decomposition is minimal, then for each $i \in \{1, \cdots, n\}$ there exists $x_i \in \bigcap_{j \neq i} Q_j \setminus Q_i$. So if $y \in (Q_i: x_i)$ we have $yx_i \in Q_i$. Thus $yx_i \in Q_i \bigcap_{i \neq j} Q_j = J$, so $y \in (J: x_i)$ and $(Q_i: x_i) \subseteq (J: x_i)$. And since $J \subseteq Q_i$ then $(J: x_i) \subseteq (Q_i: x_i)$. Therefore, $(J: x_i) = (Q_i: x_i)$, and $\rho(J: x_i) = \rho(Q_i: x_i) = P_i$. ■

Remark: The prime hyperideals P_i in the previous theorem are said to be belong to the hyperideal J or associated to the hyperideal J. The minimal elements of the set $\{P_i\}_{i=1}^{n}$ are called minimal or isolated prime hyperideals belonging to J and the other are called embedded prime hyperideals in J.

Example 4.1.2 Let $K[x, y]$ be a ring of polynomials of two variables over the field of real numbers, and let $G = \{1, -1\}$ be a subset of $K[x, y]$. Then G is a group and from Krasner's construction in Theorem 1.3.3, $R = K[x, y]/G$ is a hyperring, and

$$J = < x^2, xy >/G = \{fG | f \in < x^2, xy >\}$$

is a hyperideal of R. Then, observe that

$$J = < x >/G \cap < x^2, y >/G \tag{4.2}$$

and

$$J = < x >/G \cap < x, y >^2/G \tag{4.3}$$

Certainly, $< x >/G$ is primary, in fact it is prime since $< x >/G$ is prime in R, also $< x, y >/G$ is maximal in R, so that by Theorem 4.1.9 $< x, y >^2/G$ is primary and by Example 4.1.1 we have that $< x^2, y >/G$ is primary. Observe that, $\rho\,(< x >/G) =$

$< x >/G$ and $\rho(< x, y >^2/G) = \rho(< x^2, y >/G) = < x, y >/G$. Thus (4.2) and (4.3) are two different minimal primary decompositions of J.

Theorem 4.1.12 [11] Let J be a decomposable hyperideal of a hyperring R, and P be a prime hyperideal of R containing J. Then P contains an isolated prime hyperideal belonging to J.

Proof. Let $J = \bigcap_i Q_i$ be a minimal primary decomposition, and let $P_i = r(Q_i)$, for all i. Then $P = \rho(P) \supseteq \rho(J) = \bigcap_i \rho(Q_i) = \bigcap_i P_i$. Therefore $P \supseteq P_i$ for some i. And so P contains an isolated prime hyperideal belongs to J.■

Theorem 4.1.13 [14] Let J be a decomposable hyperideal of a hyperring R, $J = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition, and $\rho(Q_i) = P_i$. Then

$$\bigcup_{i=1}^{n} P_i = \{x \in R: (J: x) \neq J\}.$$

Particularly, if the zero hyperideal is decomposable, then the set D of the zero divisor of R is the union of prime hyperideals that belongs to 0.

Proof. If J is decomposable in R, then 0 is decomposable in R/J. Let $0 = \bigcap_i \widehat{Q_\iota}$, where $\widehat{Q_\iota} = f(Q_i)$ in R/J according to the natural homomorphism, and $\widehat{Q_\iota}$ is primary. So that it is enough to prove the last statement of the theorem. By Definition 3.1.3 we have that $D = \bigcup_{x \neq 0} \rho(0: x)$ and from the proof of Theorem 4.1.12 we have $\rho(0: x) = \bigcap_{x \notin Q_j} P_j \subseteq P_j$, for some P_j. Hence $D \subseteq \bigcup_{i=1}^{n} P_i$. But from Theorem 4.1.12, each P_i is of the form $\rho(0: x)$ for some $x \in R$. Hence $\bigcup_i P_i \subseteq D$. ■

4.2 δ-Primary Hyperideal of Commutative Semihyperring

Definition 4.2.1 [16] A semihyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1) $(R, +)$ is commutative hypermonoid, i.e.
   - (i) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$,
   - (ii) There exists $0 \in R$, such that $x + 0 = 0 + x = x$ for all $x \in R$,
   - (iii) $x + y = y + x$ for all $x, y \in R$.
2) $(R, \cdot)$ is a semigroup, i.e.
   - (i) $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.
   - (ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in R$.
3) The multiplication is distributive to the hyperaddition, so that

$$x \cdot (y+z) = x \cdot y + x \cdot z\,,\ \ (x+y) \cdot z = x \cdot y + x \cdot z \ \text{ for all } \ x, y, z \ \in R.$$

Definition 4.2.2 [16] A semihyperring $(R, +, \ \cdot)$ is called:

(i) Commutative, if $x \cdot y = y \cdot x$ for all $x, y \in R$.

(ii) Semihyperring with identity if $1_R \in R$, such that $x \cdot 1_R = 1_R \cdot x = x$, for all $x \in R$.

Definition 4.2.3 [16] A left (right) hyperideal of a semihyperring R is a nonempty subset I of R, such that:

(i) For all $x, y \in I, x + y \subseteq I$,

(ii) For all $x \in I\,, r \in R\,, r \cdot x \in I$ $(x \cdot r \in I)$, a hyperideal of a semihyperring is a non-empty subset of R, which is left and right hyperideal of R, and the set of all hyperideals of R is denoted by $Id(R)$.

Definition 4.2.4 [16] For $x \in R$, if there exist one and only one $-x \in R$ such that $0 \in x + (-x)$ then $-x$ is called the opposite of x. Denote the set of all opposite elements of R, by $V(R)$, that is $V(R) = \{x \in R | \text{there exists } y \in R, 0 \in x + y\}$.

Definition 4.2.5 [17] Let I be a hyperideal of a semihyperring R, then I is called a k-hyperideal if for $a \in I, x \in R$ we have $a + x \approx I \implies x \in I$, where $A \approx B$ means $A \cap B \neq \phi$.

Definition 4.2.6 [16] A semihyperring R is called an additively reversive if it satisfy the reversive property with respect to the hyperoperation of addition, i.e. for $a \in b + c$ implies $b \in a + (-c)$ and $c = a + (-b)$.

Definition 4.2.7 Let R be a commutative semihyperring and I be a hyperideal of R. Then the closure of I which is denoted by $cl(I)$, is defined by $cl(I) = \{a \in R\colon a + c \subseteq I$, for some $c \in I\}$.

Theorem 4.2.1 Let R be a semihyperring. Then R has at least one maximal k-hyperideal.

Proof. Let Δ be the set of all k-hyperideals of R. Since $\{0\}$ is a proper k-hyperideal of R then $\Delta \neq \phi$, and the relation of inclusion is partially order on Δ, then by Zorn's Lemma we have a maximal element in Δ. So that R has at least one maximal k-hyperideal.■

In [17], Ameri showed that for a semihyperring R, with I is a hyperideal of R, the set $(R/I\,, \oplus, \odot)$ is a semihyperring, where

$R/I = \{x + I, x \in R\}, x + I \oplus y + I = \{z + I : z \in x + y\}, x + I \odot y + I = xy + I.$

Definition 4.2.8 [15] A hyperideal expansion is a function δ, which assigns to each hyperideal I of a semihyperring R another hyperideal $\delta(I)$ of the same semihyperring such that $I \subseteq \delta(I)$ and if $I \subseteq J$ then $\delta(I) \subseteq \delta(J)$ for any hyperideal J of R.

Definition 4.2.9 [15] Let δ be a hyperideal expansion, a proper hyperideal I of a semihyperring R is called δ-primary if $ab \in I$, and $a \notin I$, then $b \in \delta(I)$. The definition can be stated as: if $b \in I, a \notin \delta(I)$, then $b \in \delta(I)$.

Remarks:

1. Let I be a hyperideal of a semihyperring R, the identity function δ_0, such that $\delta_0(I) = I$ is an expansion for each $I \in Id(R)$. A hyperideal I is δ_0-primary if and only if I is a prime hyperideal.
2. Let $\rho(I)$ be the radical of hyperideal I of a semihyperring be defined the same as on the hyperideal of hyperring. Define $\delta_1(I) = \rho(I)$ for each $I \in Id(R)$ then δ_1 is a hyperideal expansion. A hyperideal I is δ_1-primary if and only if I is a primary hyperideal.
3. Let I be a hyperideal of a semihyperring R, let $\delta_2(I) = cl(I)$. Then δ_2 is a hyperideal expansion.
4. Let B be the largest hyperideal of a semihyperring R, then $\delta_3(I) = B$ for all $I \in Id(R)$, then δ_3 is a hyperideal expansion.
5. Let δ_4 defined by the intersection of all maximal hyperideals containing I, and $\delta_4(R) = R$, then δ_4 is a hyperideal expansion.
6. If δ is a hyperideal expansion, let $E_\delta : Id(R) \longrightarrow Id(R)$ defined by $E_\delta(I) = \bigcap\{J \in Id(R) : I \subseteq J, J \text{ is } \delta\text{-primary}\}$. Then E_δ is a hyperideal expansion.

Theorem 4.2.2 If I, J are two hyperideals of a semihyperring $(R, +, \cdot)$ and δ is a hyperideal expansion, then

(i) $\delta(I) + \delta(J) \subseteq \delta(I + J)$.

(ii) $\delta(I \cdot J) \subseteq \delta(I \cap J) \subseteq \delta(I) \cap \delta(J)$.

Proof.

(i) By the definition of the hyperideal we have $I, J \subseteq I + J$ and by the definition of δ, we get $\delta(I), \delta(J) \subseteq \delta(I + J)$. Then $\delta(I) + \delta(J) \subseteq \delta(I + J)$.

(ii) Since $I \cdot J \subseteq I, I \cdot J \subseteq J$, then $I \cdot J \subseteq I \cap J$ hence $\delta(I \cdot J) \subseteq \delta(I \cap J)$, also, $I \cap J \subseteq I$ and $I \cap J \subseteq J$ then $\delta(I \cap J) \subseteq \delta(I)$ and $\delta(I \cap J) \subseteq \delta(J)$. Hence $\delta(I \cap J) \subseteq \delta(I) \cap \delta(J)$. ■

Theorem 4.2.3 [15] Suppose that R is a commutative semihyperring, δ is a hyperideal expansion, then:

1) The hyperideal I of R is δ-primary if and only if for any hyperideals J, K in R with $JK \subseteq I$, $K \nsubseteq I$ implies that $J \subseteq \delta(I)$.
2) If I is a δ-primary hyperideal of R, T is any subset of R, then

$$(I{:}T) = \{x \in R{:} xT \subseteq I\}$$

is δ-primary and if T is a hyperideal of R such that $T \nsubseteq \delta(I)$, then $(I, T) = I$.

Definition 4.2.10 [15] 1. A hyperideal expansion δ is said to be intersection preserving if for any hyperideals I, J in a semihyperring R satisfies that $\delta(I \cap J) = \delta(I) \cap \delta(J)$.
2. A hyperideal expansion δ is said to be a global if for any semihyperring homomorphism $f{:} R \longrightarrow S$ and any hyperideal I in S we have $\delta\left(f^{-1}(I)\right) = f^{-1}\left(\delta(I)\right)$.

Remark :$\delta_0(I) = I$ is intersection preserving, and $\delta(I) = \rho(I)$ is also intersection preserving because $\rho(I \cap J) = \rho(I) \cap \rho(J)$.

Theorem 4.2.4 Let $(R, +, \cdot)$ be a commutative semihyperring with a unit and $\delta(I) = \cap\{H | I \subseteq H,\ H \text{ is a maximal hyperideal}\}$. Then δ is a hyperideal expansion and δ is intersection preserving.
Proof. By the definition of δ, we have $I \subseteq \delta(I)$, and for any hyperideals P, Q in R such that $P \subseteq Q$ we have $\cap\{H | P \subseteq H, H \text{ is maximal hyperideal}\} \subseteq \cap\{H | Q \subseteq H, H$ is maximal hyperideal$\}$ which implies that $\delta(P) \subseteq \delta(Q)$ and so δ is a hyperideal expansion. To prove that δ is intersection preserving, let $\mathcal{M}_1 = \{H | I \cap J \subseteq H, H$ is maximal hyperideal$\}$, $\mathcal{M}_2 = \{H | I \subseteq H$ or $J \subseteq H, H$ is maximal hyperideal$\}$. Then $\cap\mathcal{M}_1 = \delta(I \cap J)$ and $\cap\mathcal{M}_2 = \delta(I) \cap \delta(J)$ and any H in $\mathcal{M}_2$ is directly in $\mathcal{M}_1$, so that $\mathcal{M}_2 \subseteq \mathcal{M}_1$. Now if $H \in \mathcal{M}_1$, then $I \cap J \subseteq H$, and since $I \cdot J \subseteq I \cap J$ then $I \cdot J \subseteq H$, but H is maximal and R is commutative with a unit, then by Proposition 3.3.7 in [1], H is prime hyperideal, so that $I \subseteq H$ or $J \subseteq H$, thus $H \in \mathcal{M}_2$, so that $\mathcal{M}_1 = \mathcal{M}_2$, and therefore $\delta(I \cap J) = \delta(I) \cap \delta(J)$. ■

Theorem 4.2.5 Let δ_i be hyperideal expansions for $i = 1, \cdots, n$. Then $\delta(I) = \bigcap_{i=1}^{n} \delta_i(I)$ is also a hyperideal expansion.

Proof. Since δ_i is a hyperideal expansion for all i, then $I \subseteq \delta_i(I)$ for all i, so that $I \subseteq \bigcap_{i=1}^{n} \delta_i(I) = \delta(I)$. Also, if I, J are hyperideals such that $I \subseteq J$, then $\delta_i(I) \subseteq \delta_i(J)$, for all i therefore $\delta(I) = \bigcap_{i=1}^{n} \delta_i(I) \subseteq \bigcap_{i=1}^{n} \delta_i(J) = \delta(J)$. Hence δ is a hyperideal expansion.■

Theorem 4.2.6 [15] Let δ be a hyperideal expansion of the commutative semihyperring R. If $\delta(I) \subseteq \rho(I)$ for every δ-primary hyperideal I, then $\delta(I) = \rho(I)$.

Proof. Let $x \in \rho(I)$, then $x^n \in I$ for some $n \in \mathbb{N}$. If $n = 1$, then $x \in I \subseteq \delta(I)$ and we are done. So we may assume that $n > 1$, such that $x^n \in I$ but $x^{n-1} \notin I$. Since I is δ-primary and $x^n = xx^{n-1} \in I, x^{n-1} \notin I$, then $x \in I$, therefore $\delta(I) = \rho(I)$. ■

We will define a δ-zero divisor of a semihyperring and a hyperring. This definition is an extension to the definition of a δ-zero divisor of a semiring [18].

Definition 4.2.11 Let δ be a hyperideal expansion on a semihyperring R, an element x is called δ-zero divisor in R, if there exists $y \in R$ with $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$. The set of all δ-zero divisor in R will be denoted by $Z_\delta(R)$.

Theorem 4.2.7 Let R be a semihyperring with identity and let δ be a hyperideal expansion such that $\delta(\{0\}) \neq R$. Then

1. $nil_\delta(R)$is a hyperideal of R with $nil_\delta(R) \subseteq Z_\delta(R)$.
2. If $Z_\delta(R)$ is a hyperideal, then $Z_\delta(R)$ is δ-primary.

Proof. 1. Let $x, y \in nil_\delta(R), r \in R$. Then $x, y \in \delta(\{0\})$ which is a hyperideal. So that $x + y \subseteq \delta(\{0\})$ and $rx \in \delta(\{0\})$, therefore $nil_\delta(R)$ is a hyperideal. Now for $x \in nil_\delta(R)$, we have $x = x \cdot 1_R \in \delta(\{0\})$, but $\delta(\{0\}) \neq R$, so $1_R \notin \delta(\{0\})$ and $x \in Z_\delta(R)$.
2. Suppose that $Z_\delta(R)$ is a hyperideal of R, let $x, y \in R$ such that $xy \in Z_\delta(R)$. Then there exists $r \in R$ with $xyr \in \delta(\{0\})$ but $r \notin \delta(\{0\})$. Now if $yr \in \delta(\{0\})$, then $y \in Z_\delta(R)$, and if $yr \notin \delta(\{0\})$, then $x \in Z_\delta(R)$. Hence $Z_\delta(R)$ is a δ-primary hyperideal.■

Theorem 4.2.8 Let I be a hyperideal of a semihyperring R, and let δ be a global hyperideal expansion. Then I is δ-primary if and only if $Z_\delta(R/I) \subseteq \delta(\{0_{R/I}\})$.

Proof : Suppose that $\delta(I)$ is δ-primary, let $r_1 + I$ be an element of $Z_\delta(R/I)$, then there exists $r_2 + I \notin \delta(\{0_{R/I}\})$ such that $r_1 + I \odot r_2 + I = r_1r_2 + I \in \delta(\{0_{R/I}\})$. Since δ is a global, then for natural homomorphism $f: R \longrightarrow R/I$, which defined by $f(x) = x + R$ we have $f(I) = \{0_{R/I}\}$ and $\delta(\{0_{R/I}\}) = \delta(f(I)) = f(\delta(I)) = \delta(I)/I$. Then $r_1r_2 \in$

$\delta(I), r_2 \notin \delta(I)$ so that $r_1 \in I$ and $r_1 + I \in \delta(\{0_{R/I}\})$. Conversely, suppose that $Z_\delta(R/I) \subseteq \delta(\{0_{R/I}\})$, and let $x, y \in R$ such that $xy \in I, x \notin I$, then $x + I \odot y + I = xy + I = I$, and so $y + I \in Z_\delta(R/I)$ then $y + I \in \delta(\{0_{R/I}\})$. Then $y \in \delta(I)$ and so I is δ-primary. ■

Definition 4.2.12: 1. Let R be a semihyperring with a hyperideal expansion δ, then R is called δ-semidomainlike semihyperring if $Z_\delta(R) \subseteq nil_\delta(R)$.

2. Let R be a hyperring with a hyperideal expansion δ. Then R is called a δ-domainlike hyperring if $Z_\delta(R) \subseteq nil_\delta(R)$.

Theorem 4.2.9: Let R be a semihyperring with a hyperideal expansion δ such that $\delta(\delta(I)) = \delta(I)$ for every hyperideal I in R. Then

1. $\delta(\{0\})$ is a δ-primary hyperideal if and only if $Z_\delta(R) = nil_\delta(R)$.
2. $\delta(\{0\})$ is a δ-primary if and only if R is a δ-semidomainlike semihyperring.
3. If R is a δ-semidomainlike semihyperring then $Z_\delta(R)$ is the unique minimal δ-primary hyperideal.

Proof. 1. Suppose that $\delta(\{0\})$ is a δ-primary hyperideal, by the previous theorem we proved that $nil_\delta(R) \subseteq Z_\delta(R)$, so it is enough to prove that $Z_\delta(R) \subseteq nil_\delta(R)$. Let $x \in Z_\delta(R)$ then there exists $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\})$, and since $\delta(\{0\})$ is a δ-primary hyperideal $y \notin \delta(\{0\})$ then $x \in \delta(\delta(\{0\})) = \delta(\{0\})$, and so $nil_\delta(R) \subseteq Z_\delta(R)$. Conversely, suppose that $Z_\delta(R) = nil_\delta(R)$, and let $xy \in \delta(\{0\})$, $y \notin \delta(\{0\})$, then $x \in Z_\delta(R) = nil_\delta(R)$, $x \in \delta(\{0\})$, therefore, $\delta(\{0\})$ is δ-primary.

2. Follows from (1).

3. assume that R is a δ-semidomainlike semihyperring, then by (1), (2) we have $Z_\delta(R) = nil_\delta(R)$, and since $\delta(\{0\})$ is a δ-primary hyperideal then also $Z_\delta(R)$ is a δ-primary hyperideal. Let J be any δ-primary hyperideal of R, since $\{0\} \subseteq J$, then $\delta(\{0\}) \subseteq \delta(J)$ and if $x \in Z_\delta(R)$, then there exists $y \notin \delta(\{0\})$ such that $xy \in \delta(\{0\}) \subseteq \delta(J)$ and since J is δ-primary, if $y \notin \delta(J)$, then $x \in J$, so that $Z_\delta(R) = nil_\delta(R) \subseteq J$. ■

Definition 4.2.13 1. Let R be a commutative semihyperring, and δ be a hyperideal expansion on R. Then R is called a δ-semidomain is $xy \in \delta(\{0\})$, then either $x \in \delta(\{0\})$ or $y \in \delta(\{0\})$.

2. Let R be a commutative hyperring, and δ be a hyperideal expansion on R. Then R is called a δ-domain if $xy \in \delta(\{0\})$ we have either $x \in \delta(\{0\})$ or $y \in \delta(\{0\})$.

Theorem 4.2.10 Let R be a semihyperring and δ be a global expansion such that $\delta\big(\delta(I)\big) = \delta(I)$ for every hyperideal I of R. If $\delta(I)$ is δ-primary, then R/I is a δ-semidomainlike if and only if R/I is a δ-semidomain.

Proof. Suppose that R/I is a δ-semidomainlike semihyperring, since δ is global hyperideal expansion, then $\delta\big(0_{R/I}\big) = \delta(I)/I$. Now, let $a + I, b + I \in R/I$ such that $a + I \odot b + I = ab + I \in \delta\big(0_{R/I}\big) = \delta(I)/I$. Then $ab \in \delta(I)$ and since $\delta(I)$ is δ-primary, then either $a \in \delta(I)$ or $b \in \delta\big(\delta(I)\big) = \delta(I)$. So that $a + I \in \delta(I)/I$ or $b + I \in \delta(I)/I$. Therefore R/I δ-semidomain. Conversely, assume that R/I is a δ-semidomain, then by the previous theorem it is enough to prove that $Z_\delta(R/I) \subseteq nil_\delta(R/I)$. Let $x + I \in Z_\delta(R/I)$, then there exists $y + I \notin \delta\big(0_{R/I}\big) = \delta(I)/I$ such that $x + I \odot y + I = xy + I \in \delta\big(0_{R/I}\big)$. Since R/I is a δ-semidomain, then $x + I \in \delta\big(0_{R/I}\big)$, and we have $Z_\delta(R/I) \subseteq nil_\delta(R/I)$. Hence R/I is a δ-semidomainlike semihyperring. ■

CHAPTER 5

Fuzzy Hyperideals of Commutative Semihyperrings

The Fuzzy Set Theory is a branch of mathematics. In 1965, Lotfi Zadeh defined a fuzzy subset of any set X as a function $\mu: X \longrightarrow [0, 1]$ [20]. After that the students of Zadeh introduced the basic concepts of fuzzy algebra. In this chapter we define the expansion of fuzzy hyperideal and generalize the expansion theory on hyperideals of semihyperrings to fuzzy hyperideals of semihyperrings.

5.1 δ-Primary Fuzzy Hyperideals of Commutative Semihyperrings

Definition 5.1.1 [21] Let X be a set, a fuzzy subset μ of X is a function from X to [0 , 1], such that $0 \leq \mu(x) \leq 1$ for all $x \in X$. For any two fuzzy sets μ, γ we say that $\mu \leq \gamma$ if and only if $\mu(x) \leq \gamma(x)$ for all $x \in X$, $(\mu \cap \gamma)(x) = (\mu \wedge \gamma)(x) = \min\{\mu(x), \gamma(x)\}$, and $(\mu \cup \gamma)(x) = (\mu \vee \gamma)(x) = \max\{\mu(x), \gamma(x)\}$.

Example 5.1.1 Let $\mathbb{R}$ be the set of real numbers and let $\mu: \mathbb{R} \rightarrow [0, 1]$ defined by

$$\mu(x) = \begin{cases} 1 & if\ x = 0 \\ 1 - \dfrac{1}{x} & if\ x > 0 \\ 1 + \dfrac{1}{x} & if\ x < 0 \end{cases}$$

Then, μ is a fuzzy subset of real numbers $\mathbb{R}$.

Definition 5.1.2 [22] Let R be a semihyperring and let μ be a fuzzy set of R, then μ is said to be a fuzzy hyperideal of R if:

i) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$.

ii) $\underset{z \in x+y}{inf} \mu(z) \geq \mu(x) \wedge \mu(y)$for all $x, y \in R$.

Example 5.1.2 : Let $\bar{R} = R/G$ be the hyperring defined in Example 1.3.1, and μ be defined on $\bar{R}$ by :

$$\mu\big(P(x)\big) = \begin{cases} 1/_{2} & if\ P(x) = P(0) \\ 1/_{10} & if\ P(x) \in \bar{R} \setminus \{P(0)\} \end{cases}$$

Then μ is a fuzzy hyperideal.

Definition 5.1.3 Let R be a semihyperring and let μ, γ be fuzzy hyperideals of R, let δ be a function that assigns each fuzzy hyperideal μ another fuzzy hyperideal $\delta(\mu)$ of R such that:

i) $\mu \leq \delta(\mu)$.

ii) if $\mu \leq \gamma$ then $\delta(\mu) \leq \delta(\gamma)$.

then δ is called a fuzzy hyperideal expansion.

Definition 5.1.4 [23]**:** Let μ be a fuzzy hyperideal of a semihyperring R, μ is called primary if for any $a, b \in R$, either $\mu(ab) = \mu(a)$ or $\mu(b^n) \geq \mu(ab)$ for some $n \in \mathbb{N}$.

Definition 5.1.5 : Let δ be a fuzzy hyperideal expansion, a fuzzy hyperideal μ is called δ-primary if $\mu(ab) > \mu(a)$ then $\delta(\mu(b)) \geq \mu(ab)$. Also the definition can be written as, if $\mu(ab) > \delta(\mu(a))$ then $\mu(b) \geq \mu(ab)$.

Definition 5.1.6 [21] Let μ be a fuzzy set of a semihyperring R, $t \in [0, 1]$, then $\mu_t = \{x \in R: \mu(x) \geq t\}$ is called the level set of μ.

Definition 5.1.7 [24] Let μ be a fuzzy hyperideal of a semihyperring R, then $\rho(\mu)$, called the radical of μ is defined by $\rho(\mu(x)) = \sup_{n\geq 1} \mu(x^n), n \in \mathbb{N}$.

Theorem 5.1.1 : If μ is a fuzzy hyperideal of a semihyperring R, then $\rho(\mu)$ is a fuzzy hyperideal of R.

Proof. Since μ is a fuzzy hyperideal, then $\mu(x^n y^n) \geq \mu(x^n) \vee \mu(y^n)$, so that $\rho(\mu(xy)) = \sup_{n\geq 1} \mu(x^n y^n) \geq \sup_{n\geq 1} \mu(x^n) \vee \sup_{n\geq 1} \mu(x^n) = \rho(\mu(x)) \vee \rho(\mu(y))$. Consider that $\mu(x + y) = \{\mu(z): z \in x + y\}$, so that

$$\inf_{z\in x+y} \rho(\mu(z)) = \inf_{z\in x+y} (\sup_{n\geq 1} \mu(z^n)) = inf(\sup_{n\geq 1} \mu(x + y)^n)$$

$$= inf\left(\sup_{n\geq 1} \mu \sum_{r=0}^{n} \binom{n}{r} x^r y^{y-r}\right)$$

$$\geq \left(\sup_{n\geq 1}\left(\min_{1\leq r\leq n} \mu(x^r y^{n-r})\right)\right)$$

$$\geq \left(\sup_{n\geq 1} \mu(x^{r_n} y^{n-r_n})\right),\ 0 \leq r_n \leq n,$$

where $\mu(x^{r_n}y^{n-r_n}) = \min_{1\le r\le n} \mu(x^r y^{n-r})$. If r_n bounded above i.e $r_n \le k$ for some constant k, and $n-r_n$is unbounded above then $\inf_{z\in x+y}\rho(\mu(z)) \ge \sup_{n\ge 1}\mu(x^{r_n}y^{n-r_n}) \ge \sup_{n\ge 1}\mu(y^{n-r_n})$, which is true for all values of n, so$\inf_{z\in x+y}\rho(\mu(z)) \ge \sup_{n\ge 1}\mu(y^n) = \rho(\mu(y))$. Similarly, if $n-r_n$ is bounded above and r_n is unbounded, then $\inf_{z\in x+y}\rho(\mu(z)) \ge \sup_{n\ge 1}\mu(x^n) = \rho(\mu(x))$, if none of $n-r_n, r_n$ is bounded then $\inf_{z\in x+y}\rho(\mu(z)) \ge \rho(\mu(x)) \wedge \rho(\mu(y))$, hence $\rho(\mu)$ is a fuzzy hyperideal of R. ■

Example 5.1.2

1) For any fuzzy hyperideal μ of a semihyperring R, $\delta_0(\mu) = \mu$ is a fuzzy hyperideal expansion.
2) $\delta_1(\mu) = \rho(\mu)$ is a fuzzy hyperideal expansion.

Proof. 1. It is trivial case.

2. For all $x \in R$ we have $\delta(\mu(x)) = \rho(\mu(x)) = \sup_{n\ge 1}\mu(x^n) \ge \mu(x^{n-1}x) \ge \mu(x)$, also if μ_1, μ_2 are fuzzy hyperideals such that $\mu_1 \le \mu_2$, then $\delta\big(\mu_1(x)\big) = \rho(\mu_1(x)) = \sup_{n\ge 1}\mu_1(x^n) \le \sup_{n\ge 1}\mu_2(x^n) = \rho(\mu_2) = \delta\big(\mu_2(x)\big)$. Hence $\delta(\mu) = \rho((\mu))$is a fuzzy hyperideal expansion.

Definition 5.1.8 [23] Let μ be a fuzzy hyperideal of a semihyperring R, then, μ is called prime if for all $x, y \in R$ we have $\mu(xy) = \mu(x) \; or \; \mu(xy) = \mu(y)$.

Theorem 5.1.2 Let μ be a fuzzy hyperideal of a semihyperring R, then μ is prime if and only if μ is δ_0-primary.

Proof. Assume that μ is a prime fuzzy hyperideal, let $\mu(ab) > \mu(a)$ so that $\mu(ab) = \mu(b)$, then $\delta_0\big(\mu(b)\big) = \mu(b) \ge \mu(ab)$ and μ is δ_0-primary. Conversely, let μ is δ_0-primary, and $\mu(ab) > \mu(a)$, then $\delta_0\big(\mu(b)\big) = \mu(b) \ge \mu(ab)$, but $\mu(ab) \ge \mu(b)$. So we have $\mu(ab) = \mu(b)$ and μ is a fuzzy prime hyperideal. ■

Theorem 5.1.3: Let μ be a fuzzy hyperideal of a semihyperring R, then μ is primary if and only if μ is δ_1-primary.

Proof. Assume that μ be a primary fuzzy hyperideal with $\mu(ab) \ne \mu(a)$, then $\mu(ab) > \mu(a)$. So that there exists $n \in \mathbb{N}$ such that $\mu(a^n) \ge \mu(ab)$, then $\delta(\mu(a)) = \sup_{n\ge 1}\mu(a^n)$. Therefore μ is δ_1-primary. Conversely, suppose that μ is δ_1-primary with $\mu(ab) \ne \mu(a)$, then $\mu(ab) > \mu(a)$, so that $\rho(\mu(b)) = \sup_{n\ge 1}\mu(b^n) \ge \mu(ab)$. If $\mu(b^n) < \mu(ab)$ for all $n \in$

$\mathbb{N}$, then $\sup_{n\geq 1}\mu(b^n) < \mu(ab)$ which is a contradiction, so that there exists $n \in \mathbb{N}$, such that $\mu(b^n) \geq \mu(ab)$ and so μ is a primary fuzzy hyperideal.

Theorem 5.1.4 Let μ be a fuzzy hyperideal of a semihyperring R, and δ be a fuzzy hyperideal expansion and γ be a hyperideal expansion such that $\gamma(\mu_t) = (\delta\mu)_t,\ t \in [0, 1]$. Then μ is a δ-primary fuzzy hyperideal if and only if μ_t is a γ-primary hyperideal of a semihyperring R.

Proof. Suppose that μ is a δ-primary fuzzy hyperideal of R. Let $xy \in \mu_t$ with $y \notin \mu_t$, then $\mu(xy) \geq t$ but $\mu(y) < t$. So that $\mu(xy) > \mu(y)$, then $\delta\mu(y) \geq \mu(xy) \geq t$. Therefore $y \in (\delta\mu)_t = \gamma(\mu_t)$. Hence μ_t is a γ-primary hyperideal of R.

Conversely, assume that μ_t is a γ-primary hyperideal of R. Let $\mu(xy) > \mu(y)$, then $\mu(xy) = t$ for some $t \in [0,1]$. Then $y \notin \mu_t$, but $xy \in \mu_t$ so that $y \in \gamma(\mu_t) = (\delta\mu)_t$. We get $\delta\mu(y) \geq t = \mu(xy)$. Therefore μ is a δ-primary fuzzy hyperideal.

Theorem 5.1.5 Let δ, γ be fuzzy hyperideals expansions of a semihyperring R such that $\delta(\mu) \subseteq \gamma(\mu)$ for every fuzzy hyperideal μ. If μ is δ-prrimary, then μ is γ-primary. Also every prime fuzzy hyperideal μ is a δ-primary fuzzy hyperideal.

Proof. Suppose that μ is a δ-primary fuzzy hyperideal, such that $\mu(ab) > \mu(b)$, then $\delta\mu(b) \geq \mu(ab)$. So that $\gamma\mu(b) \geq \delta\mu(b) \geq \mu(ab)$. Therefore μ is γ-primary. Now, assume that μ is a prime fuzzy hyperideal with $\mu(ab) > \mu(b)$, so that $\mu(ab) = \mu(a)$, but we know that $\delta\mu \geq \mu$, so that $\delta\mu(a) \geq \mu(a) = \mu(ab)$. Hence μ is a δ-primary fuzzy hyperideal.

Theorem 5.1.6 The intersection of two fuzzy hyperideal expansions is also a fuzzy hyperideal expansion.

Proof. Assume that δ_1, δ_2 are two fuzzy hyperideal expansions of the semihyperring R. Let δ be defined as $\delta = \delta_1 \cap \delta_2$. Then for any fuzzy hyperideal μ we have that $\mu \subseteq \delta_1\mu, \mu \subseteq \delta_2\mu$, so that $\mu \subseteq \delta_1\mu \cap \delta_2\mu = \delta(\mu)$. Now suppose σ is another fuzzy hyperideal of R such that $\mu \subseteq \sigma$. Then $\delta\mu \subseteq \delta_1\mu \subseteq \delta_1\sigma$ and $\delta\mu \subseteq \delta_2\mu \subseteq \delta_2\sigma$, which implies that $\delta\mu \subseteq \delta_1\sigma \cap \delta_2\sigma = \delta\sigma$. Hence δ is a fuzzy hyperideal expansion. ■

Remark. The previous theorem can be generalized for finite fuzzy hyperideal expansions, that is the intersection of a finite number of fuzzy hyperideal expansions is also a fuzzy hyperideal expansion.

Theorem 5.1.7 Let δ be a fuzzy hyperideal expansion of a semihyperring R, and E_δ be defined by $E_\delta(\mu) = \bigcap\{\rho : \rho$ is δ-primary a fuzzy hyperideal expansion, $\mu \subseteq \rho\}$. Then E_δ is a fuzzy hyperideal expansion.

Proof. By the definition of E_δ we have that $\mu \subseteq E_\delta(\mu)$. Now suppose that $\mu_1 \subseteq \mu_2$, then any δ-primary fuzzy hyperideal ρ that contains μ_2 is also contains μ_1. Since $\mu_1 \subseteq \mu_2$, then $E_\delta(\mu_1) \subseteq E_\delta(\mu_2)$. So that E_δ is a fuzzy hyperideal expansion.

Definition 5.1.9 [15] Let δ be a fuzzy hyperideal expansion, then δ is called intersection preserving if $\delta(\mu \cap \rho) = \delta(\mu) \cap \delta(\rho)$ for all fuzzy hyperideals μ, ρ of a semihyperring R.

Example 5.1.3 δ_0 is intersection preserving because $\delta_0(\mu \cap \rho) = \mu \cap \rho = \delta_0(\mu) \cap \delta_0(\rho)$. Also δ_1 is intersection preserving because $\delta_1(\mu \cap \rho) = \rho(\mu \cap \rho) = \sup_{n\geq 1}(\mu \wedge \rho)(x^n) = \sup_{n\geq 1}(\mu)(x^n) \wedge \sup_{n\geq 1}(\rho)(x^n) = \delta_1(\mu) \cap \delta_1(\rho)$.

Theorem 5.1.8 Let δ be an intersection preserving fuzzy hyperideal expansion and $\mu_1, \mu_2, \cdots, \mu_n$ be δ-primary fuzzy hyperideals such that $\delta(\mu_i) = \delta(\mu_j) = \rho$ for $1 \leq i, j \leq n$. Let

$$\mu = \bigcap_{i=1}^{n} \mu_i.$$

Then μ is a δ-primary fuzzy hyperideal.

Proof. Suppose that $\mu(ab) > \mu(b)$, then $\mu(ab) = \wedge_{i=1}^{n} \mu_i(ab) > \wedge_{i=1}^{n} \mu_i(a)$. So that $\mu(ab) > \mu_j(a)$ for some j. So we have $\mu_j(ab) > \wedge_{i=1}^{n} \mu_i(a) = \mu(a) \geq \mu_j(a)$ and since μ_j is δ-primary. Then $\delta\mu_j(b) \geq \mu(ab)$. Since δ is intersection preserving, then

$$\delta(\mu) = \delta\left(\bigcap_{i=1}^{n} \mu_i\right) = \left(\bigcap_{i=1}^{n} \delta\mu_i\right) = \bigcap_{i=1}^{n} \rho = \rho.$$

Therefore, $\mu(b) = \rho(b) = \delta\mu_j(b) \geq \mu(ab)$. Hence μ is a δ-primary fuzzy hyperideal. ■

Definition 5.1.10 [25] Let R, S be nonempty subsets , and $\theta \in [0, 1)$, f be a fuzzy subset of the Cartesian cross product $R \times S$. Then f is called a fuzzy function if:

1) For all $x \in R$ there exists $y \in S$, such that $f(x, y) > \theta$.
2) For all $x \in R$, for all $y_1, y_2 \in S$, $f(x, y_1) > \theta$ and $f(x, y_2) > \theta$ implies that $y_1 = y_2$.

Definition 5.1.11 [25] Let R, S be nonempty subsets, $\theta \in [0,1)$ and f be a fuzzy function defined on $R \times S$. Then;

1) f is onto if for all $y \in S$ there exists $x \in R$ such that $f(x, y) > \theta$.
2) f is one-to-one if for all $x_1, x_2 \in R$, for all $y \in S$, $f(x_1, y) > \theta, f(x_2, y) > \theta$ implies that $x_1 = x_2$.

Definition 5.1.12 Let R, S be commutative semihyperrings, and f be a fuzzy function defined on $R \times S$. Then f is said to be a fuzzy homomorphism if and only if for all $x_1, x_2 \in R$ and for all $y \in S$

1) $\inf_{z \in x_1 + x_2} f(z, y) \geq \sup\{\inf\{f(x_1, y_1), f(x_2, y_2)\} | \; y \in y_1 + y_2, y_1, y_2 \in S\}$
2) $f(x_1 x_2, y) \geq \sup\{\inf\{f(x_1, y_1), f(x_2, y_2)\} | \; y = y_1 y_2, \; y_1, y_2 \in S\}$.

The homomorphism f is said to be a fuzzy isomorphism if it is one-to-one and onto.

Definition 5.1.13 [25] Let f be a fuzzy subset of $R \times S$, then the inverse of f is f^{-1} on $S \times R$ is defined by $f^{-1}(s, r) = f(r, s)$ for all $(s, r) \in S \times R$.

Definition 5.1.13 [25] Let R, S be semihyperrings and f be a fuzzy function of R into S.

1) If μ is a fuzzy subset of R, then the fuzzy subset $f(\mu)$ of S defined by for all $s \in S$, $f(\mu)(s) = \sup\{\mu(r) : f(r, s) > \theta\}$.
2) If λ is a fuzzy subset of S, then the fuzzy subset $f^{-1}(\lambda)$ of R defined by for all $r \in R$, $f^{-1}(\lambda)(r) = \lambda(s)$ where $f(r, s) > \theta$.

Theorem 5.1.9 Let R, S be semihyperrings, and f be a fuzzy homomorphism from R onto S. Then;

1) If μ is a fuzzy hyperideal of R then $f(\mu)$ is a fuzzy hyperideal of S.
2) If λ is a fuzzy hyperideal of S then $f^{-1}(\lambda)$ is a fuzzy hyperideal of R.

Proof. 1) Assume that μ is a fuzzy hyperideal of R. Since f is onto homomorphism, then for each $x, y \in S$ there exists $r_1, r_2 \in R$ such that $f(r_1, x) > \theta, f(r_2, y) > \theta$. So that $f(\mu)(xy) = \sup\{\mu(r) : f(r, xy) > \theta\} \geq \sup\{\mu(r_1 r_2) : f(r_1 r_2, xy) > \theta\}$

$$\geq \sup\{\mu(r_1) \vee \mu(r_2) : f(r_1 r_2, xy) > \theta\}$$

$$\geq \sup \{\mu(r_1) : f(r_1, x) > \theta\} \vee \sup\{\mu(r_2) : f(r_2, y) > \theta\}$$

$$= f(\mu)(x) \vee f(\mu)(y).$$

Also, $\underset{s\in x+y}{inf} f(\mu)(s) = \inf\left(\underset{s\in x+y}{\sup}\{\mu(r_1r_2): f(r_1r_2, s) > \theta\}\right)$

$$\geq \inf(\sup\{\mu(r_1): f(r_1, x) > \theta\} \wedge \sup\{\mu(r_2): f(r_2, x) > \theta\})$$

$$= \sup\{\mu(r_1): f(r_1, x) > \theta\} \wedge \sup\{\mu(r_2): f(r_2, x) > \theta\}$$

$$= f(\mu)(x) \wedge f(\mu)(y).$$

Therefore $f(\mu)$ is a fuzzy hyperideal of R.

2) Suppose that λ is a fuzzy hyperideal of S, then for $x, y \in R$ we have,

$$f^{-1}(\lambda)(xy) = \lambda(s_1s_2) \text{ where } f(xy, s_1s_2) > \theta$$

$$\geq \{\lambda(s_1) \vee \lambda(s_2): f(xy, s_1s_2) \geq \sup\{f(x, s_1) \wedge f(y, s_2)\}$$

$$\geq \{\lambda(s_1): f(x,\ s_1) > \theta\} \vee \{\lambda(s_2): f(y, s_2) > \theta\}$$

$$= f^{-1}(\lambda)(x) \vee f^{-1}(\lambda)(y).$$

$$\underset{z\in x+y}{inf} f^{-1}(\lambda)(z) = \inf\{\lambda(s_1s_2): f(z, s_1s_2) > \theta\}$$

$$\geq \inf\{\lambda(s_1) \vee \lambda(s_2): f(z, s) \geq f(x, s_1) \vee f(y, s_2), s \in s_1 + s_2\}$$

$$\geq \inf\{\{\lambda(s_1): f(x, s_1) > \theta\} \vee \{\lambda(s_2): f(y, s_2) > \theta\}$$

$$= \lambda(s_1) \wedge \lambda(s_2) = f^{-1}(\lambda)(x) \wedge f^{-1}(\lambda)(y).$$

Hence, $f^{-1}(\lambda)$ is a fuzzy hyperideal of R. ■

Definition 5.1.14 Let f be a fuzzy semihyperring homomorphism from R into S, and δ be a fuzzy hyperideal expansion. Then δ is said to be a global expansion if $\delta\left(f^{-1}(\lambda)\right) = f^{-1}(\delta(\lambda))$ for any fuzzy hyperideal λ of S.

Theorem 5.1.10 Let R, S be semihyperrings, and f be a fuzzy homomorphism from R into S and δ be a global fuzzy hyperideal expansion. If μ is a δ-primary fuzzy hyperideal of S, then $f^{-1}(\mu)$ is a δ-primary fuzzy hyperideal of R.

Proof. By Theorem 5.1.9, we have that $f^{-1}(\mu)$ is a fuzzy hyperideal of R whenever μ is a fuzzy hyperideal of S. Assume that $f^{-1}(\mu)(ab) > f^{-1}(\mu)(a)$, then $\mu(xy) > \mu(x)$ where $f(ab, xy) > \theta$ and $f(a, x) > \theta$, but μ is δ-primary, so that $\delta(\mu(y)) \geq \mu(xy)$

which implies that $f^{-1}(\delta(\mu(b))) \geq f^{-1}(\mu(ab))$ and since δ is global, we have $\delta(f^{-1}(\mu(b))) \geq f^{-1}(\mu(ab))$. Therefore $f^{-1}(\mu)$ is a δ-primary fuzzy hyperideal of R.■

Definition 5.1.14 [25] Let f be a fuzzy semihyperring homomorphism from R to S, $\theta \in [0,1)$, then the kernel of f is $Ker(f) = \{x \in R: f(x,0) > \theta\}$.

Theorem 5.1.11 Let f be a fuzzy semihyperring homomorphism from R onto S and δ be a global fuzzy hyperideal expansion. The fuzzy hyperideal μ of R is δ-primary then $f(\mu)$ is a δ-primary fuzzy hyperideal of S.

Proof. By Theorem 5.1.9 μ is a fuzzy hyperideal of R then $f(\mu)$ is a fuzzy hyperideal of S. Now, suppose that $f(\mu)(sr) = \mu(ab), f(ab,sr) > \theta$, since f is homomorphism, then $f(\mu)(s) = \mu(a)$ and $f(\mu)(r) = \mu(b)$. Assume that μ is δ-primary and $f(\mu)(sr) > f(\mu)(s)$. Then $\mu(ab) > \mu(a)$, so $\delta(\mu(b)) \geq \mu(ab)$ and $f(\delta(\mu(r))) \geq f(\mu(sr))$, and since δ is global. Then $\delta(f(\mu(r))) \geq f(\mu(sr))$. So $f(\mu)$ is δ-primary fuzzy hyperideal of S.■

Theorem 5.1.12 Let f be a fuzzy semihyperring homomorphism from R onto S and δ be a global fuzzy hyperideal expansion. Then f induces one to one corresponding between δ-primary fuzzy hyperideals of R and δ-primary fuzzy hyperideals of S.

Proof. The prove is a direct result of Theorem 5.1.10 and Theorem 5.1.11.■

CHAPTER 6

Conclusion and Discussions

In this study, we studied many well-known properties and concepts of the commutative hyperrings in the sense of Krasner. Also we constructed some examples and defined hyperideals of commutative hyperrings. The homomorphism and the hyperring isomorphism theorems are analogue to those in classical algebra. On primary hyperideals, we gave an example to show that the primary decomposition is not unique. Also we extended the definition of δ-primary ideal expansion to the hyperideal and fuzzy hyperideal of the commutative semihyperring. We showed that if f is a fuzzy semihyperring homomorphism from R onto S and δ be a global fuzzy hyperideal expansion. Then f induces one to one corresponding between δ-primary fuzzy hyperideals of R and δ-primary fuzzy hyperideals of S. Further work is about the δ-primary co-hyperideals of commutative semihyperrings. We hope that we can arise new results and concepts.

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CURRICULUM VITAE

PERSONAL INFORMATION

Name Surname	: Ashraf ABUMGHAISEEB
Date of birth and place	: 23.11.1978 in Deir Alballah- Palestine
Foreign Languages	: English
E-mail	: ashrafgaza@gmail.com

EDUCATION

Degree	**Department**	**University**	**Date of Graduation**
Master	Mathematics	Islamic University of Gaza	2007
Undergraduate	Mathematics	Islamic University of Gaza	2001

WORK EXPERIENCE

Year	**Corporation/Institute**	**Enrollment**
2007-2011	Mathematics Department, Islamic University of Gaza, Palestine	Instructor
2002-2007	Mathematics Department, Islamic University of Gaza, Palestine	Teacher Assistant

PUBLISHMENTS

Papers

1. Abumghaiseeb A. and Ersoy B.A., (2017)."On δ-Primary Hyperideals of Commutative Semihyperrings". Sigma Journal of Engineering and Natural Sciences.

Conference Papers

1. Abumghaiseeb A. and Ersoy B.A., (2017). "δ-Primary Hyperideals of Commutative Hyperrings". International Conference on Mathematics and Engineering. 10-12 May, Istanbul, Turkey.