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ZERO-PRODUCT PRESERVING OPERATORS AND PRODUCT-FACTORABILITY OF BILINEAR MAPS

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Ezgi ERDOĞAN

TABLE OF CONTENTS

Page

LIST OF SYMBOLS

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{K}$	tscalar field $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{T}$	real line mod 2π
χ_A	characteristic function defined on the set A
e_i	ith coordinate function
X^*	topological dual of the Banach space X
$X'(\mu)$	Köthe dual of the Banach function space $X(\mu)$
U_X	open unit ball of X
B_X	closed unit ball of X
$\bar{A}$	closure of the set A
E^+	positive cone of the lattice E
$\langle .,. \rangle$	duality bracket
$\sigma(E, E^*)$	Weak topology on E
$L(X, Y)$	space of all Y- valued linear operators on X
$\mathcal{L}(X, Y)$	space of all Y- valued linear and continuous operators on X
$X \times Y$	topological product of the spaces X, Y
$B(X \times Y, Z)$	space of all Z- valued bilinear operators on the topological product space $X \times Y$
$\mathcal{B}(X \times Y, Z)$	space of all Z- valued bilinear and continuous operators on the topological product space $X \times Y$
$C(K)$	space of continuous functions $f: K \to \mathbb{K}$, where K is compact
$\ell^p(\Delta)$	space of the all scalar valued p-summable functions
$\hookrightarrow$	continuous embedding (inclusion)
$\Pi_{p,q}(X, Y)$	set of all (p, q)-summing operators
(Ω, Σ, μ)	measure space
$\mathfrak{B}(\Omega)$	Borel σ-algebra on Ω
$\mathcal{N}_0(\mu)$	collection of μ-null sets
$\mathrm{Sim}(\Sigma)$	set of Σ-simple functions
$L^0(\mu)$	space of all μ-measurable functions
$L^p(\mu)$	space of classes of p-integrable functions
$L^p(E)$	space of (classes of) Bochner p-integrable functions $f: \Omega \to E$
$L^{p,q}$	Lorentz function space
M_ψ	multiplication operator
$*$	convolution product
$\odot$	pointwise product

G	topological group
Γ	set of the characters of a topological group
$\mathcal{W}(G)$	algebra of functions with absolutely convergent Fourier series
$\mathcal{W}(\mathbb{T})$	Wiener algebra
$\mathcal{J}(G)$	space of all trigonometric polynomials on G
$\hat{f}$	Fourier transform of the function f
$\check{g}$	inverse Fourier transform of the function g
$E^{(p)}$	p-convexification of the space E
B_L	linearization of the bilinear map B
$x \otimes y$	elementary tensor
$X \otimes Y$	tensor product of the spaces X, Y
$X \otimes_\alpha Y$	tensor product $X \otimes Y$ equipped with α
$X \widehat{\otimes}_\alpha Y$	completion of $X \otimes_\alpha Y$
$x \vee y$	$\sup(x, y)$
$x \wedge y$	$\inf(x, y)$
$\lvert x \rvert$	$x \vee (-x)$
$x \perp y$	x and y disjoint

LIST OF ABBREVIATIONS

B.f.s.	Banach function space
LCA	Locally compact Abelian group
l.u.b.	least upper bound
μ-a.e.	μ-almost everywhere
n.p.	norm preserving
o.c.	order continuous
σ-o.c.	σ-order continuous
ppp	positive product preserving
r.i.	rearrangement invariant
zpp	zero product preserving

ABSTRACT

ZERO-PRODUCT PRESERVING OPERATORS AND PRODUCT-FACTORABILITY OF BILINEAR MAPS

Ezgi ERDOĞAN

Department of Mathematics
PhD Thesis

Adviser: Prof. Dr. Ömer GÖK
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The present dissertation deals with bilinear operators acting in pairs of Banach spaces that factor through a canonical product. We find similar situations in different contexts of the functional analysis, including abstract vector lattices −orthosymmetric maps−, C^*-algebras −zero product preserving operators−, and classical and harmonic analysis −integral bilinear operators. We purpose the use of a generic product as a linearizing tool for bilinear maps.

Concretely, in this dissertation we introduce a certain bilinear map, called *product,* by some inclusion and norm equality requirements and present a factorization through the product given in terms of a summability condition for bilinear continuous operators acting in topological product of Banach spaces. If we specialize the product and the domain space of the bilinear map, this factorization also concerns about zero product preserving bilinear maps.

In a second step, we center our attention to the pointwise product and convolution product particularly. In the case of pointwise product, we consider the bilinear maps acting in couples of Banach function spaces and sequence spaces. We obtain that a bilinear map can be pointwise product factorable if and only if it is zero product preserving. In the sequel, we notice that the same result works if we take into account convolution product and the bilinear maps acting in a product of Hilbert spaces of integrable functions, respectively, a product of Banach algebras of integrable functions. In this case, we get that all bilinear maps that are 0-valued for couples of functions whose convolution equals zero have a factorization through convolution.

The other objective of the dissertation is to apply these factorizations to provide new descriptions of some classes of bilinear integral operators, and to obtain integral representations for abstract classes of bilinear maps by some concavity properties of operators. In addition to them, we give also compactness and summability properties for these operators under the assumption of some classical properties for the range spaces,we adapt and apply our results to the case of some particular classes of integral bilinear operators and kernel operators and explain some consequences in a more applied context.

Key words: Factorization, zero product preserving map, bilinear operators, symmetric operators, pointwise product and convolution.

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ÖZET

SIFIR ÇARPIM KORUYAN OPERATÖRLER VE BİLİNEER OPERATÖRLERİN ÇARPIM-ÇARPANLANABİLMESİ

Ezgi ERDOĞAN

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Bu tez çalışması Banach uzay çiftleri üzerinde tanımlı olan ve kanonik bir çarpım aracılığıyla çarpanlanabilen ikilineer operatörler ile ilgilidir. Benzer durumlar fonksiyonel analizin soyut vektör latisleri —ortosimetrik dönüşümler—, C^*-cebirleri —sıfır çarpım koruyan dönüşümler—, ve klasik ve harmonik analiz —integral ikilineer operatörler— gibi farklı içeriklerinde bulunabilir. Bir jenerik çarpımın ikilineer operatörler için bir lineerleştirme aracı olarak kullanılması amaçlanmıştır.

Temel olarak, bu tez çalışmasında bazı kapsama ve norm eşitlikleri şartları ile *çarpım* adı verilen bir ikilineer dönüşüm tanımlanmış ve Banach uzaylarının topolojik çarpımında hareket eden ikilineer sürekli operatörlerin bu çarpım vasıtasıyla çarpanlanması bir toplanabilirlik koşulu ile verilmiştir. Özel olarak belirli bir çarpım ve ikilineer operatör için belirli bir tanım uzayı ele alındığında, bu çarpanlama sıfır çarpım koruyan ikilineer operatörlerle yakından ilgilidir.

İkinci bir adımda, özel olarak nokta çarpım ve konvolüsyon çarpım ele alınmıştır. Nokta çarpım durumunda, Banach fonksiyon uzayı ve dizi uzayı çiftleri üzerinde tanımlı ikilineer operatör göz önünde bulundurulmuştur. Bir ikilineer operatörün nokta çarpım çarpanlanabilir olması ancak ve ancak bu ikilineer operatörün sıfır çarpım koruyan operatör olması ile mümkün olduğu görülmüştür. Ardından, konvolüsyon çarpım ve integrallenebilir fonksiyonların Hilbert uzaylarının, sırasıyla Banach cebirlerinin çarpımı üzerinde tanımlı olan ikilineer operatörler incelendiğinde de aynı sonuç elde edilmiştir. Bu durumda, sıfır konvolüsyona sahip fonkiyon çiftleri için sıfır değerli olan tüm ikilineer operatörlerin konvolüsyon aracılığıyla bir çarpanlamaya sahip olduğu gösterilmiştir.

Tezin bir diğer amacı da bu çarpanlamaları kullanarak ikilineer integral operatörlerin bazı sınıfları için yeni tanımlamalar sağlamak ve operatörlerin konkavlık özellikleri ile ikilineer operatörlerin soyut sınıfları için integral temsilleri elde etmektir. Bunlara ek olarak, operatörün görüntü uzayına klasik özellikler yüklenerek ikilineer operatörlerin kompaktlık ve toplanabilirlik özellikleri verilmiştir. Son olarak, sonuçlar integral ikilineer operatörlerin belirli sınıflarına ve çekirdek operatörlerine uyarlanıp uygulanmış ve sonuçlar uygulamalı olarak açıklanmıştır.

Anahtar Kelimeler: Çarpanlama, sıfır çarpım koruyan dünüşüm, ikilineer operatörler, simetrik operatörler, nokta çarpım ve konvolüsyon.

CHAPTER 1

INTRODUCTION

A particular class of bilinear operators that play a fundamental role in Functional Analysis is the one defined by what we can call, broadly speaking, *a product*. We are thinking of for example, the internal product of a Banach algebra, but also the pointwise or convolution product acting on a couple of Banach spaces.

Consider a bilinear operator acting in a couple of Banach spaces in which a product is defined. If the bilinear map factors through such a product, some of the good properties of the factorization operator are preserved, so it is interesting to know which bilinear operators satisfy such a factorization. This general philosophy —factorization of maps— is one of the main techniques that inspires classical and current developments in mathematical analysis.

In this study, we will give the factorization for the class of the bilinear maps acting on the Cartesian product of some particular Banach spaces that are 0-valued for the couples of elements whose pointwise or convolution product is zero. These bilinear maps are commonly called *zero product preserving*, also *orthosymmetric*, in the literature.

1.1 Literature Review

In the current literature, most of the studies of zero product preserving bilinear maps are related with Banach algebras and vector lattices.

The first reference that we found on the literature related notions due to Lamperti [1]. However, we can find similar definitions in completely different mathematical setting. For example, the notion of orthosymmetric was already used in the paper [2], and it is not connected with convolution but lattice type properties.

In paper [3], Fremlin showed that given two Archimedean Riesz spaces E and F, there is an Archimedean Riesz space $E\overline{\otimes}F$, called *the Archimedean vector lattice tensor product of E and F*, in which the linear space tensor product $E \otimes F$ is embedded. He proved that for Archimedean Riesz spaces E, F and any positive bilinear map $\psi : E \times F \to H$, where H is a relatively uniformly complete Archimedean Riesz space, there is a unique increasing linear map $T : E\overline{\otimes}F \to H$ such that $T\otimes = \psi$.

In the same paper Fremlin proved that for any positive bilinear functional $B : C(X) \times C(Y) \to \mathbb{R}$ there exists a unique increasing continuous linear functional $T : C(X \times Y) \to \mathbb{R}$ such that $T(f \otimes g) = B(f, g)$ for all $f \in C(X)$, $g \in C(Y)$, by considering the partial ordering on $C(X) \otimes C(Y)$ induced by the embedding $C(X) \otimes C(Y)$ in $C(X \times Y)$ [3, Corollary 3.6.].

Zero product preserving bilinear maps were studied by Buskes and van Rooij in [2] with the term "*orthosymmetric*". For an Archimedean Riesz space E, they called the vector space-valued bilinear map $B : E \times E \to F$ as orthosymmetric if $f \wedge g = 0$ implies $B(f, g) = 0$ for all $f, g \in E$. To obtain the commutativity of f-algebras, they proved that every positive orthosymmetric bilinear operator defined on a sublattice of an f-algebra can be factored through a positive linear operator and the algebra multiplication [2].

The same authors noticed these results gave rise to the concept of the square of vector lattices given in [4] and they introduced the relation between orthosymmetric maps and squares of Riesz spaces [4]. A certain quotient of the Fremlin's Archimedean tensor product $E\overline{\otimes}E$ is also a square of E. The authors defined the notion of square of Riesz spaces and showed (via tensor products as introduced by Fremlin in [3]) the existence and uniqueness of squares (see Definitinon 3 and Theorem 4 in [4]).

In the sequel, Buskes and Kusraev proved that symmetry is necessary and sufficient condition for being orthosymmetric of a bilinear map [5]. By using the commutators, Ben Amor gave a generalization of the symmetry theorem given by Buskes and Kusraev for order bounded orthosymmetric bilinear maps in [5] to the class of orthosymmetric bilinear maps $B : X \times X \to Y$ that are continuous with respect to the relatively uniform topologies of X and Y (r.u. continuous for short)[6, Theorem 14]. A more detailed information on the orthosymmetric bilinear maps defined on vector lattices can be found in the survey paper [7].

On the other hand, some authors studied zero product preserving bilinear maps defined on product of Banach algebras and C^*-algebras to obtain a characterization for (weighted) homomorphisms and derivations (see [8, 9, 10, 11]).

Alaminos et al showed that any normed space valued zero product preserving bilinear map defined on the Cartesian product of the C^*-algebra $C(I)$ of continuous functions on a compact interval I of $\mathbb{R}$ is symmetric and this result was the main tool for their method used for the characterization of homomorphisms [8].

The same authors have obtained a class of Banach algebras A that satisfy the equality $\phi(ab,c) = \phi(a,bc)$ $(a,b,c \in A)$ for every continuous zero product preserving bilinear map $\phi : A \times A \to B$. By adding some conditions to the algebra, they have proved that $\phi(ab,c) = \phi(a,bc)$ gives a factorization for the bilinear operator ϕ as $\phi(a,b) := P(ab)$ for a certain linear map $P : A \to B$ (It is obvious that this factorization can be obtained for all unital algebras by defining $P(a) = \phi(a,1)$) and this factorization gives the symmetry of the bilinear map whenever the initial Banach algebra is commutative [9].

Alaminos et al have shown that there are some Banach algebras that do not satisfy the equality $\phi(ab,c) = \phi(a,bc)$ such as the algebra $C^1[0,1]$ of continuously differentiable functions from $[0,1]$ to $\mathbb{C}$, although the bilinear operator ϕ is zero product preserving map [10]. This shows that any bilinear operator cannot be factored through a product. In the same paper, the authors obtained that a zero product preserving bilinear map B defined on $C^1[0,1]$ can be expressed as $B(f,g) = T(fg) + S(fg') + R(f'g')$ for all $f,g \in C^1[0,1]$, where T,S,R are linear operators [10, Theorem 2.1].

More recently, same authors investigated zero product preserving bilinear maps defined on the algebra $M_n(F)(n \geqslant 2)$ of all $n \times n$ matrices over a field F of characteristic not 2 and obtained that the bilinear maps defined on $M_n(F) \times M_n(F)$ such that for any rank one idempotents f,g satisfying $fg = 0$ implies $B(f,g) = 0$ can be factored through $M_n(F)$ via a linear map and algebraic multiplication of matrices [11, Corollary 2.3].

1.2 Objective of the Thesis

As the mentioned above, the factorization of a bilinear map via a product has been investigated for Banach algebras and vector lattices. As we know, a factorization for the zero product preserving bilinear maps through a product defined on arbitrary Banach spaces has not been introduced yet.

The purpose of this study is to obtain a class of Banach valued continuous bilinear maps defined on the Cartesian product of Banach spaces that can be factored through a bilinear map, called *product*, and a linear map. Besides, we will consider the continuous bilinear maps acting on particular Banach spaces, as Banach function spaces, sequence spaces or group algebras. In these particular cases, the product will be considered as the pointwise product or convolution. In addition to these factorization results, we aim to show that the initial bilinear map inherits some properties of the linear operator factored through.

The systematic analysis of some classes of bilinear operators that factor through relevant product –pointwise product of functions and convolution of functions– will be the subject of the thesis. We will provide fundamental structure results for characterization of the general case and representation theorems for the above mentioned cases.

All these information will be used for obtaining the main obective of our work factorization theorems and integral representations of families of classical and recently introduced operators.

In this way, we will provide factorization theorems for convolution bilinear maps, integral transforms and kernel bilinear operators.

1.3 Findings

In this study, we introduce a class of continuous bilinear maps defined on the Banach spaces that can be factor through a spesific bilinear map, called *product*, typically with some special properties and being canonical in some sense. The bilinear maps having such a factorization are concidered in algebraic manner, thus we firstly give a summary of existing theorems and results in the literature. In the sequel, we establish that holding a particular summability condition is a necessary and sufficient condition for having such a factorization for a continuous bilinear map defined on a product of the Banach spaces. Following, we notice that this result can be improved if we specialize the mentioned product as pointwise product or convolution product, in parallel with the domain space of the bilinear map.

We consider these two product separately. Firstly, we obtain a factorization through point-

wise product for bilinear maps defined on a couple of Banach function spaces and sequence spaces, respectively. Secondly, we take into account the convolution operation and give a factorization of bilinear maps defined on a couple of Hilbert spaces of integrable functions, respectively, group algebras through convolution. For both cases it is seen that such a factorization is equal to zero product preserving property and it implies the symmetry condition. Using these factorizations, we investigate the compactness and summability properties of bilinear maps inherited from their linearizations under the assumption of some clasical properties for the range spaces as the cotype-related properties, the Schur property or the Dunford–Pettis property. As a result, we obtain integral representations of zero product preserving bilinear maps by using vector measures. Since this factorization allows us to linearize a class of bilinear maps, we get some applications of bilinear maps that factors through a linear map as integral transform or generalized convolution.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

Throughout the thesis, $\mathbb{K}$ represents the scalar field of all real or complex numbers. $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{T}$ denote the set of integers, natural numbers and real line mod 2π, respectively. As standard notation, the letters X, Y, Z are used to denote Banach spaces and X^* is dual space of the Banach space X with respect to its norm topology. The duality between a Banach space X and its topological dual X^* is denoted by $\langle x, f \rangle$ for $x \in X$ and $f \in X^*$. U_X and B_X represent the open and closed unit balls of the Banach space X, respectively. For a subset A of X, the $\overline{A}$ will denote the closure of A with respect to norm topology. We write χ_A and e_i to denote characteristic function for a given set A and ith unit sequence, respectively. For a positive real number p, $\ell^p(\Delta)$ is the space of all scalar valued functions ξ on Δ such that $\sum_{\gamma \in \Delta} |\xi(\gamma)|^p < \infty$. It is a Banach space with the norm $\|\xi\| = (\sum_{\gamma \in \Delta} |\xi(\gamma)|^p)^{1/p}$, for $p \geqslant 1$.

Recall that a set Λ is called *partially ordered* if there is a reflexive, antisymmetric, transitive relation $\preccurlyeq$ on the set Λ. The partially ordered set Λ is called *directed* whenever every pair of elements has an upper bound. A *net* $(x_\lambda)_{\lambda \in \Lambda}$ in a Banach space X is a function of a directed set Λ into X.

Let K be a compact set. $C(K)$ denotes the space of all continuous scalar valued functions on K. It is a Banach space with the uniform norm $\|f\|_{C(K)} = \sup_{x \in K} |f(x)|$.

The support of a real valued function $f : A \to \mathbb{R}$ is the set $\{x \in A : f(x) \neq 0\}$. A function is said to have compact support if this set is compact.

The space of all (bounded) linear operators between Banach spaces X, Y is denoted by $L(X, Y)$ $(\mathcal{L}(X, Y))$. If $Y = X$, then it will denote by $L(X)$ $(\mathcal{L}(X))$.

Since we will be concerned with bilinear maps through the thesis, we give a detailed description for them. Consider the Banach spaces X,Y,Z over the same scalar field $\mathbb{K}$. A Z-valued map $B : X \times Y \to Z$ is called *bilinear* if

$$\text{for any } y \in Y, \quad x \longmapsto B(x,y)$$

$$\text{for any } x \in X, \quad y \longmapsto B(x,y)$$

are linear maps from X to Z and Y to Z, respectively. That is, it is a map such that it is linear in each of variables. The vector space of all bilinear maps is a normed space with the norm

$$\|B\| = \sup\left\{\|B(x,y)\|_Z : (x,y) \in (B_X \times B_Y)\right\}.$$

We say a bilinear map is bounded if $\|B\| < \infty$. Similar to linear case, a bilinear map is continuous if and only if it is bounded (see [12, Proposition 1.2]).

The vector space of all (bounded) bilinear operators defined on the topological product space $X \times Y$ into the Banach space Z is denoted by $B(X \times Y, Z)$ (respectively $\mathcal{B}(X \times Y, Z)$). A bilinear continuous map is separately continuous, that is, continuous in each coordinate.

The handicap of working with the bilinear maps is the fact that the range and the null sets of a bilinear map are not linear spaces in general. As a natural consequence of this rough the well-known relations between the dimensions of the kernel and the range in the linear case do not hold necessarily in the bilinear case. The general theory of bilinear maps can be found in [12] or [13].

If the range of a linear (bilinear) operator is the field $\mathbb{K}$, then this linear (bilinear) operator is called *functional* or *form*.

Let X and Y be Banach spaces. $X \hookrightarrow^K Y$ means that $\|x\|_Y \leqslant K\|x\|_X$ for any $x \in X$, i.e. the embedding $X \subset Y$ is continuous. $X \hookrightarrow Y$ means $X \hookrightarrow^K Y$ for some $K > 0$.

The Banach spaces X and Y are said to be *(isometrically) isomorphic* if there exists a (isometric) isomorphism X onto Y, where the (isometric) isomorphism referes a (norm protect) bijective linear operator such that both it and its inverse are continuous; [14, Definitions 1.4.13].

The ***uniform boundedness principle*** or ***Banach–Steinhaus theorem*** that is one of the fundamental results in functional analysis states that if $\mathscr{A}$ is a nonempty family of bounded linear operators $T : X \to Y$ such that $\sup\{\|Tx\| : T \in \mathscr{A}\} < \infty$ for each $x \in X$, then $\sup\{\|T\| : T \in \mathscr{A}\}$ is finite ([14, Theorem 1.6.9]).

A linear operator $T : X \to Y$ is called ***(weakly) compact*** if it maps the unit ball of X onto a relatively (weakly) compact set –that is, a set having a (weakly) compact norm closure– in Y. An equivalent definition for these operators is the following; an operator is (weakly) compact if and only if, given any bounded sequence $(x_i)_{i=1}^{\infty} \in X$, the image $(T(x_i))_{i=1}^{\infty}$ has a (weakly) convergent subsequence in Y (see [14, Section 3.4]).

A (weakly) compact bilinear operator is defined similarly. We say a bilinear map $B : X \times Y \to Z$ is *(weakly) compact* if it takes the unit ball of $X \times Y$ onto a relatively (weakly) compact set in Z.

It is well known that, every linear operator with a reflexive domain or range space is weakly compact. Other useful theorem known as ***Pitt's theorem*** states that every linear operator from ℓ^q into ℓ^p is compact whenever $1 \leqslant p < q < \infty$ (see [15, Chapter 12]).

A linear operator is called ***completely continuous (or Dunford–Pettis operator)*** if it takes weakly compact sets into norm compact sets. Due to the Eberlein-Smulian Theorem this happens when it takes weakly convergent sequences to norm convergent sequences (see [15, Chapter 2]). This concept is original definition of compact operators given by Hilbert in [16], but these two definitions do not coincide exactly. Completely continuity exists between compactness and boundedness; see [14, Chapter 3].

A Banach space E is said to have the ***Dunford–Pettis property*** if each weakly compact linear operator from E into F is completely continuous for every Banach space F, that is, every weakly compact operator acting on E maps weakly compact sets to norm compact ones (see [14, Definition 3.5.15]).

An operator $T : X \to Y$ is said to be (p,q)***-summing*** if there is a constant $k > 0$ such that for every $x_1, \ldots, x_n \in X$ (regardless of the choise of the natural number n),

$$\left(\sum_{i=1}^{n} \|T(x_i)\|_Y^p\right)^{1/p} \leqslant k \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^q\right)^{1/q}.$$

This means that it images weakly q-summable sequences $(x_i)_{i=1}^{\infty} \in X$ to absolutely p-

summable sequences $(T(x_i))_{i=1}^{\infty} \in Y$. We will write $\Pi_{p,q}(X,Y)$ —$\Pi_p(X,Y)$ if $p = q$ —for the set of all (p,q)-summing operators; see [15, Chapter 10].

A characterization of a p-summing operator is given by classical ***Pietsch's domination theorem*** by virtue of a norm domination inequality. This theorem states that an operator $T : X \to Y$ is p-summing (for $1 \leqslant p < \infty$) if and only if there exists a constant c and a regular probabiliy measure μ on B_{X^*} –equipped with the compact topology $\sigma(X^*,X)$– such that the inequality $\|Tx\| \leqslant c\big(\int_{B_{X^*}} |\langle x,x^*\rangle|^p d\mu(x^*)\big)^{1/p}$ holds for each $x \in X$ [15, Theorem 2.12] (see page 10 for regular measure).

Recall that a Banach space is said to have ***Schur property*** if weakly and norm convergent sequences coincide in it. Namely, for a sequence $(x_n)_{n=1}^{\infty}$ and an element x in the space, $(x_n)_{n=1}^{\infty}$ converges weakly to x if and only if $(x_n)_{n=1}^{\infty}$ converges to x in norm ([14]). The sequence space ℓ^1 and some other spaces have the Schur property (for example, some discrete Nakano spaces, see (IV) in [17]).

The *Rademacher functions* $r_n(t)$ is defined on the interval $[0,1]$ by $r_n(t) := sign(\sin 2^n \pi t)$, and for each subinterval $\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$, where $i = 0,\ldots,2^n-1$, $n \in \mathbb{N}$ and *sign* is signum function, we get $r_n(t) := (-1)^i$ ([18, §8.5]).

A Banach space E is said to be of ***type p*** for some $p \in [1,2]$ if there exists a positive constant K so that, for every finite set of vectors $(x_k)_{k=1}^{n} \in E$

$$\left(\int_0^1 \Big\| \sum_{k=1}^{n} r_k(t) x_k \Big\|_E^2 dt\right)^{1/2} \leqslant K\left(\Big\| \sum_{k=1}^{n} x_k \Big\|_E^p\right)^{1/p}.$$

We say that E has ***cotype q*** for some $2 \leqslant q$ if there exists a positive constant K so that, for every finite set of vectors $(x_k)_{k=1}^{n} \in E$

$$\left(\Big\| \sum_{k=1}^{n} x_k \Big\|_E^q\right)^{1/q} \leqslant K\left(\int_0^1 \Big\| \sum_{k=1}^{n} r_k(t) x_k \Big\|_E^2 dt\right)^{1/2}.$$

For these definitions, see [19, Definition 1.e.12] and also [18, Section 7.7]. It is well-known that every space is of type 1 and cotype ∞, and Hilbert spaces have both type 2 and cotype 2. Moreover, infinite dimensional $C(K)$ and $L^1(\mu)$ do not have proper type/cotype, that is they do not have neither cotype $< \infty$ nor type > 1.

Now we will recall a property that is named as ***Orlicz property***, particularly for Banach spaces being of cotype 2. If a Banach space X is of cotype 2, it implies that every weak

ℓ^1 sequence in X is a strong ℓ^q sequence. In other words, identity map on X is $(q,1)$-summing; see [15, Corollary 11.17]. Note that a Banach space has the Orlicz property if it is of cotype 2; see [18, §8.9].

2.1 Measures and Spaces

Consider a set Ω. A collection Σ of subsets of the set Ω is called a *σ-algebra* (or *σ-field*) if it contains the Ω and is closed under the operations of difference and countable union. The dual (Ω,Σ) is called a *measurable space* and every element of the Σ is a *measurable set*. A measure μ on a σ-algebra Σ is a set operation such that it is extended real valued, non negative, and countably additive with the condition $\mu(\varnothing) = 0$. The triple (Ω,Σ,μ) will denote a *measure space*. The measure μ is said to be *complete* if any subset of a set E with zero measure is measurable. It is said that the measure μ is *regular* if every $E \in \Sigma$ can be approximated by a class of the open measurable sets from above (outer regularity) and a class of the compact measurable sets from below (inner regularity). A set $A \subset \Omega$ is said to be *μ-null set* if $A \in \Sigma$ and $\mu(A) = 0$. The collection of μ-null sets is denoted by $\mathscr{N}_0(\mu)$. These definititions can be found in [20].

Let Ω be a topological space. The smallest σ-algebra that contains all open sets is called the Borel σ-algebra and denoted by $\mathfrak{B}(\Omega)$. A measure defined on the σ-algebra of Borel sets is called *Borel measure* ([20, §52]). A *Radon measure* is a Borel meause which is inner regular.

Let us consider a measurable set E in a measure space, we say the measure of the set E is finite if $\mu(E) < \infty$ and E has σ-finite measure if there is a sequence $(E_n)_{n=1}^{\infty}$ of sets in Σ such that $E \subset \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all $\mathbb{N}$. A measure μ on Σ is said to be *finite (σ-finite)*, if every set E in Σ has finite (σ-finite) measure; see [20, Chapter 2].

An *atom* $A \in \Sigma$ in a measure space (Ω,Σ,μ) is a measurable set with a positive measure such that if $B \subset A$ then $\mu(B) = 0$. A measure μ with no atoms is called *non-atomic* ([20, §40]).

A function f, defined on the measurable space Ω, is called ***simple function*** if it takes a finite number of values and it can be written as $f = \sum_{j=1}^{n} \alpha_j \chi_{\mathbb{E}_j}$, where $\{\alpha_1,...,\alpha_n\}$ is a finite set of numbers and $\{\mathbb{E}_1,...,\mathbb{E}_n\}$ is a finite, disjoint class of measurable sets; see [20, §20]. We will show the set of simple functions by $\mathrm{Sim}(\Sigma)$. A function $f : \Omega \to X$ is said to be μ-*measurable* if there is a sequence $(f_n)_{n=1}^{\infty} \in \mathrm{Sim}(\Sigma)$ such that $\lim_{n\to\infty} \|f_n - f\|_X = 0$ μ-almost everywhere.

Let $f : \Omega \to X$ be a μ-measurable function. The function f is called *Bochner integrable* if there exists a sequence of simple functions $(f_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \int_{\Omega} \|f_n - f\|_X d\mu = 0$. In this case, $\int_E f d\mu$ is defined for each $E \in \Sigma$ by $\int_E f d\mu = \lim_n \int_E f_n d\mu$; see [21, Chapter II].

Let (Ω, Σ, μ) be a complete σ-finite measure space. $L^0(\mu)$ denotes the space of (equivalence classes of) all μ-measurable functions on Ω. $L^p(\mu)\,(p \geqslant 1)$ is the Banach space of functions for which the p-th power of the absolute value is μ-integrable equipped with its standard norm $\|f\| = (\int_{\Omega} |f|^p d\mu)^{1/p}$.

A Banach space $(X(\mu), \|\cdot\|)$ of (equivalence classes of) μ-measurable functions is a ***Banach function space*** —sometimes called also a ***Köthe function space***— (briefly B.f.s) over μ (or over (Ω, Σ, μ) if

(i) if $g \in X(\mu)$ and f is a measurable function such that $|f| \leqslant |g|$ $\mu-$a.e., then $f \in X(\mu)$ and $\|f\| \leqslant \|g\|$,

(ii) for all $A \in \Sigma$ with positive measure there exists $B \in \Sigma$ such that $B \subset A$, $\mu(B) > 0$ and $\chi_B \in X(\mu)$.

The assumption (ii) is equivalent to saturation property, that is, there is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ a.e. for all $f \in X(\mu)$. Since the measure space is assumed to be σ-finite, this is also equivalent to $X(\mu)$ having a weak unit, i.e. a function $g \in X(\mu)$ such that $g > 0$ a.e. (see [22]).

It is worth noting that if the measure μ is finite, the requirement (i) simply means integrability, and also that all simple functions must be contained in $X(\mu)$.

We shortly write X instead of $X(\mu)$ if the measure is clear in the context.

Since every function $f \in X$ is locally integrable by the definition of the Banach function space, it follows that for every measurable set $E \in \Sigma$, the functional $f \to \int_{\Omega} f(x)\chi_E(x) d\mu$ is an element of topological dual X^* of X. These functionals are called *integral* and the space of all integrals is denoted by X' that is known as ***Köthe dual space***, also called ***associate dual space***, of the X and $X' \subset X^*$. Namely,

$$X' = \left\{ f \in L^0(X,\mu) : \int_X |fg| d\mu < \infty \, \forall g \in X \right\}.$$

It is known that the topological dual X^* is a Banach lattice and associate dual X' is a Banach function space; see [23, Lemma 2.8(i) and Proposition 2.16].

For a linear continuous operator $T : E \to F$ between Banach function spaces E and F, T^*, respectively, T' will denote its adjoint operator, respectively, Köthe adjoint operator –the restriction of adjoint operator to Köthe dual.

A Banach function space $X(\mu)$ is ***order continuous*** (briefly *o.c.*) if downward directed nets converging μ-a.e. to 0 converge also in the norm, i.e. any $(f_\alpha)_\alpha \searrow 0$, we have $\lim_\alpha \|f_\alpha\|_{X(\mu)} = 0$ ([19, Definition 1.a.6]). If the limit is defined by sequences, it is said $X(\mu)$ is ***σ-order continuous*** (shortly *σ-o.c.*). It is shown that these two concepts coincide in Banach function spaces (see [23, Remark 2.5]). One of the important result of the order continuity is that; the order continuity of the norm of the B.f.s. $X(\mu)$ implies the density of the set $\mathrm{Sim}(\Sigma)$ (see [22, Lemma 3] or [23, Remark 2.6]). Another important characterization: a B.f.s. $X(\mu)$ has o.c. norm if and only if its Köthe and topological duals coincide, that is $X'(\mu) = X^*(\mu)$.

A Banach function space $X(\mu)$ has the ***Fatou property*** if any increasing positive sequence $(f_n)_{n\in\mathbb{N}}$ converging μ-a.e. to a measurable function f with $\sup_{n\in\mathbb{N}} \|f_n\|_{X(\mu)} < \infty$ implies $f \in X(\mu)$ and $\|f_n\|_{X(\mu)} \nearrow \|f\|_{X(\mu)}$ ([19, Section 1.b]). A B.f.s. $X(\mu)$ has Fatou property if and only if $X''(\mu) = X(\mu)$, where X'' denotes the space $(X')'$ that can be defined for every Banach function space. It is known that the Köthe dual X' of a Banach function space X has Fatou property.

Given Banach function space $E \subset L^0(\mu)$ and $p \geqslant 1$, we will denote its ***p-convexification*** by $E^{(p)}$ in the sense of [19, Section 1.d] (see also the equivalent notion of $1/p$-th power in [23, Chapter 2] for a more explicit description). Recall that, when E is a Banach function space, $E^{(p)}$ is the space of μ-measurable functions such that the pth power of its modulus belongs to E itself. That is,

$$E^{(p)} = \{ f \in L^0(\mu) : |f|^p \in E \}.$$

In this case, $E^{(p)}$ is also a Banach function space with the norm $\|f\|_{E^{(p)}} = \||f|^p\|_E^{1/p}$, for $f \in E$ (see [24, Proposition 1]).

It is known that the p-convexification $E^{(p)}$ $(0 < p < \infty)$ of E is order continuous, if E is so. In this case the set $\mathrm{Sim}(\Sigma)$ is dense in $E^{(p)}$, for $1 \leqslant p < \infty$.

In the paper [24], authors obtained the following version of well-known Hölder inequality;

Hölder-Rogers inequality: Let E be a B.f.s. and $p, q > 0$. For $x \in E^{(p)}$, $y \in E^{(q)}$, $xy \in E^{(r)}$ and $\|xy\|_r \leqslant \|x\|_p \|y\|_q$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$; see [24, Lemma 1].

The following definitions can be found in [19, Chapter 1.d.]. Let us consider a pair of Banach function spaces X and Y. A linear operator $T : X \to Y$ is said to be ***p-convex*** if there is a constant C_p such that regardless of the choise of the n and regardless of the choise of the vectors $x_1, x_2, \ldots, x_n \in X$

$$\Big\|\Big(\sum_{j=1}^n |Tx_j|^p\Big)^{1/p}\| \leqslant C_p\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}, \quad \text{if } 1 \leqslant p < \infty,$$

or

$$\Big\|\bigvee_{j=1}^n |Tx_j|\Big\| \leqslant C_\infty \max_{1 \leqslant j \leqslant n} \|x_j\|, \quad \text{if } p = \infty.$$

A linear operator $T : X \to Y$ is ***p-concave*** if there is a constant C^p such that for every choise of the elements $x_1, \ldots, x_n \in X$

$$\Big(\sum_{j=1}^n \|Tx_j\|^p\Big)^{1/p} \leqslant C^p \Big\|\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}\Big\|, \quad \text{if } 1 \leqslant p < \infty,$$

or

$$\max_{1 \leqslant j \leqslant n} \|Tx_j\| \leqslant C_\infty \Big\|\bigvee_{j=1}^n |x_j|\Big\|, \quad \text{if } p = \infty.$$

A Banach function space X is ***p-convex (p-concave)*** if the identity operator defined on X is p-convex (p-concave). Every operator and every Banach function space is 1-convex.

For a couple of Banach function spaces (E_0, E_1) and a concave function $\varphi : [0,\infty) \times [0,\infty) \to [0,\infty)$ which is positively homogeneous –that is, $\varphi(ka, kb) = k^2\varphi(a,b)$ for all $k > 0$– such that $\varphi(a,b) = 0 \Leftrightarrow a = b = 0$, the **Calderón- Lozanovskii space** $\varphi(E_0, E_1)$ generated by the couple (E_0, E_1) and the function φ is defined as all $z \in L^0(\Omega)$ such that for some $a \in E_0$, $b \in E_1$ with $\|a\|_{E_0} \leqslant 1$, $\|b\|_{E_1} \leqslant 1$ and for $\alpha > 0$ we have $|z| \leqslant \alpha\varphi(|a|,|b|)$ a.e. on Ω. The norm of an element $z \in \varphi(E_0,E_1)$ is the infimum of α satisfying the above inequality. If the power function $\varphi_\theta(a,b) = a^\theta b^{1-\theta}(0 < \theta < 1)$, $\varphi_\theta(E_0,E_1)$ is the well-known *Calderón space* $E_0^\theta E_1^{1-\theta}$ ([25]).

2.2 Group Algebras and $L^1(G)$-Modules

In this section we are going to set out the basic terminology of abstract harmonic analysis and we establish the terminology concerning group algebra $L^1(G)$ and its modules that are essential tools for Chapter 5. See the Appendix A-3 for the basic information on Banach algebras.

A *topological group* is a group with a topology such that the group operations –multiplication and inversion– are continuous. A topological group G is called *locally compact group*, respectively, *Abelian group* if it is at the same time a locally compact Hausdorff space, respectively, an Abelian group. We will use the notation *LCA* for a locally compact Abelian group.

Let (G,Σ,μ) be a measure space, where G is a topological group.

On every locally compact group G, there exists a non-zero, positive, regular (see page 10 for definition of regular measure), left-translation invariant –that is, $\mu(E+x) = \mu(E)$ for all $E \in \Sigma$ and $x \in G$– measure μ on G. This measure, called *Haar measure*, is uniquely determined up to multiplication by a positive constant. If we consider G as the circle group $\mathbb{T}$, then the Haar measure is normalized Lebesgue measure.

Let us denote the Haar measure on LCA group G by $d\mu(t)$. $L^p(G)\,(p \in [0,\infty])$ denotes the space $L^p(\mu)$ on G corresponding to Haar measure. One defines the integral of a function f on G by $\int_G f(t)d\mu(t)$ with respect to the Haar measure $d\mu(t)$. Convolution of the elements $f,g \in L^1(G)$ on G is defined by $f*g(x) = \int_G f(t^{-1}x)g(t)d\mu(t)$ and $f*g \in L^1(G)$. The Banach space $L^1(G)$ is a non-unital Banach algebra under convolution product by $\|f*g\| \leqslant \|f\|\|g\|$. These concepts can be found in references [26, 27, 28] more detailed.

A *character* of a LCA group G is a continuous group homomorphism from G to the circle group $\mathbb{T}$. We will denote by Γ the set of all *continuous* characters of the LCA group G.

The set Γ forms also a group and it is called *dual group, or Pontryagin dual* of G. By the duality between G and Γ, we denote the value of a character ξ at a point $x \in G$ by (x, ξ). The space of all *trigonometric polynomials* on G is defined as the following set;

$$\mathfrak{I}(G) = \operatorname{span}\{(\cdot, \xi) : \xi \in \Gamma\}.$$

The set $\mathfrak{I}(G)$ is an algebra and consists of all finite linear combinations of the characters of an Abelian group G.

In reference [26, pp. 204] we can find the following characterizations: if Γ is dual of G, then G is dual group of Γ. The dual of a compact group is discrete and the dual of a discrete group is compact.

The very well-known examples of LCA groups are $\mathbb{R}$, $\mathbb{T}$ and $\mathbb{Z}$. The circle group $\mathbb{T}$ –the real line mod 2π– will be essential on Chapter 5, so we establish its dual. Any character on $\mathbb{T}$ with the usual topology has the form $t \mapsto e^{-int}$ for an integer n and the dual group of $\mathbb{T}$ is the discrete group $\mathbb{Z}$; see [26, §VII]. Therefore, $\mathfrak{I}(\mathbb{T})$ consists of all functions in the form $\sum_{k=-n}^{n} \alpha_k e^{ikt}$, namely, all trigonometric polynomials in ordinary sense. By the Pontryagin duality theorem the dual group of discrete group $\mathbb{Z}$ is isomorphic to $\mathbb{T}$ ([26, §VII]).

Let G be a LCA group and $f \in L^1(G)$. The ***Fourier transform*** of f denoted by $\hat{f}$ is the map $\hat{f} : \Gamma \to \mathbb{C}$ defined by

$$\hat{f}(\xi) = \int_G f(x)\overline{\langle x, \xi \rangle} dx = \int_G f(x)\langle -x, \xi \rangle dx, \quad \xi \in \Gamma; \text{ see } [26, \S VII].$$

The Fourier transform satisfies the equalities $\widehat{(f+g)} = \hat{f} + \hat{g}$ and $\widehat{f * g} = \hat{f}\hat{g}$ ([26, §VII.4]).

Since the Haar measures on G and Γ are properly normalized, inversion formulas states that $f(-x)$ is the Fourier transform of $\hat{f} \in L^1(\Gamma)$. The inverse Fourier transform of $g \in L^1(\Gamma)$ will be denoted $\check{g}$; see [26, §VII.4].

We give below some examples of well known LCA groups with their dual groups and Fourier transforms;

$$\begin{array}{llll}
G = \mathbb{R}, & \Gamma = \mathbb{R}: & \hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-iyx} dx & (y \in \mathbb{R}), \\
G = \mathbb{T}, & \Gamma = \mathbb{Z}: & \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} dx & (n \in \mathbb{Z}), \\
G = \mathbb{Z}, & \Gamma = \mathbb{T}: & \hat{f}(e^{i\alpha}) = \int_{-\infty}^{\infty} f(n) e^{-in\alpha} dx & (e^{i\theta} \in \mathbb{T}).
\end{array}$$

The well-known ***Plancherel's Theorem*** states that Fourier transformation defines a linear isometry from $L^2(G)$ onto $L^2(\Gamma)$ and inverse Fourier transformation is an linear isometry from $L^2(\Gamma)$ onto $L^2(G)$. Moreover, the Fourier transformation and inverse Fourier transformation are inverse of each other. That is, $(\hat{f})\check{} = f$ and $(\check{g})\hat{} = g$, for $f \in L^2(G)$ and $g \in L^2(\Gamma)$, respectively. These notions with their proof can be found in references [26, 27, 28, 29].

For a compact group G, the Banach function space $L^p(G)\,(1 \leqslant p \leqslant \infty)$ with normalized Haar measure is a Banach algebra under convolution. Thus, $f * g \in L^p(G)$ and the norm satisfies $\|f * g\|_p \leqslant \|f\|_p \|g\|_p$ for all $f, g \in L^p(G)$. Moreover, the function space $C(G)$ of scalar valued continuous functions is a Banach algebra under convolution multiplication endowed with uniform norm ([29, Theorem 28.46]).

The Banach algebra $L^1(G)$ is not unital however it has a bounded approximate identity for any LCA group (see the Appendix A-3 for approximate identities). If the group G is compact, then we can construct a special approximate identity for $L^1(G)$. For a compact group G, there is a bounded left approximate identity (h_α) for $L^1(G)$ such that $h_\alpha \in \mathcal{T}^+(G)$ –that is, h_α is a positive trigonometric polynomial– and $\|h_\alpha\|_1 \leqslant 1$ for each α; see [29, Theorem 28.53].

Recall that a function ϕ defined on the group G is said to be *positive-definite*, also called *function of positive type,* if the inequality $\sum_{n,m=1}^{N} a_n \overline{a_m} \phi(x_n - x_m) \geqslant 0$ holds for every choice of $x_1, x_2, .., x_N$ of the distinct elements of G and for every choice of complex numbers $a_1, a_2, ..., a_N$ [29, Definition 32.1]. Every character is positive definite and it follows that if the coefficients are positive, then any finite linear combination of characters is so. $\mathcal{W}(G)$ will denote the complex space of functions defined on G spanned by all continuous positive-definite functions on G. It is a unital algebra of functions under pointwise operations [29, Theorem 32.10]. If G is a compact group then $\mathcal{W}(G)$ coincides with the set of functions with absolutely convergent Fourier series. For a function $f \in \mathcal{W}(G)$, we denote the norm $\|f\|_W = \|\hat{f}\|_1$, where $\hat{f}$ denotes the Fourier transform of f. For a compact Abelian group G with character group Γ, $\mathcal{W}(G)$ is the space of all functions f on G of the form $f = \sum_{n=1}^{\infty} a_n \xi_n$, where ξ_n in the dual space Γ and (a_n) is a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$ ([29, 34.13]).

The convolution of two functions in $L^2(G)$ is positive definite if G is compact Abelian, besides it is continuous. This means that the set $L^2(G) * L^2(G)$ gives the Banach algebra $\mathcal{W}(G)$; see [29, 34.16]. Consequently, for a compact Abelian group G with the character group Γ we get that $\mathcal{W}(G)$ is isomorphic to the Banach algebra $\ell_1(\Gamma)$ via the Fourier transform and any function $f \in \mathcal{W}(G)$ can be written as factorization of two functions of $L^2(G)$ (see [26] or [29, 34.34(a)]).

Particularly, we will deal with the compact Abelian group $\mathbb{T}$. It is useful to note that, for the group $\mathbb{T}$, the Banach algebra $\mathcal{W}(\mathbb{T})$ is the algebra known as Wiener algebra. It is isomorphic to the Banach algebra $\ell^1(\mathbb{Z})$ by the isomorphism given by Fourier transform and it is endowed with the norm $\|f\|_{\mathcal{W}} = \|\hat{f}\|_1$ for $f \in \mathcal{W}(\mathbb{T})$, where $\hat{f}$ denotes the Fourier transform of f.

Now we will give a characterization for Banach algebras and Banach modules, the definition of modules can be found in Appendix A-3. Remark 38.6 in [29] states that if a subalgebra $U(G)$ of the algebra $L^1(G)$ with the norm $\|.\|_{U(G)}$ is a left Banach $L^1(G)$-module with respect to convolution for a compact group G such that $\mathcal{T}(G)$ is dense in $U(G)$, then we get $L^1(G) * U(G) = U(G)$. Moreover, a left bounded approximate identity of $L^1(G)$ is also a left bounded approximate identity for $U(G)$, i.e. $\lim_\alpha \|h_\alpha * g - g\|_{U(G)} = 0$ for all $g \in U(G)$, where (h_α) is a left bounded approximate identity of $L^1(G)$.

The algebras $L^p(G)\,(1 \leqslant p < \infty)$, $C(G)$, $\mathcal{W}(G)$ enjoy the properties ascribed to $U(G)$ written above. Each of these spaces contains $\mathcal{T}(G)$ as a dense subspace and each of them is a left Banach $L^1(G)$-module with respect to convolution (see [29, Remark 38.6]).

CHAPTER 3

THE NOTION OF PRODUCT FACTORABILITY FOR BILINEAR MAPS

In this chapter, firstly we will gather the existed results both for showing the importance and currentness of the topic and for giving a general aspect to the reader. In the sequel, we introduce the notions of product and product factorability with examples and we will finish the chapter by giving a necessary and sufficient condition for product factorability.

3.1 A Brief Glance at Zero Product Preserving Maps

The notion of zero product preserving map appeared second half of the last century. This concept was introduced by Lamperti to give a complete proof of isometries on $L^p(X)$, $(1 \leqslant p \leqslant \infty, p \neq 2)$ firstly characterized by Banach for $L^p([0,1])$ in the monograph [30] without a full proof. Lamperti showed that any isometry on $L^p(X)$ must image functions with disjoint support, that is, their supports do not intersect, to functions with disjoint support ([1]). These operators called Lamperti operators gave rise to various situations and were investigated widely in literature, for instance for operators defined on vector lattices and function spaces. Recall that any elements x_1, x_2 in a vector lattice is called *disjoint* (in symbols $x_1 \perp x_2$) if $|x_1| \wedge |x_2| = 0$. In the context of the vector lattices, several authors have considered the linear operator T defined from a vector lattice to a vector lattice satisfying $Tx_1 \perp Tx_2$ whenever $x_1 \perp x_2$. Such kind of operators are called *disjointness preserving operators* or *d-homomorphisms* and their inverses with spectral properties have extensively been investigated by various authors (see [31, 32] and references therein). This notion was conveyed to the function algebras to establish a

Stone-Banach theorem by Beckenstein and Narici in [33]. They called *separating maps* the linear maps $T : C(X) \to C(Y)$ (with X and Y compact Hausdorff spaces) acting between the Banach spaces of continuous functions that satisfies the Lamperti's property. This property shows that if T is a separating map, then $fg = 0$ implies $T(f)T(g) = 0$ for the functions $f, g \in C(X)$ ([34]). Zero product preserving maps, whose typical examples are weighted composition maps, are especially studied in algebraic manner and the standard aim of the studies is to characterize them as weighted homomorphisms (see [35, 36, 37, 38]). Since zero product structure of a Banah algebra determines the full algebraic structure, zero product preserving linear operators are used as a tool for investigation of the algebraic properties. One of the known result in this direction is that two Abelian C^*-algebras are $*$-isomorphic whenever there exists a bijective zero product preserving linear map between them (see [39] and references therein). Also, Araujo and Jarosz obtained that two operator algebras are isomorphic as Banach algebras if there is a bijective zero product preserving linear map with a zero product preserving inverse [38]. In addition, compactness properties of separating maps have been investigated by some authors. Kamowitz proved that any compact separating map defined on the space of all continuous functions on a compact Hausdorff space is of finite rank [40]. Lin and Wong showed that if ones considers locally compact space, the situation become richer [41]. The necessary and sufficient conditions for the compactness of the separating maps acting on the space of all vector-valued continuous functions were given by Jamison and Rajagolalan in [42].

In recent years, this notion was exported the bilinear maps. In this study we will concern with the bilinear maps preserving zero product, so let us give the results that are found in the current literature. Note that the notions that will be mentioned below –as bimorphism, regular or positive map, relatively uniform topology– can be found in Appendix A-2.

In paper [3], Fremlin showed that given two Archimedean Riesz spaces E and F, there is an Archimedean Riesz space $E\overline{\otimes}F$, called *the Archimedean vector lattice tensor product of E and F*, in which the linear space tensor product $E \otimes F$ is embedded. He defined the following universal property to introduce this space.

Theorem 3.1 [3, Theorem 4.2] Let E, F, H be Archimedean Riesz spaces and $\psi : E \times F \to H$ be a Riesz bimorphism. Then there is a unique Archimedean Riesz space G and Riesz bimorphism $\phi : E \times F \to G$ such that

i. there is a unique Riesz homomorphism $T : G \to H$ satisfying $T\phi = \psi$,

ii. ϕ induces an embedding of $E \otimes F$ in G such that $E \otimes F$ is dense in G.

By this new tensor product space, Fremlin obtained the following factorization;

Theorem 3.2 [3, Theorem 5.3] Let E, F be Archimedean Riesz spaces and $\psi : E \times F \to H$ be a positive bilinear map, where H is a relatively uniformly complete Archimedean Riesz space. Then there is a unique increasing linear map $T : E\overline{\otimes}F \to H$ such that $T\otimes = \psi$.

This shows that for the same E, F, H, the correspondence $T \leftrightarrow T\otimes$ is an isomorphism between the vector space of the regular linear operators from $E\overline{\otimes}F$ to H and the vector space of the regular bilinear operators from $E \times F$ to H.

For the topological spaces X and Y, $C(X) \otimes C(Y)$ can be considered as a subspace of $C(X \times Y)$. The following corollary was obtained by Fremlin by considering the partial ordering on $C(X) \otimes C(Y)$ induced by the embedding $C(X) \otimes C(Y)$ in $C(X \times Y)$.

Corollary 3.1 [3, Corollary 3.6.] Let X and Y be compact Hausdorff spaces, and $B : C(X) \times C(Y) \to \mathbb{R}$ is a positive bilinear functional. Then

i. there exists a unique increasing continuous linear functional $T : C(X \times Y) \to \mathbb{R}$ such that $T(f \otimes g) = B(f, g)$ for all $f \in C(X)$, $g \in C(Y)$,

ii. consequently there is a Radon measure μ on $X \times Y$ such that $B(f, g) = \int f(t)g(s)d\mu$ for $f \in C(X)$, $g \in C(Y)$.

Zero product preserving bilinear maps defined on vector lattices were studied by Buskes and van Rooij in [2] with the term "*orthosymmetric*". For an Archimedean Riesz space E, they called the vector space-valued bilinear map $B : E \times E \to F$ as orthosymmetric if $f \wedge g = 0$ implies $B(f, g) = 0$ for all $f, g \in E$. To obtain the commutativity of f-algebras, they proved that every positive orthosymmetric bilinear operator defined on a sublattice of an f-algebra can be factored through a positive linear operator and the algebra multiplication; see [2].

Theorem 3.3 [2, Theorem 1] Let K be a compact Hausdorff space, E is a uniformly dense Riesz subspace of $C(K)$, F is an Archimedean Riesz space and B is a positive orthosymmetric bilinear map $B : E \times E \to F$. Let E^2 be the linear hull of $\{fg : f, g \in E\}$. Then there exists an increasing linear map $T : E^2 \to F$ such that $B(f, g) = T(fg)$ for all

$f, g \in C(K)$.

Corollary 3.2 [2, Corollary 2] For the Archimedean Riesz spaces E and F, any orthosymmetric positive bilinear map $B : E \times E \to F$ is symmetric, that is, $B(f,g) = B(g,f)$ for $f, g \in E$.

These results gave rise to the concept of the square of the vector lattices. The same authors noticed the relation between orthosymmetric maps and squares of Riesz spaces defined as $E^2 = \{fg : f, g \in E\}$ for a Riesz space E ([4]). A certain quotient of the Fremlin's Archimedean tensor product $E\overline{\otimes}E$ is also a square of E. The authors gave the definition of the square of vector lattices and showed, the existence and uniqueness of squares as follows by the tensor products as introduced by Fremlin;

Definition 3.1 [4, Definition 3.] The square of a Riesz space E denoted by $(E^{\odot}, \odot)$ is a Riesz space such that

i) $\odot : E \times E \to E^{\odot}$ is an orthosymmetric bimorphism.

ii) For any Riesz space valued orthosymmetric bimorphism $T : E \times E \to F$, there is a unique Riesz homomorphism $T^{\odot} : E^{\odot} \to F$ such that $T^{\odot} \circ \odot = T$.

Theorem 3.4 [4, Theorem 4.] Every Riesz space has a unique square.

Finally, Buskes and Kusraev proved that being symmetric is necessary and sufficient condition that for being orthosymmetric and they also obtained the following factorizations [5].

Theorem 3.5 [5, Theorem 3.1.] Let X be a vector lattice and H is a relatively uniformly complete vector lattice. For any orthoregular bilinear operator $B : X \times X \to H$, there is a unique regular linear operator $\phi : X^{\odot} \to H$ such that $B(f,g) = \phi(f \odot g)$.

The Fremlin's construction is essential for this theorem. To obtain the result, the authors used the commutativity of the following diagram;

$$\begin{tikzcd}
X \times X \arrow[dr, "\odot"] \arrow[drr, "B"] \arrow[dd, "\otimes"'] & & \\
& X^{\odot} \arrow[r, dotted, "\phi"] & H, \\
X\overline{\otimes}X \arrow[ur, "\psi"] \arrow[urr, "T"'] & &
\end{tikzcd}$$

where $T : X\overline{\otimes}X \to H$ is the linear map obtained by the Fremlin's factorization theorem and ψ is lattice homomorphism implementing an isomorphism of $X\overline{\otimes}X$ and $X^{\odot}$.

By using the commutators, Ben Amor gave a generalization of the symmetry theorem given for order bounded orthosymmetric bilinear maps by Buskes and Kusraev in [5] to the class of orthosymmetric bilinear maps $B : X \times X \to Y$ that are continuous with respect to the relatively uniform topologies of X and Y (r.u. continuous for short).

Theorem 3.6 [6, Theorem 14] Let X and Y be Archimedean vector lattices. Any r.u. continuous orthosymmetric bilinear map $B : X \times X \to Y$ is symmetric.

The symmetry condition of a positive orthosymmetric bilinear operator defined on Archimedean Riesz spaces was proved using analitic methods by Buskes and van Rooij [2]. For this reason, Chil considered the question by an algebraic approach and he proved that for Archimedean vector lattices X and Y, any orthosymmetric lattice bimorphism $B : X \times X \to Y$ is symmetric [43, Theorem 3].

On the other hand, some authors studied zero product preserving bilinear maps defined on product of Banach algebras and C^*-algebras to obtain a characterization of (weighted) homomorphisms and derivations (see [8, 9, 10, 11]).

Alaminos et al gave the following symmetry theorem for the zero product preserving bilinear maps defined on the Cartesian product of the C^*-algebra $C(I)$ of continuous functions on an interval I that was the main tool for their method used for the characterization of homomorphism.

Theorem 3.7 [8, Lemma 2.1] Let Y be a normed space and let I be a compact interval of $\mathbb{R}$. Consider a bounded bilinear map $B : C(I) \times C(I) \to Y$ in which $fg = 0$ implies $B(f,g) = 0$. Then B is symmetric for all $f,g \in C(I)$.

Same authors of the paper [8] investigated the zero product preserving bilinear maps to show that there is a close relation with Lamperti maps –zero product preserving *linear*– and homomorphisms. It is already clear that any homomorphism $T : A \to B$ is a Lamperti map where A and B are Banach algebras, since $ab = 0$ implies $T(a)T(b) = T(ab) = 0$ for all $a,b \in A$. Their aim was to obtain the inverse: how close is a zero product preserving linear map defined on Banach algebras to being a homomorphism? To get the answer, they considered zero product preserving bilinear maps and proved the following factorization and symmetry theorems.

Theorem 3.8 [9, Lemma 2.3] Let X be a Banach algebra with a bounded left approximate identity and let $B : X \times X \to Y$ be a zero product preserving bilinear map satisfying $B(f, gh) = B(fg, h)$ for all $f, g, h \in X$. Then there is a continuous linear map $T : X^2 \to Y$ such that B is of the form $B(f, g) = T(fg)$.

Theorem 3.9 [9, Corollary 2.4] Let X be a commutative Banach algebra with a bounded left approximate identity. Any zero product preserving bilinear map $B : X \times X \to Y$ satisfying $B(f, gh) = B(fg, h)$ is symmetric.

It is worth saying that the authors obtained a class of Banach algebras X that for any bilinear map B defined on $X \times X$ to Y, the zero product preserving property implies $B(f, gh) = B(fg, h)$ for all $f, g, h \in X$. If we consider a unital Banach algebra X with this property, we can define a linear map $T : X \to Y$ for any bilinear map $B : X \times X \to Y$ by $B(a, b) = T(ab)$. Indeed, $B(a, b) = B(a, b\mathbf{1}) = B(ab, \mathbf{1}) = T(ab)$, where $\mathbf{1}$ denotes the unital element of X. This class of Banach algebras is rather large, it includes group algebras, C^*-algebras and Banach algebras generated by idempotents (see [9]).

The algebra $C^1[0, 1]$ of continuously differentiable functions from the interval $[0, 1]$ to $\mathbb{C}$ can be given as an example of Banach algebra that is not included the mentioned class of algebras. This means that there exists a zero product preserving bilinear map $B : C^1[0, 1] \times C^1[0, 1] \to Y$ which does not hold the equality $B(f, gh) = B(fg, h)$ for all $f, g, h \in C^1[0, 1]$. Indeed, if we consider the bilinear map $B : C^1[0, 1] \times C^1[0, 1] \to C[0, 1]$ defined by $B(f, g) = fg'$, where g' denotes the derivative of the function g, it is seen that this map is zero product preserving but $B(f, gh) \neq B(fg, h)$ ([10]). This shows that it is hard to find a general form for zero product preserving bilinear maps. They obtained the following characterization for zero product preserving bilinear maps defined on $C^1[0, 1]$.

Theorem 3.10 [10, Theorem 2.1] A Banach valued zero product preserving bilinear map $B : C^1[0, 1] \times C^1[0, 1] \to Y$ can be expressed as $B(f, g) = T(fg) + S(fg') + R(f'g')$ for all $f, g \in C^1[0, 1]$, where T, S, R are linear operators such that $T : C^1[0, 1] \to Y$ and $R, S : C[0, 1] \to Y$.

In the paper [11], Alaminos et al obtained a factorization for zero product preserving bilinear maps defined on the algebra $M_n(F)(n \geqslant 2)$ of all $n \times n$ matrices over a field F of characteristic not 2, that is $2 \cdot \mathbf{1_F} \neq \mathbf{0_F}$, where $\mathbf{1_F}$ and $\mathbf{0_F}$ denote the multiplicative and additive identity of the algebra $M_n(F)$.

Theorem 3.11 [11, Corollary 2.3] Let $B : M_n(F) \times M_n(F) \to Y$ be a bilinear map such that for any rank one idempotents f, g satisfying $fg = 0$ we have $B(f,g) = 0$. Then the map B has the form $B(fg) = T(fg)$ for all $f, g \in M_n(F)$, where $T : M_n(F) \to Y$ is a linear operator.

3.2 A Generic Map: Products and Properties

Let us introduce some definitions that will be used in the rest of the study. We will use the term *product* for a generic continuous bilinear map having some specific properties that will play a central role in each fixed situation of the ones that follows.

Definition 3.2 Consider a bilinear operator $\circledast : X \times Y \to Z$, $(x,y) \rightsquigarrow \circledast(x,y) =: x \circledast y$, where X, Y, Z are Banach spaces. We say that the bilinear operator $\circledast$ is a *norming product* if it satisfies the inclusion $U_Z \subseteq \circledast(U_X \times U_Y)$.

Definition 3.3 We say that a bilinear operator $\circledast : X \times Y \to Z$, $(x,y) \rightsquigarrow \circledast(x,y) = x \circledast y$ is a *norm preserving product* (*n.p. product* for short) if it is a norming product and satisfies

$$\|x \circledast y\|_Z = \inf\{\|x'\|_X \|y'\|_Y : x' \in X, y' \in Y, x \circledast y = x' \circledast y'\}, \text{ for every } (x,y) \in X \times Y.$$

Example 3.1 Let E, F be normed spaces and $E \otimes F$ denote their algebraic tensor product. Projective norm π and injective norm ε on $E \otimes F$ are calculated by

$$\pi(z) = \inf\Big\{ \textstyle\sum_{i=1}^n \|x_i\| \|y_i\| : z = \sum_{i=1}^n x_i \otimes y_i \Big\},$$

and

$$\varepsilon(z) = \sup\Big\{ \langle x' \otimes y', z \rangle : x' \in B_{E'}, y' \in B_{F'} \Big\},$$

respectively (see Appendix B). It is well-known that $\varepsilon \leqslant \pi$. By definition, any *reasonable* tensor norm α on the tensor product $E \otimes F$ satisfies the inequality $\varepsilon \leqslant \alpha \leqslant \pi$. For every $(x,y) \in E \times F$, it is seen that by the definitions of these norms

$$\begin{aligned}\varepsilon(x \otimes y) \leqslant \alpha(x \otimes y) \leqslant \pi(x \otimes y) &= \inf\Big\{ \sum_{i=1}^n \|x_i\| \|y_i\| : x \otimes y = \sum_{i=1}^n x_i \otimes y_i \Big\} \\ &\leqslant \inf\{\|x'\| \|y'\| : x' \otimes y' = x \otimes y\}.\end{aligned}$$

Besides, for every elemantary tensor $x \otimes y$ it is known that for any reasonable tensor norm

α we have $\alpha(x \otimes y) = \|x\|_E \|y\|_F$ (see [18, Section 12]). Then, any reasonable tensor norm satisfies the equality involving the norm in Definition 3.3. But the tensor product does not satisfy the norming property, since clearly it is not surjective. So, it is not a norm preserving product.

Example 3.2 Let us define the following seminorm on $X \otimes \mathcal{L}(X,Y)$. If $z = \sum_{j=1}^{n} x_j \otimes T_j$ is such that $\sum_{j=1}^{n} T_j(x_j) = y_z \in Y$, we define

$$\pi_\bullet(z) = \inf\left\{\pi(z') : z' = \textstyle\sum_{j=1}^{m} x_j' \otimes T_j', \text{ such that } \sum_{j=1}^{m} T_j'(x_j') = y_z\right\}.$$

That is, $\pi_\bullet$ is the quotient norm given by the tensor contraction c associated to the following factorization.

$$\begin{tikzcd} X \times \mathcal{L}(X,Y) \arrow[d, "\otimes"'] \arrow[dr, dotted, "\bullet"] & \\ X \hat{\otimes}_\pi \mathcal{L}(X,Y) \arrow[r, "c"] & Y. \end{tikzcd}$$

The description of this seminorm can be found in [44]. It defines a norm if we construct a quotient space $X \hat{\otimes}_{\pi_\bullet} \mathcal{L}(X,Y)$ by identifying the equivalence classes of the projective tensor product $X \hat{\otimes}_\pi \mathcal{L}(X,Y)$ with the range of c in Y, i.e. $\bullet(X \hat{\otimes}_\pi \mathcal{L}(X,Y)) \subset Y$. Thus, for $z = \sum_{j=1}^{n} x_j \otimes T_j$ and $z' = \sum_{j=1}^{m} x_j' \otimes T_j'$, $z \sim z'$ if and only if $\sum_{j=1}^{m} T_j(x_j') = \sum_{j=1}^{n} T_j(x_j)$. The norm of a class $[z] = \{z' : z \sim z'\}$, for $z = \sum_{j=1}^{n} x_j \otimes T_j$, is given by

$$\pi_\bullet(z) = \inf\{\pi(z') : z \sim z'\}.$$

Let us show that $\bullet$ is a norm preserving product.

Fix $T \in \mathcal{L}(X,Y)$ and $x \in X$ and consider $y_z = T(x)$; clearly the inequality $\|y_z\| \leqslant \|T\| \|x\|$ holds. Now, consider another tensor $z = \sum_{i=1}^{n} x_i \otimes T_i$ such that $y_z = \sum_{i=1}^{n} T_i(x_i)$. Since $\|y_z\| = \|\sum_{i=1}^{n} T_i(x_i)\| \leqslant \sum_{i=1}^{n} \|T_i\| \|x_i\|$, we obtain that $\|x \bullet T\| = \|y_z\| \leqslant \pi_\bullet(z)$.

In the opposite direction, for $y \in Y$ there are elements $T_0 \in \mathcal{L}(X,Y)$ and $x_0 \in X$ such that $T_0(x_0) = y$ and $\|y\| = \|T_0\| \|x_0\|$. To see this, just take a couple (x_0, x_0^*) of norm one elements $x_0 \in X$ and $x_0^* \in X^*$ such that $\langle x_0, x_0^* \rangle = 1$. Now define $T_0(x) := \langle x, x_0^* \rangle y$, $x \in X$, and note that $\|T_0\| = \|y\|$. Therefore, if $z = x_0 \otimes T_0$, we have that $y = y_z$. So, this gives in particular that $U_Y \subseteq \bullet(U_X \times U_{\mathcal{L}(X,Y)})$, since $\pi_\bullet(z) \leqslant \|y_z\|$. Together with the inequality in the previous paragraph this also gives $\|x_0 \bullet T_0\| = \|y_z\| = \pi_\bullet(z)$. More precisely, we have proven that

$$\|x \bullet T\|_Y = \inf\{\|x_0\|_X \|T_0\|_{\mathcal{L}(X,Y)} : x_0 \in X,\, T_0 \in \mathcal{L}(X,Y),\, x \bullet T = x_0 \bullet T_0\}$$

for all $T \in \mathcal{L}(X,Y)$ and $x \in X$. Thus, $\bullet$ is a norm preserving product.

Example 3.3 A non zero Banach algebra A endowed with the norm $\|.\|$ is called *absolute-valued* if the product $x, y \to xy$ satisfies $\|xy\| = \|x\|\|y\|$ for all x, y. Most of the classical Banach spaces such as c_0 and ℓ^p are absolute-valued under suitable products (see [45] and references therein). By the definition, the product operation of an absolute-valued algebra satisfies norm property given in Definition 3.3, however it does not need to be norming. But we can find a class of absolute-valued Banach algebras whose multiplication operation is an n.p. product. A Banach algebra A is said to factor weakly if $A = A^2$. Thus, the multiplication operation of any algebra factoring weakly is a norming product. As a result, if A is absolute-valued and factors weakly, then the algebra multiplication is an n.p. product. Note that the well-known Cohen factorization theorem states that any Banach algebra with a bounded left approximate identity factors weakly. Therefore, for an absolute-valued Banach algebra with a bounded left approximate identity, the multiplication $x, y \to xy$ is a norm preserving product.

Absolute-valued algebras was introduced by Ostrowski in 1918 and then this notion exported to Banach spaces by the name *absolute-valuable*. A Banach space X is called *absolute-valuable* if there is a bilinear map $\diamond$, called a *product*, $x \times y \to x \diamond y$ on X satisfying $\|x \diamond y\| = \|x\|\|y\|$ for every x, y. Becerra et al showed that every infinite dimensional Hilbert space are absolute valuable and also the classical Banach spaces c_0 and $\ell^p\,(1 \leqslant p \leqslant \infty)$ are so as mentioned above; see [46, Theorem 2.3 and Corollary 2.5]. However, the Banach space c of convergent sequences is an example for non-absolute-valuable space ([46, Proposition 2.8]).

Let A be a Banach space that is absolute-valuable for the product $\diamond$ satisfying surjectivity and ϕ_1, ϕ_2, ϕ_3 are any isometries from A onto A. The space A is again an absolute-valuable space with respect to a new product $\triangle$ defined by $a \triangle b = \phi_3^{-1}(\phi_1(a) \diamond \phi_2(b))$; see [47]. Indeed, A is absolute-valuable with respect to $\triangle$, that is seen by the isometries and the absolute-valuability of A with respect to the product $\diamond$ as follows;

$$\|a \triangle b\| = \|\phi_3^{-1}(\phi_1(a) \diamond \phi_2(b))\| = \|\phi_1(a) \diamond \phi_2(b)\| = \|\phi_1(a)\|\|\phi_2(b)\| = \|a\|\|b\|.$$

Moreover, this new product is also surjective. To show the surjectivity, assume that it is not onto. Then there is not a couple $(a,b) \in A \times A$ satisfiying $z = a \triangle b = \phi_3^{-1}(\phi_1(a) \diamond \phi_2(b))$ for at least one element $z \in A$. That implies the equality $\phi_3(z) = \phi_1(a) \diamond \phi_2(b)$ does not hold. Since the ϕ_1, ϕ_2, ϕ_3 are onto isometries, it follows that there exists an element $z' = \phi_3(z)$ such that it cannot be obtained as a product with respect to $\diamond$. This contradicts the surjectivity of the $\diamond$.

Since to find the factors of a function space is a current problem in the mathematical literature, there are found many examples of the norm preserving products including the Lorentz and Cesàro function spaces (see [48, 49, 50, 51, 52, 53, 54]). We will already mention and use some of them in Chapter 4, now we will give an example for Cesàro function spaces.

Example 3.4 The Cesàro function spaces $Ces^p = Ces(I) = Ces^p([0,\infty))(1 \leqslant p \leqslant \infty)$ are classes of all Lebesgue measurable functions f such that

$$\|f\|_{Ces^p} = \Big[\int_I \Big(\frac{1}{t}\int_0^t |f(x)|dx\Big)^p\Big]^{1/p} < \infty \text{ for } 1 \leqslant p < \infty,$$

and

$$\|f\|_{Ces^\infty} = \sup_{t \in I, t>0} \frac{1}{t}\int_0^t |f(x)|dx < \infty \text{ for } p = \infty$$

(see [49] and references therein). The dual space $(Ces^p)^* = (Ces^p)'$ of the Cesàro function space Ces^p is defined by the norm $\|f\|_{(Ces^p)'} = \|\tilde{f}\|_{L^{p'}}$, where $\tilde{f}(x) = \mathrm{esssup}_{t \in [x,\infty)}|f(t)|$ and $1/p + 1/p' = 1$. The p-convexification $(Ces^\infty)^{(p)}(1 \leqslant p < \infty)$ of the space Ces^∞ is the space with the norm

$$\|f\|_{(Ces^\infty)^{(p)}} = \||f|^p\|_{Ces^\infty}^{1/p} = \sup_{t \in I, t>0} \big(\frac{1}{t}\int_0^t |f(x)|^p dx\big)^{1/p}.$$

Proposition 1 in [48] asserts that $Ces^p = L^p \cdot (Ces^\infty)^{(p')}$ and

$$\|f\|_{Ces^p} = \inf\{\|g'\|_{L^p}\|h'\|_{(Ces^\infty)^{(p')}} : g' \in L^p, h' \in (Ces^\infty)^{(p')}, g \cdot h = g' \cdot h'\}.$$

This implies that $\odot : L^p \times (Ces^\infty)^{(p')} \to Ces^p$ is an n.p. product.

Again, Proposition 2 in [48] indicates that $(Ces^p)' \cdot (Ces^\infty)^{(p)} = L^p$ and

$$\|f\|_{L^p} = \inf\{\|g'\|_{(Ces^p)'}\|h'\|_{(Ces^\infty)^{(p)}} : g' \in (Ces^p)', h' \in (Ces^\infty)^{(p)}, g \cdot h = g' \cdot h'\}.$$

Thus we get that $\odot : (Ces^p)' \times (Ces^\infty)^{(p)} \to L^p$ is an n.p. product.

3.3 Product-Factorable Bilinear Maps in Banach Spaces

Now, we state our fundamental tools. Using the terminology coming from Banach algebras and vector lattices, we will define *zero product preserving* bilinear maps.

Definition 3.4 We say that a bilinear map $B : X \times Y \to Z$ is *zero product preserving* (shorthly *zpp*) with respect to the norming product $\circledast$ if

$$x \circledast y = 0 \quad \text{implies} \quad B(x,y) = 0$$

for all $(x,y) \in X \times Y$.

Example 3.5 A bilinear continuous map $B : A \times A \to Z$ defined on the product of an absolute-valued algebra A is always a zero product preserving map. Indeed, by the boundedness of the map B, it follows that if $xy = 0$, then

$$\|B(x,y)\|_Z \leqslant \|B\|\|x\|\|y\| = 0$$

since $\|xy\| = \|x\|\|y\| = 0$. This holds even without the boundedness of the operator B, due to

$$xy = 0 \Leftrightarrow \|xy\| = 0 \Leftrightarrow \|x\|\|y\| = 0 \Leftrightarrow x \text{ or } y \text{ is zero.}$$

This gives $B(x,y) = 0$ whenever $xy = 0$.

A bilinear operator $B : A \times A \to Z$ defined on the Cartesian product of absolute-valuable Banach space A with respect to multiplication $\diamond$ is zero product preserving for the product $\diamond$ if and only if it is zero product preserving for any product $\triangle$ defined in Example 3.3 by the isometries. It is seen from the equality $a \diamond b = 0 \iff a \triangle b = 0$, since $\|a \triangle b\| = \|a\|\|b\| = \|a \diamond b\|$.

Definition 3.5 A Banach valued continuous bilinear operator $B : X \times Y \to Z$ will be called $\circledast$*-factorable* through the norming product $\circledast : X \times Y \to G$ if it can be written as $B = T \circ \circledast$ for a linear bounded map $T : G \to Z$.

Lemma 3.1 A bilinear operator B is $\circledast$-factorable through the norming product $\circledast$ if and only if there is a constant K such that for all $x_1, x_2, \ldots, x_n \in X$ and $y_1, y_2, \ldots, y_n \in Y$, we have

$$\|\sum_{i=1}^{n} B(x_i, y_i)\|_Z \leqslant K \|\sum_{i=1}^{n} x_i \circledast y_i\|_G. \tag{3.1}$$

In this case, the following triangular diagram commutes;

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[dr, dotted, "\circledast"'] & Z \\ & G. \arrow[u, "T"'] \end{tikzcd}$$

Proof. If the map B is $\circledast$-factorable with the product $\circledast$, then by definition of factorability we have the factorization $B = T \circ \circledast$ where T is a linear continuous operator. By the continuity and the linearity of the operator T, it follows that

$$\begin{aligned} \|\sum_{i=1}^{n} B(x_i, y_i)\|_Z &= \|\sum_{i=1}^{n} T \circ \circledast(x_i, y_i)\|_Z = \|\sum_{i=1}^{n} T(x_i \circledast y_i)\|_Z \\ &= \|T(\sum_{i=1}^{n} x_i \circledast y_i)\|_Z \leqslant \|T\| \|\sum_{i=1}^{n} x_i \circledast y_i\|_G. \end{aligned}$$

For the converse, define the map $T : X \circledast Y \to Z$ such that

$$T\big(\sum_{i=1}^{n} x_i \circledast y_i\big) = \sum_{i=1}^{n} B(x_i, y_i) = B_L\big(\sum_{i=1}^{n} x_i \otimes y_i\big)$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is a tensor in the projective tensor product space $X \hat{\otimes}_\pi Y$ and B_L denotes linearization of B from $X \hat{\otimes}_\pi Y$ to Z (see Appendix B). To prove that the mapping T is well defined, let us assume that $\sum_{i=1}^{n} x_i \circledast y_i = 0$. Then $\|\sum_{i=1}^{n} x_i \circledast y_i\|_G = 0$ and by the inequality (3.1) we have $\|\sum_{i=1}^{n} B(x_i, y_i)\|_Z = 0$. This shows that $T(\sum_{i=1}^{n} x_i \circledast y_i) = 0$, then the mapping T is well defined. The map T is linear since it is defined by the linear operator B_L. Finally, the inequality (3.1) gives the boundedness of the linear map T as follows;

$$\|T\big(\sum_{i=1}^{n} x_i \circledast y_i\big)\|_Z = \|\sum_{i=1}^{n} B(x_i, y_i)\|_Z \leqslant K \|\sum_{i=1}^{n} x_i \circledast y_i\|_G.$$

$\square$

Definition 3.6 A subset $A \subseteq X \times Y$ is called $\circledast$-*(relatively) compact*, respectively, $\circledast$-*(relatively) weakly compact* if the set $\{x \circledast y : (x,y) \in A\}$ is (relatively) compact, respectively, (relatively) weakly compact.

Definition 3.7 We will say that a bilinear map $B : X \times Y \to Z$ is *equivalently zero product preserving*

$$\text{if } x \circledast g = 0 \text{ if and only if } B(x,y) = 0 \text{ for every } x,y \in X \times Y.$$

Finally, recall the definition of symmetric bilinear map. As usual, we will say a bilinear map $B : X \times X \to Y$ is *symmetric* if it satisfies $B(x,y) = B(y,x)$ for all $x,y \in X$.

Notice that there is a relation between $\circledast$-factorability and symmetry or compactness of a bilinear map.

Indeed, let us consider a $\circledast$-factorable bilinear map $B : X \times X \to Y$ with respect to *commutative* norming product $\circledast$. By the $\circledast$-factorability of the map B and commutativity of the product, we get $B(x,y) = T(x \circledast y) = T(y \circledast x) = B(y,x)$.

Consider a $\circledast$-factorable bilinear map $B : X \times Y \to Z$ to give a characterization for the (weak) compactness, therefore the following is seen

$$\begin{aligned}
\circledast\text{-factorable map } B \text{ is (weakly) compact} &\Longleftrightarrow B(U_X \times U_Y) \text{ is relatively (weakly) compact} \\
&\Longleftrightarrow T(U_X \circledast U_Y) \text{ is relatively (weakly) compact} \\
&\Longleftrightarrow T(U_G) \text{ is relatively (weakly) compact} \\
&\Longleftrightarrow T \text{ is (weakly) compact.}
\end{aligned}$$

This shows that the (weak) compactness of a $\circledast$-factorable bilinear map is possible only with the (weak) compactness of the linear map that appears in the factorization.

CHAPTER 4

FACTORABILITY THROUGH POINTWISE PRODUCT

In this chapter we will center our attention in the case when the product $\circledast$ is the pointwise product $\odot$ among Banach function spaces and sequence spaces. It is useful to recall that the pointwise product $\odot$ is defined as $f \odot g = f(x) \cdot g(x)\,(\forall x)$, respectively, $a \odot b = (a_n \cdot b_n)\,(\forall \mathbb{N})$ for the functions f, g, respectively, the sequences $a = (a_n)$, $b = (b_n)$. In the case of non-atomic measures (see page 10) the pointwise product can be changed by the μ-almost everywhere pointwise product in the usual manner. Notice that this product is commutative and associative. Together with the specific structure of the Banach function spaces and sequence spaces, pointwise product will allow us to improve the basic characterization of $\circledast$-factorable operators given by Lemma 3.1.

Note that the results given in Section 4.1 were accepted for publishing in Positivity, see [55].

4.1 Product Factorability of Symmetric Operators on Function Spaces

Now, we will give a factorization for zero product preserving bilinear maps acting on Banach function spaces via pointwise product and we will show that Lemma 3.1 gives rise to characterize the family of $\odot$-factorable operators as the class of *symmetric operators* defined below. Moreover, this class of operators coincides with the class of zero product preserving bilinear maps acting in B.f.s.

The reader can find —versions of— the definition of symmetric operators in different articles. We follow the one given below, introduced in [56].

Definition 4.1 [56] Let $X(\mu)$, $Y(\mu)$ and $Z(\mu)$ be Banach function spaces over the (σ-

finite) measure μ. A continuous bilinear map $B : X(\mu) \times Y(\mu) \to Z(\mu)$ is called symmetric if the equality $B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C})$ is satisfied for every $A, C \in \Sigma$.

Remark 4.1 It is clearly seen that a symmetric bilinear map B satisfies $B(\chi_A, \chi_C) = B(\chi_C, \chi_A)$ for every $A, C \in \Sigma$, since

$$B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C}) = B(\chi_{C\cap A}, \chi_{C\cup A}) = B(\chi_C, \chi_A).$$

The inverse is not true in general. To show that consider the bilinear continuous form $B : L^1(\mu) \times L^1(\mu) \to \mathbb{R}$ defined by $B(f, g) = \int f d\mu \cdot \int g d\mu$ for all $f, g \in L^1(\mu)$. It holds $B(\chi_A, \chi_C) = B(\chi_C, \chi_A)$, indeed

$$B(\chi_A, \chi_C) = \mu(A) \cdot \mu(C) = \mu(C) \cdot \mu(A) = B(\chi_C, \chi_A).$$

However, it does not satisfy the equality $B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C})$, since in general

$$B(\chi_A, \chi_C) = \mu(A) \cdot \mu(C) \neq \mu(A \cap C) \cdot \mu(A \cup C) = B(\chi_{A\cap C}, \chi_{A\cup C}).$$

As a result, the above remark shows that Definition 4.18 is not equivalent to the usual symmetry condition $B(f, g) = B(g, f)$ for all $f, g \in X$, where $B : X \times X \to Z$ is a bilinear continuous operator.

Theorem 4.1 Let (Ω, Σ, μ) be a $\sigma-$finite measure space and let $X(\mu), Y(\mu)$ be B.f.s. over μ such that the set Sim(Σ) of simple functions is dense in both $X(\mu)$ and $Y(\mu)$. Let B be a continuous bilinear map $X(\mu) \times Y(\mu) \to E$, where E is a Banach space. Suppose that there is a Banach function space $G(\mu)$ such that the pointwise product $\odot : X(\mu) \times Y(\mu) \to G(\mu)$ is an n.p product. Then the following assertions are equivalent.

(1) B is a symmetric operator.

(2) B is $\odot$-factorable, that is, there is a continuous linear operator $R : G(\mu) \to E$ such that $B = R \circ \odot$.

(3) For all $f_1, \ldots, f_n \in X(\mu)$ and $g_1, \ldots, g_n \in Y(\mu)$ there exists a positive real number K such that

$$\Big\| \sum_{i=1}^{n} B(f_i, g_i) \Big\|_E \leqslant K \Big\| \sum_{i=1}^{n} f_i \odot g_i \Big\|_{G(\mu)}. \tag{4.1}$$

(4) The operator B is zero product preserving. That is, $B(f, g) = 0$ whenever $f \odot g = 0$

for all $f \in X(\mu)$ and $g \in Y(\mu)$.

Proof. Let us assume that $B : X(\mu) \times Y(\mu) \to E$ is a symmetric operator. Thus, we get $B(\chi_A, \chi_C) = 0$ whenever $\mu(A \cap C) = 0$. Indeed, $B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C}) = 0$ holds for a bilinear map under the assumption $\mu(A \cap C) = 0$.

Since Ω is $\sigma-$finite, then there exists a sequence $(E_k)_{k\in\mathbb{N}}$ in Σ such that $\Omega = \bigcup_{k=1}^{\infty} E_k$ and $\mu(E_k) < \infty$ for all $k \in \mathbb{N}$. Let us define the sequence of increasing sets $Y_m = \bigcup_{k=1}^{m} E_k$. Consider a couple of simple functions $f = \sum_{i=1}^{p} \lambda_i \chi_{A_i}$ and $g = \sum_{j=1}^{r} \gamma_j \chi_{C_j}$, where (A_i) and (C_j) are sequences of pairwise disjoint measurable sets. Definition of the simple functions gives rise to define a common partition for each couple (f, g) of simple functions. Let us rewrite them by a common partition $f = \sum_{i=1}^{r} \lambda_i \chi_{D_i}$ and $g = \sum_{j=1}^{r} \gamma_j \chi_{D_j}$, where (D_i) is the sequence of pairwise disjoint measurable sets.

By the properties of a characteristic function, the pointwise product of a simple function f and χ_{Y_m} is obtained as $f \odot \chi_{Y_m} = \sum_{i=1}^{r} \lambda_i \chi_{D_i} \odot \chi_{Y_m} = \sum_{i=1}^{r} \lambda_i (\chi_{D_i} \odot \chi_{Y_m}) = \sum_{i=1}^{r} \lambda_i \chi_{D_i \cap Y_m}$. For every $m \in \mathbb{N}$, let us define the bilinear operator $B_m : X(\mu) \times Y(\mu) \to E$, $B_m(f, g) = B(f \odot \chi_{Y_m}, g \odot \chi_{Y_m})$. Then $(B_m)_{m\in\mathbb{N}}$ is a sequence of well-defined, continuous, bilinear maps. The symmetry and bilinearity properties of the operator B give the following equality with some set operations;

$$
\begin{aligned}
B_m(f,g) &= B(f \odot \chi_{Y_m}, g \odot \chi_{Y_m}) \\
&= B\Big(\sum_{i=1}^{r} \lambda_i \chi_{D_i\cap Y_m}, \sum_{j=1}^{r} \gamma_j \chi_{D_j \cap Y_m}\Big) \\
&= \sum_{i=1}^{r}\sum_{j=1}^{r} \lambda_i \gamma_j B(\chi_{D_i\cap Y_m}, \chi_{D_j\cap Y_m}) \\
&= \sum_{i=1}^{r} \lambda_i \gamma_i B(\chi_{D_i\cap Y_m}, \chi_{D_i\cap Y_m}) \\
&= \sum_{i=1}^{r} \lambda_i \gamma_i \Big[B(\chi_{D_i\cap Y_m}, \chi_{D_i\cap Y_m}) + B(\chi_{D_i\cap Y_m}, \chi_{Y_m \setminus D_i}) + B(\chi_{D_i\cap Y_m}, \chi_{(D_i\cap Y_m)\cap Y_m\setminus D_i})\Big] \\
&= \sum_{i=1}^{r} \lambda_i \gamma_i \big[B(\chi_{D_i\cap Y_m}, \chi_{Y_m})\big] \\
&= B(\sum_{i=1}^{r} \lambda_i \gamma_i \chi_{D_i\cap Y_m}, \chi_{Y_m}) = B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m}).
\end{aligned}
$$

Thus, $B_m(f, g) = B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m})$ holds for every couple of simple functions.

Now let us show $B_m(f,g) = B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m})$ holds for the elements of $f \in X(\mu)$ and $g \in Y(\mu)$ which are not simple function. By density of Sim(Σ), there are sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ of simple functions such that $f = \lim_{n\to\infty} f_n$ and $g = \lim_{n\to\infty} g_n$. Applying the separate continuity of both the bilinear map B and the product $\odot$, we obtain

$$\begin{aligned}
B_m(f,g) &= B(\lim_{n\to\infty} f_n \odot \chi_{Y_m}, \lim_{n\to\infty} g_n \odot \chi_{Y_m}) \\
&= \lim_{n\to\infty} B(f_n \odot \chi_{Y_m}, g_n \odot \chi_{Y_m}) \\
&= \lim_{n\to\infty} B(f_n \odot g_n \odot \chi_{Y_m}, \chi_{Y_m}) \\
&= B(\lim_{n\to\infty} f_n \odot \lim_{n\to\infty} g_n \odot \chi_{Y_m}, \chi_{Y_m}) \\
&= B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m}).
\end{aligned}$$

Therefore, we get that for every $m \in \mathbb{N}$ the bilinear operator B_m can be written as $B_m(f,g) = B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m})$ for all $f \in X(\mu)$, $g \in Y(\mu)$.

Now define the map $R_m : G(\mu) \to E$ by $R_m(h) = B_m(f,g) = B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m})$ for every function $h = f \odot g$ and every $m \in \mathbb{N}$. $(R_m)_{m\in\mathbb{N}}$ is a sequence of well-defined continuous linear operators from $G(\mu) = X(\mu) \odot Y(\mu)$ to E. Since it is easy to see that it is well-defined linear, we only show the continuity. By the continuity of B, and taking into account that $\odot$ is an n.p. product, we have that

$$\sup_{h\in B_{G(\mu)}} \|R_m(h)\|_E = \sup_{(f,g)\in B_{X(\mu)}\times B_{Y(\mu)}} \|B(f \odot g \odot \chi_{Y_m}, \chi_{Y_m})\|_E < \infty.$$

Indeed, note that the supremun over all the pairs (f,g), where the functions f and g are in the corresponding unit balls, coincides with the supremum for all functions h in the ball of $G(\mu)$ as a direct consequence of the definition of n.p. product.

It follows that $(R_m)_{m\in\mathbb{N}}$ is a sequence of bounded, linear operators. Moreover, it is pointwise convergent. Indeed, for each $f \odot g$, the sequence $(R_m(f \odot g))_{m\in\mathbb{N}}$ satisfies the following

$$\begin{aligned}
\lim_{m\to\infty} R_m(f \odot g) &= \lim_{m\to\infty} B(f \odot \chi_{Y_m}, g \odot \chi_{Y_m}) \\
&= B(\lim_{m\to\infty} f \odot \chi_{Y_m}, \lim_{m\to\infty} g \odot \chi_{Y_m}) \\
&= B(f, g).
\end{aligned}$$

Consequently, we have the pointwise limit operator $R := \lim_{m\to\infty} R_m$. It is clear that this operator is well-defined and linear. As a result of the Uniform Boundedness theorem, it is obtained that R is a continuous operator. This shows that the map $R : G(\mu) \to E$ is a linear continuous operator such that $R(f \odot g) = B(f,g)$ for all $f \in X(\mu)$, $g \in Y(\mu)$.

Moreover, it is independent of the representation of $f \odot g$. Assume that $h = f_1 \odot g_1 = f_2 \odot g_2$. Then, $R(f_1 \odot g_1) - R(f_2 \odot g_2) = B(f_1, g_1) - B(f_2, g_2) = 0$. Therefore, we obtain the required factorization for a symmetric operator. The equivalence of (2) and (3) is proved in Lemma 3.1 and it is obvious that (3) implies (4).

Finally, it only remains to show that for every $A, C \in \Sigma$, the symmetry condition $B(\chi_A, \chi_C) = B(\chi_{A\cap C}, \chi_{A\cup C})$ holds when the operator is zero product preserving. The characteristic functions corresponding to the sets $A, C \in \Sigma$ satisfy $\chi_A \odot \chi_C = \chi_{A\cap C}$. It follows that $\chi_A \odot \chi_C = 0$ μ-a.e. if and only if $\mu(A \cap C) = 0$. By considering the assumption (4), we conclude that $B(\chi_A, \chi_C) = 0$ whenever $\mu(A \cap C) = 0$. It is already trivial that $B(\chi_{A\cap C}, \chi_{A\cup C}) = 0$ if $\mu(A \cap C) = 0$. Thus, we get that the symmetry condition holds for disjoint sets. To see that it is satisfied for arbitrary sets, consider $M, N \in \Sigma$ such that $\mu(M \cap N) \neq 0$. By the fact that $B(\chi_A, \chi_C) = 0$ whenever $\mu(A \cap C) = 0$, the following equality holds for the sets $M, N \in \Sigma$ by set operations and properties of the characteristic functions;

$$
\begin{aligned}
B(\chi_M, \chi_N) &= B(\chi_{(M\cap N^c)\cup(M\cap N)}, \chi_N) \\
&= B(\chi_{(M\cap N^c)} + \chi_{(M\cap N)} - \chi_{(M\cap N^c)\cap(M\cap N)}, \chi_N) \\
&= B(\chi_{(M\cap N^c)}, \chi_N) + B(\chi_{(M\cap N)}, \chi_N) - B(\chi_{(M\cap N^c)\cap(M\cap N)}, \chi_N) \\
&= B(\chi_{(M\cap N)}, \chi_N) = B(\chi_{(M\cap N)}, \chi_{(N\cap M^c)\cup(N\cap M)}) \\
&= B(\chi_{(M\cap N)}, \chi_{(N\cap M^c)}) + B(\chi_{(M\cap N)}, \chi_{(N\cap M)}) - B(\chi_{(M\cap N)}, \chi_{(N\cap M^c)\cap(N\cap M)}) \\
&= B(\chi_{(M\cap N)}, \chi_{(N\cap M)}) \\
&= B(\chi_{(M\cap N)}, \chi_{(N\cap M)}) + B(\chi_{(M\cap N)}, \chi_{(N\cap M^c)\cup(M\cap N^c)}) \\
&\quad + B(\chi_{(M\cap N)}, \chi_{[(N\cap M^c)\cup(M\cap N^c)]\cap(N\cap M)}) \\
&= B(\chi_{(M\cap N)}, \chi_{(N\cup M)}).
\end{aligned}
$$

Thus, the equality $B(\chi_M, \chi_N) = B(\chi_{(M\cap N)}, \chi_{(M\cup N)})$ is obtained for arbitrary sets $M, N \in \Sigma$ and it follows that B is a symmetric operator. This completes the proof. □

Remark 4.2 If we have a finite measure space (Ω, Σ, μ), the factorization is obtained more easily. Indeed, $\chi_\Omega \in X(\mu)$ and $\chi_\Omega \in Y(\mu)$ since the measure μ is finite. We can obtain in this case an equivalent definition of symmetric operators as below;

$$
\begin{aligned}
B(\chi_A,\chi_C) &= B(\chi_{A\cap C},\chi_{A\cup C}) \\
&= B(\chi_{A\cap C},\chi_{A\cap C}) + B(\chi_{A\cap C},\chi_{(A\cup C)\cap(A\cap C)^c}) \\
&= B(\chi_{A\cap C},\chi_{A\cap C}) \\
&= B(\chi_{A\cap C},\chi_{A\cap C}) + B(\chi_{A\cap C},\chi_{(A\cap C)^c}) \\
&= B(\chi_{A\cap C},\chi_{(A\cap C)\cup(A\cap C)^c}) \\
&= B(\chi_{A\cap C},\chi_{\Omega}).
\end{aligned}
$$

This means that the map B is symmetric if and only if $B(\chi_A,\chi_C) = B(\chi_{A\cap C},\chi_\Omega)$ for all $A, C \in \Sigma$ for a finite measure μ. Using the density as in the proof of Theorem 4.1, we get that a symmetric operator B is of the form $B(f,g) = B(fg,\chi_\Omega)$ for all $f \in X(\mu)$, $g \in Y(\mu)$. If we define a map $T : G(\mu) \to E$, $T(h) = T(f \odot g) = B(f \odot g, \chi_\Omega)$, we get the desired bounded linear continuous operator T satisfying $B := T \circ \odot$.

Corollary 4.1 Under the assumptions of Theorem 4.1, a bilinear map $B : X(\mu) \times X(\mu) \to Y$ is symmetric in the manner that $B(f,g) = B(g,f)$ for all $f,g \in X(\mu)$ if the map B is zero product preserving.

Proof. Let us assume that B is zero product preserving, then it has factorization operator R sayisfying $B(f,g) = R(f \odot g)$ for all $f,g \in X(\mu)$. Thus, $B(f,g) = R(f \odot g) = R(g \odot f) = B(g,f)$ holds. $\square$

Converse of the above corollary is not provided in general, i.e. being symmetric in the manner that $B(f,g) = B(g,f)$ does not give rise to be zero product preserving. Indeed, we obtain in Theorem 4.1 that zero product preservation holds only if the map satisfy the equality $B(\chi_A,\chi_C) = B(\chi_{A\cap C},\chi_{A\cup C})$ for all $A, C \in \Sigma$. But it was noted in Remark 4.1 that the condition $B(f,g) = B(g,f)$ does not involve the condition $B(\chi_A,\chi_C) = B(\chi_{A\cap C},\chi_{A\cup C})$ for all $A, C \in \Sigma$. Therefore, it is seen that a symmetric map in the manner $B(f,g) = B(g,f)$ does not give the zero product preservation property.

4.1.1 Factorization Through Particular Function Spaces

As a consequence of Theorem 4.1, it is desirable to know what a symmetric operator is factored through. In this section, we establish some results for bilinear operators defined on some particular spaces.

We shortly write X instead of $X(\mu)$ if the measure is clear in the context.

Let $X(\mu)$ and $Y(\mu)$ be Banach function spaces over the measure μ. We will say that they are *compatible* —or that they form a compatible couple— if the product space

$X(\mu)\cdot Y(\mu) := \{f\cdot g \in L^0(\mu) : f \in X(\mu), g \in Y(\mu)\}$

is a Banach function space again when it is endowed with the norm

$\|h\| = \inf\{\|f\|\,\|g\| : f\cdot g = h\}.$

Remark 4.3 The fact that the pointwise product $X \odot Y$ of Banach function spaces X and Y is an n.p. product is related to the Fatou property of the spaces involved. In the case that

$G = X \odot Y = \{f\cdot g : f \in X, g \in Y\}$

is a Banach function space with the norm

$\|h\|_{X\odot Y} = \inf\{\|f\|_X \|g\|_Y : h = fg, f \in X, g \in Y\},$

we have that the Fatou property of both X and Y implies the Fatou property of G (see [51, Corollary 1] or [52, Theorem 2.3]). By Theorem 2.4 in [52], we have that for all $h \in G$ there are $f \in X$ and $g \in Y$ such that $f = g\cdot h$ and $\|f\|_G = \|f\|_X\,\|g\|_Y$, what means that $B_G \subseteq \odot(B_X \times B_Y)$.

The factorization of Banach function spaces are considered by some authors to answer the questions of when the pointwise product of Banach function spaces are again a B.f.s. and it is possible to factor a B.f.s. through some Banach function spaces. The product of Banach function spaces, called *product Banach function space* or *generalized Köthe dual* in the literature, is indeed a Banach function space with the mentioned norm under some assumptions (see [50, 51, 52]). Since this gives rise to a factorization of a Banach function space through a couple of Banach function spaces by definition, it is closely related with the notion of compatibility of Banach function spaces. Now, by using the results of the factorization of Banach function spaces obtained in the references [50, 51, 52], we will investigate some particular cases for the domain of the linear operator that the symmetric map is factored through.

4.1.1.1 Factorization Through r-Convexification In this section, we will take into account the Banach function spaces with their r-convexifications. Recall that the r-convexification of a B.f.s. E is denoted by $E^{(r)}$ that is defined in page 12.

Remark 4.4 Let us consider the bilinear operator defined by the (μ-a.e.) pointwise product $\odot : E^{(p)} \times E^{(q)} \to E^{(r)}$, $(f,g) \rightsquigarrow f \cdot g$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $1 \leqslant r < p,q < \infty$. This bilinear map is *a norm preserving product* (cf. [51, Example 1], [24, Lemma 1] or [23, Lemma 2.21(i)]). In particular, it is a norming product.

Proof. For any function $f \in U_{E^{(r)}}$, we can define the functions $h := |f|^{r/p} sgnf \in E^{(p)}$ and $g := |f|^{r/q} \in E^{(q)}$, where $sgnf$ denotes the sign function of f. By the definition of the norm of the p-convexification, it follows that $\|h\|_{E^{(p)}} = \left\| \left| |f|^{r/p} sgnf \right|^p \right\|_E^{1/p} = \||f|^r\|_E^{1/p} = \|f\|_{E^{(r)}}^{r/p} < 1$. Similarly, $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q} < 1$. Therefore, $U_{E^{(r)}} \subseteq \odot(U_{E^{(p)}} \times U_{E^{(q)}})$ is obtained.

Let us show now that $\|h \cdot g\|_{E^{(r)}} = \inf\{\|h'\|_{E^{(p)}} \|g'\|_{E^{(q)}} : h' \in E^{(p)}, g' \in E^{(q)}, h \cdot g = h' \cdot g'\}$ for $h \in E^{(p)}$ and $g \in E^{(q)}$. By the Hölder-Rogers inequality we have that $h \cdot g \in E^{(r)}$ and $\|h \cdot g\|_{E^{(r)}} \leqslant \|h\|_{E^{(p)}} \|g\|_{E^{(q)}}$ (see page 13 or [24, Lemma 1]). Since this inequality holds for all couples (h', g') such that $f = h \cdot g = h' \cdot g'$, we obtain $\|h \cdot g\|_{E^{(r)}} \leqslant \inf\{\|h'\|\|g'\| : h \cdot g = h' \cdot g'\}$. Conversely, consider an arbitrary element $f \in E^{(r)}$. Then f has the following factorization: $h = |f|^{r/p} sgnf \in E^{(p)}$, $g = |f|^{r/q} \in E^{(q)}$ and $h \cdot g \in E^{(r)}$. Moreover, $\|h\|_{E^{(p)}} = \|f\|_{E^{(r)}}^{r/p}$ and $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q}$. Therefore $\|h\|_{E^{(p)}} \|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/p} \|f\|_{E^{(r)}}^{r/q} = \|f\|_{E^{(r)}}$. This proves

$$\|f\|_{E^{(p)}} = \|h \cdot g\|_{E^{(r)}} = \inf\{\|h'\|_{E^{(p)}} \|g'\|_{E^{(q)}} : h \cdot g = h' \cdot g'\},$$

thus the pointwise product $\odot$ from $E^{(p)} \times E^{(q)}$ to $E^{(r)}$ is an n.p. product. □

It is known that the p-convexification $E^{(p)}$ $(0 < p < \infty)$ of E is order continuous, if E is so. Consequently, $\mathrm{Sim}(\Sigma)$ is dense in $E^{(p)}$, $1 \leqslant p < \infty$ whenever E is o.c. Therefore, we get the following corollaries;

Corollary 4.2 Let E be an order continuous Banach function space. Let $1 \leqslant r < p,q < \infty$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. For a Banach space valued bilinear operator $B : E^{(p)} \times E^{(q)} \to Y$, the following statements imply each other.

(i) The operator B is symmetric.

(ii) The bilinear operator B is $\odot$-factorable, that is, there exists a bounded operator $T : E^{(r)} \to Y$ such that $B := T \circ \odot$.

(iii) The operator B is zero product preserving.

Note that if we consider the Banach function space $E = L^1(\mu)$ we obtain that the pointwise product is an n.p. product from $L^p(\mu) \times L^q(\mu)$ to $L^r(\mu)$ for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where $1 \leqslant r < p, q < \infty$, by the definition of p-convexification.

Corollary 4.3 Let (Ω, Σ, μ) be a σ-finite measure space and let $1 \leqslant r < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Consider a Banach space valued bilinear operator $B : L^p(\mu) \times L^q(\mu) \to Y$. Then the following statements imply each other.

(i) The bilinear operator B is symmetric.

(ii) The operator B is $\odot$-factorable through a linear bounded operator $T : L^r(\mu) \to Y$.

(iii) The operator B is zero product preserving.

4.1.1.2 Factorization Through the Duality Map Let E be an order continuous Banach function space over μ and consider its Köthe dual space E'. In this section we will show the case when we consider the pointwise product $\odot : E \times E' \to L^1(\mu)$ associated to the duality map, as product. Several well-known results allow to assert that it is in fact an n.p. product.

Remark 4.5 Recall that the well-known factorization theorem of Lozanovskii states that for any Banach function space with the Fatou property E and its associate space E', the product space $E \odot E' := E \cdot E'$ is a product Banach function space that is isometrically equal to $L^1(\mu)$ (see [53], also [54]). In other words, E and E' always form a compatible couple.

By Theorem 4.1 we immediately obtain the following.

Corollary 4.4 Let the set $\mathrm{Sim}(\Sigma)$ be dense in both E and its associate space E', and assume that E has the Fatou property Y is a Banach space. Then, for any bilinear continuous operator $B : E \times E' \to Y$ the following statements are equivalent.

(i) The bilinear operator $B : E \times E' \to Y$ is $\odot$-factorable.

(ii) The operator B is zero product preserving, that is, for each pair of elements $f \in E$ and $h \in E'$ we have that

$$\langle f,h\rangle = \int_\Omega f h \, d\mu = 0 \quad \Rightarrow \quad B(f,h) = 0.$$

(iii) The operator B is symmetric.

Unifying the classical setting for the relation among the Calderón construction and the pointwise product, Kolwicz et al have considered the product spaces with the Calderón construction in [51, Theorem 1]. For example, for a couple of Banach function spaces $X(\mu)$ and $Y(\mu)$, they obtained the following isometric equalities in [51]:

- $X(\mu)^{(p)} \odot Y(\mu)^{(p')}$ is equal to the Calderón space $X(\mu)^{1/p}Y(\mu)^{1/p'}$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,
- $X(\mu)^{(p)} \odot Y(\mu)^{(p)}$ gives the p-convexification of the product B.f.s. $X(\mu) \odot Y(\mu)$ for $0 < p < \infty$, i.e. $X(\mu)^{(p)} \odot Y(\mu)^{(p)} = (X(\mu) \odot Y(\mu))^{(p)}$,
- $X(\mu) \odot Y(\mu)$ can be represented as $\frac{1}{2}$-convexification of the Calderón space $X(\mu)^{1/2}Y(\mu)^{1/2}$, that is, $X(\mu) \odot Y(\mu) = (X(\mu)^{1/2}Y(\mu)^{1/2})^{(1/2)}$.

By this compatible couples, we get the following corollary;

Corollary 4.5 Let $X(\mu)$ and $Y(\mu)$ be order continuous Banach function spaces. Then

i) If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, the Banach space valued symmetric operator $B : X(\mu)^{(p)} \times Y(\mu)^{(p')} \to Z$ factors through a linear operator $T : X(\mu)^{1/p}Y(\mu)^{1/p'} \to Z$, where $X(\mu)^{1/p}Y(\mu)^{1/p'}$ is the corresponding Calderón space.

ii) If $X(\mu)^{(p)}$ and $Y(\mu)^{(p)}$ form a compatible couple, then every symmetric operator $B : X(\mu)^{(p)} \times Y(\mu)^{(p)} \to Z$ factors through a linear operator $T : (X(\mu) \odot Y(\mu))^{(p)} \to Z$, where $0 < p < \infty$.

iii) If $X(\mu)$ and $Y(\mu)$ form a compatible couple, then every symmetric operator $B : X(\mu) \times Y(\mu) \to Z$ factors through a linear operator $T : (X(\mu)^{1/2}Y(\mu)^{1/2})^{(1/2)} \to Z$.

Remark 4.6 The abstract requirement on the space $G(\mu)$ in Theorem 4.1 is clearly fulfilled when $G(\mu)$ is the product space of $X(\mu)$ and $Y(\mu)$, when they form a compatible couple. This can be easily checked just by considering its definition. Order continuity of this space will be relevant through the paper, in particular because it implies density of the set $\mathrm{Sim}(\Sigma)$. When a product space is order continuous in terms of the properties of the factor spaces is nowadays well-known. The reader can find complete characterizations or sufficient conditions for this property to hold in several recent papers. For example, the reader can find in Section 5 of [57] (Corollary 5.3) the following result: if $X(\mu)$ and $Y(\mu)$

define a compatible couple of order continuous B.f.s. over a finite measure μ such that $X(\mu) \subseteq Y(\mu)'$, then the product $X(\mu) \cdot Y(\mu)$ is order continuous. Another result in this direction is the following. Take $1 < p, p' < \infty$ such that $1/p + 1/p' = 1$, and consider two Banach function spaces $X(\mu)$ and $Y(\mu)$. The Calderón space $X(\mu)^{1/p} Y(\mu)^{1/p'}$ is order continuous if at least one of the spaces $X(\mu)$ and $Y(\mu)$ is order continuous; see [52].

A necessary and sufficient condition for the order continuity of the Banach function space $X(\mu) \cdot Y(\mu)$ can be given by means of the notion of *jointly order discontinuity*. Let X_a denote the subspace of all order continuous elements of the Banach function space X, that is, the space of the elements $f \in X$ such that for any sequence $(f_n)_{n=1}^{\infty} \subset X$ satisfying $0 \leqslant f_n \leqslant |f|$ and $f_n \to 0$ $\mu-$a.e. one has $\|f\|_X \to 0$. A couple of Köthe spaces (X, Y) is said to be *jointly order discontinuous* if there are elements $f \in X \backslash X_a$, $g \in Y \backslash Y_a$ and a sequence of measurable sets $A_n \searrow \varnothing$ such that for each sequence $(B_n)_{n=1}^{\infty} \in \Sigma$ with $B_n \subset A_n$ for all $n \in \mathbb{N}$ there are a number $a > 0$ and a subsequence $(n_k) \in \mathbb{N}$ such that either

$\|f \chi_{B_{n_k}}\|_X \geqslant a$ and $\|g \chi_{B_{n_k}}\|_Y \geqslant a$ for all $k \in \mathbb{N}$,

or

$\|f \chi_{B'_{n_k}}\|_X \geqslant a$ and $\|g \chi_{B'_{n_k}}\|_Y \geqslant a$ for all $k \in \mathbb{N}$,

where $B'_n = A_n \backslash B_n$ (see [58, Definition 12]). Corollary 1 in the paper [51] states that the Banach function space $X(\mu) \cdot Y(\mu)$ is order continuous if and only if $X(\mu)$ and $Y(\mu)$ are not jointly order discontinuous.

Corollary 4.6 Consider order continuous Banach function spaces E, F and G over the same measure space μ. Suppose that E and F have the Fatou property, G' is order continuous, and $E \odot F = G$. Then if the bilinear operator $B : E \times F \to F$ is symmetric with factorization operator T_B, the bilinear operator $A : E \times G' \to G'$ given by $A = T'_B \circ \odot$ is symmetric also. Conversely, if a bilinear operator $A : E \times G' \to G'$ is symmetric with factorization operator T_A, then the operator $B : E \times F \to F$ given by $B = T'_A \circ \odot$ is also symmetric.

Proof. Assume that B is a symmetric operator. Then there is a linear operator $T_B : G \to F$ defined by $B(e, f) = T(e \odot f)$, $e \in E$, $f \in F$. The linear operator T_B has an adjoint operator T'_B that can be defined having the image in G', due to the order continuity of G, and so $T'_B : F' \to G'$ is defined by $\langle g, T'_B(f') \rangle = \langle T_B(g), f' \rangle$. Theorem 3.7 in [52] states that if E, F have the Fatou property and $E \odot F = G$, then $E \odot G' = F'$ holds, and so F' is also a product Banach function space, namely E and G' are a compatible couple. Thus, we can write $f' = e_1 \odot g'$ for every $f' \in F'$, where $e_1 \in E$ and $g' \in G'$. It is clear that for the linear adjoint operator T_B' there is a symmetric operator $A : E \times G' \to G'$ defined by $T_B'(f') = T_B'(e_1 \odot g') = A(e_1, g')$. Moreover by the definition of the adjoint operator we obtain the symmetric operator A such that $A(e_1, g')g = \langle T_B(g), e_1 \odot g' \rangle$. Conversely, consider a symmetric operator $A : E \times G' \to G'$. Since the space G' is order continuous, $\overline{\mathrm{Sim}(\Sigma)} = G'$. Thus, it allows us to get a factorization for the bilinear operator A such that $T_A : F' \to G'$. Therefore, by using adequately the duality properties of the spaces F, G we obtain a well-defined adjoint operator $T_A' : G \to F$ and we conclude that the operator $B = T_A' \circ \odot$ is also a symmetric operator. □

Corollary 4.7 Suppose that the order continuous Banach function spaces E, F, G defined over the same measure space have the Fatou property, and E forms a compatible couple with both F and G such that $E \odot F = E \odot G$ isomorphically. Then, a bilinear operator $B_1 : E \times F \to Y$ is symmetric if and only if there is a symmetric bilinear operator $B_2 : E \times G \to Y$ and an isomorphism $\phi : F \to G$ such that $B_2(\cdot, \cdot) = B_1(\cdot, \phi(\cdot))$.

Proof. Let us assume that B_1 is symmetric, then it has a linear factorization $T_1 : E \odot F \to Y$ such that $B_1(e, g) = T_1(e \odot g)$. Since $E \odot F = E \odot G$ isomorphically, it follows that $F = G$ isomorphically (see [52, Corollary 2.6]). Therefore, we obtain a bilinear operator $B_2 : E \times G \to Y$ defined by $B_2(e, g) = T_1 \circ \odot \circ (Id \times \phi)(e, g)$, where Id denotes the identity operator defined on E and the ϕ is the isomorphism between the function spaces F and G. Conversely, assume that B_2 is symmetric. Then, there is a linear operator $T_2 : E \odot G \to Y$ such that $T_2(e \odot g) = B_2(e, g)$. Define the map $B_1(e, f) = T_2 \circ \odot \circ (Id^{-1} \times \phi^{-1})(e, f) = B_2(e, g)$. It is easily seen that this is a bilinear map and symmetric. □

4.1.2 Properties of Symmetric Bilinear Operators

Now, we will investigate the compactness and summability properties of zero product preserving map acting on Banach function spaces.

4.1.2.1 Compactness Properties of Symmetric Maps

Let us consider an n.p. product defined from $X(\mu) \times Y(\mu)$ to $G(\mu)$ and a symmetric map $B : X(\mu) \times Y(\mu) \to Z$.

Assume that simple functions are dense in both $X(\mu)$ and $Y(\mu)$.

It is easily seen that the symmetric map B is (weakly) compact if and only if the linear operator T appearing in its factorization is (weakly) compact, due to the definition of the product. Indeed,

$$\begin{aligned}
\text{the zpp map } B \text{ is (weakly) compact} &\iff B(U_{X(\mu)} \times U_{Y(\mu)}) \text{ is relatively (weakly) compact} \\
&\iff T \circ \odot(U_{X(\mu)} \times U_{Y(\mu)}) \text{ is relatively (weakly) compact} \\
&\iff T(U_{G(\mu)}) \text{ is relatively (weakly) compact} \\
&\iff T \text{ is (weakly) compact.}
\end{aligned}$$

Now, we will give more specific results for compactness.

Corollary 4.8 If $1 < r < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, each symmetric bilinear operator $B : L^p(\mu) \times L^q(\mu) \to Z$ is weakly compact.

Proof. By Corollary 4.3, B factors through a linear factorization operator $T : L^r(\mu) \to Z$. Since a linear operator with a reflexive domain is weakly compact, the linear operator T is weakly compact by the reflexivity of $L^r(\mu)$ for $1 < r < \infty$. Therefore, the map B is weakly compact. □

Corollary 4.9 Let $X(\mu)$ and $Y(\mu)$ be an order continuous compatible couple with the Fatou property, and assume that $(X(\mu) \cdot Y(\mu))'$ is order continuous. Then, any symmetric bilinear continuous operator $B : X(\mu) \times Y(\mu) \to Z$ is weakly compact.

Proof. Since both $X(\mu)$ and $Y(\mu)$ have order continous norm and Fatou property, the Banach function space $X(\mu) \cdot Y(\mu)$ has order continous norm and Fatou property, too (see [51, Corollary 1]). Then, direct dual spaces computations show that the assumption on the product $X(\mu) \cdot Y(\mu)$ implies that it is reflexive as a Banach space. By the symmetry of B, it factors through a reflexive space, and so it is weakly compact. □

For a range space Z with the Schur property this result can be improved.

Corollary 4.10 Under the assumptions on the compatible couple defined by $X(\mu)$ and $Y(\mu)$ given in Corollary 4.9, we have that every symmetric bilinear map $B : X(\mu) \times Y(\mu) \to Z$ is compact if Z has the Schur property.

Corollary 4.11 Let us consider the weakly compact symmetric bilinear map $B : E \times E' \to Z$ and the set $A \in E \times E'$ be a $\odot$-weakly compact set (see Definition 3.6 in page 29), then

$B(A)$ is compact.

Proof. Since B is weakly compact and symmetric map, it factors through a weakly compact linear map $T : L^1(\mu) \to Z$ defined by $B(f,g) = T(f \odot g)$ for all $f \in E$, $g \in E'$. By the Dunford–Pettis property of $L^1(\mu)$, the weakly compact operator T maps weakly compact sets to norm compact ones. Thus, we get $B(A) = T(\{f \odot g : (f,g) \in A\})$ is compact. □

The following theorem is a consequence of some well-known results on integral representation of weakly compact linear operators defined on $L^1(\mu)$ and our previous arguments.

Theorem 4.2 Let (Ω, Σ, μ) be a finite measure space and let the set $\mathrm{Sim}(\Sigma)$ be dense in both $E(\mu)$ and its associate space $E'(\mu)$. A symmetric bilinear operator $B : E(\mu) \times E'(\mu) \to Z$ is weakly compact if and only if it has a representation as $B(f,g) = \int_\Omega fghd\mu$ for all $f \in E(\mu)$, $g \in E'(\mu)$, where h is an essentially bounded Z-valued Bochner integrable function defined on μ with a $\mu-$essentially relatively weakly compact range.

Proof. The symmetric map B has a linear factorization through $L^1(\mu)$, that is, there is an operator T such that $B(f,g) = T(f \cdot g)$, $T : L^1(\mu) \to Z$. On the other hand, B is weakly compact if and only if T is weakly compact by the definition of the product acting in B.f.s. Dunford–Pettis–Philips' theorem states that a linear operator T defined on $L^1(\mu)$ to Z is weakly compact if and only if there exists an essentially bounded Z-valued Bochner integrable function h defined on μ with a $\mu-$essentially relatively weakly compact range such that $T(k) = \int_\Omega khd\mu$ for all $k \in L^1(\mu)$ (see [21, Ch. III, Theorem 2.12]). Since $T(k) = T(f \cdot g) = B(f,g)$, we get $B(f,g) = \int_\Omega fghd\mu$ for all $f \in E(\mu)$, $g \in E'(\mu)$. This gives the desired representation. □

Let us assume that the space Z is also a Banach lattice. We will say that a bilinear operator $B : X(\mu) \times Y(\mu) \to Z$ is *positive product preserving (ppp for short)* if

$$B(f,g) \geqslant 0 \text{ whenever } f \odot g \geqslant 0 \text{ for } f \in X(\mu) \text{ and } g \in Y(\mu).$$

Remark 4.7 It is natural to ask for a relation between positive bilinear maps and positive product preserving bilinear maps. (Recall that a bilinear map $B : X(\mu) \times Y(\mu) \to Z$ is positive if $B(f,g) \geqslant 0$ for all $f \in X(\mu)^+$, $g \in Y(\mu)^+$, see Appendix A-2 or [7]). The answer is the question; every positive product preserving bilinear map is positive bilinear map but the inverse is not true in general. Indeed, let us assume that B is positive product preserving. The order relation of an arbitrary B.f.s. E is given as $f_1 \geqslant f_2$ if $f_1(x) \geqslant f_2(x)$ a.e. on Ω, for $f_1, f_2 \in E$. For $f \in X(\mu)^+,\ g \in Y(\mu)^+$, we get $f(x) \geqslant 0$ and $g(x) \geqslant 0$ a.e. on Ω and it follows that

$$\{x \in \Omega : f(x)g(x) < 0\} \subset \{x \in \Omega : f(x) < 0\} \cup \{x \in \Omega : g(x) < 0\}.$$

By the monotonicity of the measure, we get

$$0 \leqslant \mu(\{x \in \Omega : f(x)g(x) < 0\}) \leqslant \mu(\{x \in \Omega : f(x) < 0\}) + \mu(\{x \in \Omega : g(x) < 0\}) = 0.$$

This shows that $f(x) \geqslant 0$ and $g(x) \geqslant 0$ a.e. on Ω implies $f(x)g(x) \geqslant 0$ a.e. on Ω, i.e. $f \cdot g \geqslant 0$ whenever $f \geqslant 0$ and $g \geqslant 0$. By the assumption of positive product preservation we get that $B(f,g) \geqslant 0$ for all $f \in X(\mu)^+,\ g \in Y(\mu)^+$. Thus B is a positive bilinear map.

To show the inverse is not true in general consider the measure space $([0,1], \mathfrak{B}([0,1]), dx)$ and an arbitrary positive bilinear map $B : L^1([0,1]) \times L^1([0,1]) \to Z$. For the functions $f = -c$ and $g = -\dfrac{1}{c}$, where $c \in (0,1]$, notice that $f \cdot g = 1$ and f and g is not in $L^1([0,1])^+$. This shows that we can find a couple of functions that are not in positive cone with a positive product. Thus, we can not say $B(f,g) \geqslant 0$ for the functions $f = -c$ and $g = -\dfrac{1}{c}$ even if B is positive and $f \cdot g \geqslant 0$. Therefore, we get that the positive product preservation of a bilinear map can not be decided by the positivitiy of the map.

Recall that a linear operator T between Banach lattices is called *positive* if $T(x) \geqslant 0$ whenever $x \geqslant 0$ (see Appendix A-2 or [59, Chapter 1]). Now we will give a factorization of positive product preserving maps through a positive linear map.

It is clear that a symmetric bilinear map $B : X(\mu) \times Y(\mu) \to Z$ with an order continuous compatible couple $X(\mu), Y(\mu)$ is positive product preserving if and only if its factorization operator $T : G(\mu) \to Z$ defined by $T(f \odot g) = B(f,g)$ is a positive linear operator. Indeed,

$$\begin{aligned} \text{the symmetric map } B \text{ is ppp} &\Longleftrightarrow f \odot g \geqslant 0 \text{ implies } B(f,g) \geqslant 0 \\ &\Longleftrightarrow f \odot g \geqslant 0 \text{ implies } T(f \odot g) \geqslant 0 \\ &\Longleftrightarrow T \text{ is positive linear map.} \end{aligned}$$

Corollary 4.12 Let $X(\mu)$ and $Y(\mu)$ be a compatible couple such that $\mathrm{Sim}(\Sigma)$ is dense in both. A symmetric positive product preserving bilinear map $B : X(\mu) \times Y(\mu) \to \ell^1$ is weakly compact –hence, compact– if and only if the associate space $(X(\mu) \cdot Y(\mu))'$ of $X(\mu) \cdot Y(\mu)$ has order continuous norm.

Proof. By Theorem 4.1, a symmetric map B is weakly compact, positive product preserving if and only if it has a weakly compact positive linear factorization operator $T : X(\mu) \cdot Y(\mu) \to \ell^1$ defined by $T(f \odot g) = B(f,g)$. It is known that a positive linear operator from a Banach function space to ℓ^1 is weakly compact if and only if the associate space of its domain has order continuous norm (see [60, pp 275]). Therefore, T is weakly compact and compact by the Schur property of ℓ^1 if and only if the associate space $(X(\mu) \cdot Y(\mu))'$ of $X(\mu) \cdot Y(\mu)$ has order continuous norm, what implies that B is compact too. □

For example, if $1 < r < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, every symmetric positive product preserving bilinear map $B : L^p(\mu) \times L^q(\mu) \to \ell^1$ is compact. Finally, we show another result for $C(K)$-type spaces. Recall that a Banach lattice Z has a strong order unit if there is an element e in Z with the property that for every $z \in Z$ there exists a real number α such that $|z| \leqslant \alpha e$.

Corollary 4.13 Let Z be a Dedekind complete Banach lattice with a strong order unit. Then every symmetric bilinear operator $B : X(\mu) \times Y(\mu) \to Z$ can be written as a difference of two positive product preserving symmetric bilinear operators.

Proof. Since B is symmetric, there is a linear operator acting in the factorization space $T : X(\mu) \cdot Y(\mu) \to Z$ such that B factors through T. The operator T is regular, that is, it can be written as a difference of two positive linear operators, say $T_1 - T_2$, since the space Z is a Dedekind complete Banach lattice with a strong order unit [61, Theorem 4.1]. Thus, the operator B can be written as $B = T \circ \odot = (T_1 - T_2) \circ \odot = T_1 \circ \odot - T_2 \circ \odot$. Since T_1 and T_2 are positive linear operators, it follows that $T_1 \circ \odot$ and $T_2 \circ \odot$ are positive product preserving symmetric bilinear operators. Therefore, B is written as a difference of two positive product preserving symmetric bilinear operators. □

4.1.2.2 Summability Properties Consider a symmetric Banach space-valued bilinear operator $B : X(\mu) \times Y(\mu) \to Z$, where $X(\mu)$, $Y(\mu)$ are Banach function spaces over μ such that the set of simple functions is dense and the pointwise product $\odot : X(\mu) \times Y(\mu) \to G(\mu)$ is an n.p. product. In this case, by the factorization given in Theorem 4.1 we obtain that for all $f_1, f_2, ..., f_n \in X(\mu)$ and $g_1, g_2, ..., g_n \in Y(\mu)$,

$$\sum_{i=1}^{n} \|B(f_i, g_i)\|_Z \leqslant k \sum_{i=1}^{n} \|f_i \cdot g_i\|_{G(\mu)}. \tag{4.2}$$

If we ask Z to have some particular properties, we get a domination –even an integral domination– for a symmetric bilinear map. The first case that we explain it is associated to some cotype-related properties of the space Z. Recall that being of cotype 2 implies the Orlicz property (see page 9 or [18, §8.9]). So, for a Banach space Z with the Orlicz property, we get the following domination of the symmetric map B. There exists $k > 0$ such that for $f_1, f_2, ..., f_n \in X(\mu)$ and $g_1, g_2, ..., g_n \in Y(\mu)$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_Z^2\Big)^{1/2} \leqslant \Big(\sum_{i=1}^{n} \|T\| \|(f_i \cdot g_i)\|_Z^2\Big)^{1/2} \leqslant k \sup_{\varepsilon_i = \mp 1} \Big\| \sum_{i=1}^{n} \varepsilon_i f_i \cdot g_i \Big\|_{G(\mu)}.$$

As a second example, let us provide some direct applications on summability of certain bilinear operators. Suppose that E is a Banach function space over a measure μ with associate space E' and let H be a Hilbert space. By Grothendieck's Theorem, we know that $\mathcal{L}(L^1(\mu), H) = \Pi_1(L^1(\mu), H)$. As a consequence of Pietsch Domination Theorem (see page 9), we directly obtain the next.

Corollary 4.14 Let the set of simple functions be dense in both E and E' and assume that E has the Fatou property. For any symmetric Hilbert space valued bilinear continuous map $B : E \times E' \to H$, there is a positive constant c such that the following equivalent statements hold.

i) For $f_1, f_2, ..., f_n \in E$ and $g_1, g_2, ..., g_n \in E'$,

$$\sum_{i=1}^{n} \big\|B(f_i, g_i)\big\|_H \leqslant c \sup_{\phi \in B_{L^\infty}} \sum_{i=1}^{n} |\langle f_i \cdot g_i, \phi \rangle|. \tag{4.3}$$

ii) For $f \in E$ and $g \in E'$,

$$\|B(f, g)\|_H \leqslant c \int_{\phi \in B_{L^\infty}} |\langle f \cdot g, \phi \rangle| d\nu(\phi), \tag{4.4}$$

where ν is regular probability measure on the unit ball of $L^\infty(\mu)$.

In particular by the Dunford–Pettis property of $L^1(\mu)$ the bilinear map factors through a completely continuous linear operator.

4.1.3 Lattice Geometric Inequalities for $\odot$-Factorable Maps

We are going to apply some classical arguments on factorization of operators for giving a particular integral representation for linear maps with good concavity properties.

Firstly, recall some concepts. We will utilise the vector measures that are a generalization of the notion of measure which are (countably additive) set functions taking vector values instead of nonnegative real numbers only, i.e. $\nu : \Sigma \to E$ is a vector measure if it is finitely additive, where E is a Banach space and Σ is a σ-algebra. It is called countably additive vector measure if ν is countably additive function; see [21, Chapter I].

Let $\nu : \Sigma \to E$ be a vector measure. The *variation* of ν is the extended nonnegative function $|\nu|$ which is defined by

$$|\nu|(A) = \sup \sum_{i=1}^{n} \|\nu(A_i)\|$$

for a set $A \in \Sigma$. The supremum is taken over all partitions $A = \bigcup_{i=1}^{n} A_i$ of A into a finite number of pairwise disjoint members of Σ. If $|\nu|(\Omega) < \infty$, it is said that the variation $|\nu|$ is finite or the vector measure ν is called *measure of bounded variation* ([21, Section I.1]).

Let μ be a nonnegative real-valued measure and ν is a vector measure on the same σ-field Σ. The vector measure ν is said to be μ-continuous, if $\lim_{\mu(E)\to 0} \nu(E) = 0$ for every $E \in \Omega$ and this is signified by $\nu \ll \mu$; see [21, Section I.2].

For a vector measure $\nu : \Sigma \to E$, a finite scalar measure $\mu : \Sigma \to [0,\infty)$ is called a *control measure* if μ and ν are mutually continuous, that is, $\mu(A) \to 0$ if and only if $\nu(A) \to 0$. By one of the theorems of Pettis, this equals to the identity $\mathscr{N}_0(\mu) = \mathscr{N}_0(\nu)$ ([21, Section I.2]).

For a $x^* \in E^*$, $\langle \nu, x^* \rangle$ is the scalar measure defined by $\langle \nu, x^* \rangle : A \to \langle \nu(A), x^* \rangle$, where $A \in \Sigma$. $|\langle \nu, x^* \rangle|$ denotes the variation of $\langle \nu, x^* \rangle$.

We can obtain a class of control measures for a vector measure $\nu : \Sigma \to E$ by using the Rybakov's Theorem. This theorem states that there is a functional $x^* \in E^*$ such that the $|\langle \nu, x^* \rangle| = \Sigma \to [0,\infty)$ is a control measure for ν, i.e. $|\langle \nu, x^* \rangle|$ has the same null sets as ν; see [21, Section IX.2]. The functional x^* is called *Rybakov functional*. It is known that for a Rybakov functional x^*, $L^1(\nu) \subseteq L^1(|\langle \nu, x^* \rangle|)$ and the natural inclusion map is continuous ([23, Chapter 3, Thm 3.7.(iv)]).

As usual, we write $[f]_\mu$ for the equivalence class of almost everywhere equal measurable functions that are associated with f. Recall that, if $X(\mu)$ and $Z(\eta)$ are Banach function spaces such that $\eta \ll \mu$ and the identification $[f]_\mu \mapsto [f]_\eta$ is well-defined, then it is automatically continuous, since it is a positive map between Banach lattices (see [19, p. 2]). Thus, we can use this assignation to define a (continuous) inclusion/quotient operator $X(\mu) \hookrightarrow Z(\eta)$ (see [62, p. 90]).

An operator $M_\psi : X(\mu) \to Y(\mu)$ between Banach function spaces is called a multiplication operator if the value of the M_ψ at a function f is given by multiplication by a fixed function ψ. That is, $M_\psi(f) = \psi \cdot f$ for all $f \in X(\mu)$, where $\psi \in L^0(\mu)$.

Let $T : X(\mu) \to Y$ be a continuous linear operator. The *optimal domain* $Z(\mu)$ of T is the *largest* Banach function space satisfying $X(\mu) \hookrightarrow Z(\mu)$ such that there exists a bounded linear operator $\tilde{T} : Z(\mu) \to Y$ which is the maximal extension of the operator T; see [23, Chapter 1].

Recall that a Banach space X has the Radon-Nikodym property if for any finite measure μ and any linear operator $T : L^1(\Omega,\mu) \to X$ there is a bounded μ-measurable function $f : \Omega \to X$ such that $Tg = \int_\Omega fgd\mu$ for all $g \in L^1(\Omega,\mu)$ ([18, §16.4]).

Theorem 4.3 Consider a compatible couple of Banach function spaces $X(\mu)$ and $Y(\mu)$ having order continuous norms. Suppose that the product space $X(\mu)\cdot Y(\mu)$ is p-convex for $1 \leqslant p < \infty$. Consider a bilinear (continuous) Banach-space-valued operator $B : X(\mu) \times Y(\mu) \to E$. The following statements are equivalent.

(i) For $f_1,...,f_n \in X(\mu)$ and $g_1,...,g_n \in Y(\mu)$,

$$\Big(\sum_{i=1}^n \|B(f_i,g_i)\|^p\Big)^{1/p} \leqslant \Big\|\Big(\sum_{i=1}^n |f_i\cdot g_i|^p\Big)^{1/p}\Big\|_{X(\mu)\cdot Y(\mu)}. \tag{4.5}$$

(ii) There are a multiplication operator $M_h : X(\mu)\cdot Y(\mu) \to L^p(\mu)$ such that $\|M_h\| = 1$ and a linear operator $T : L^p(\mu) \to E$ such that B factors as $B = T \circ M_h \circ \odot$, that is, it factors through the scheme

$$\begin{tikzcd} X(\mu)\times Y(\mu) \arrow[r, "B"] \arrow[d, dotted, "\odot"'] & E. \\ X(\mu)\cdot Y(\mu) \arrow[r, dotted, "M_h"] & L^p(\mu) \arrow[u, dotted, "T"'] \end{tikzcd}$$

(iii) There is an E-valued vector measure ν such that $L^p(\mu) \hookrightarrow L^1(\nu)$, and

$$B(f,g) = \int_\Omega f(t)g(t)h(t)\,d\nu(t), \tag{4.6}$$

where h defines a multiplication operator $M_h : X(\mu)\cdot Y(\mu) \to L^p(\mu)$.

Proof. (i) $\Rightarrow$ (ii) Note that the inequality in (i) directly implies that B is 0-product preserving. This means by Theorem 4.1 that it factors through a linear continuous map $S : X(\mu)\cdot Y(\mu) \to E$ such that

$$\Big(\sum_{i=1}^n \|S(f_i\cdot g_i)\|^p\Big)^{1/p} \leqslant \Big\|\big(\sum_{i=1}^n |f_i\cdot g_i|^p\big)^{1/p}\Big\|_{X(\mu)\cdot Y(\mu)}$$

for $f_1,\dots,f_n \in X(\mu)$ and $g_1,\dots,g_n \in Y(\mu)$ by the inequality in (i) again. Consequently, we get that S is p-concave. By hypothesis, we have that the product space $X(\mu)\cdot Y(\mu)$ is p-convex. A standard application of Maurey-Rosenthal argument ([63, Corollary 5]) states that any r-concave ($1 \leqslant r \leqslant \infty$) linear operator from a r-convex order continuous B.f.s. to a B.f.s is factored through $L^r(\mu)$ via a multiplication operator and a linear operator. This gives the existence of a norm one multiplication operator $M_h : X(\mu)\cdot Y(\mu) \to L^p(\mu)$ such that $S = T\circ M_h$, where $T : L^p(\mu) \to E$ is a linear continuous map. Composing all the elements, we get the desired diagram: $B = T\circ M_h \circ \odot$.

(ii) $\Rightarrow$ (iii) Note that the space $L^p(\mu)$ is order continuous, so we get that the operator $T : L^p(\mu) \to E$ defines a (countably additive) vector measure $\nu(A) := T(\chi_A)$ called *the vector measure associated to* T where $A \in \Sigma$. Moreover, if we assume that T is μ-determined, that is $\mathcal{N}_0(\mu) = \mathcal{N}_0(\nu)$, we get that the optimal domain of T exists and it is $L^1(\nu)$, thus $L^p(\mu) \hookrightarrow L^1(\nu)$ due to the optimality of the space $L^1(\nu)$. Besides, the optimal extension of T is the integral operator $I_\nu(s) = \int_\Omega s\,d\nu$, for $s \in L^1(\nu)$ and the following commutative diagram is valid:

$$\begin{tikzcd}
X(\mu)\times Y(\mu) \arrow[rr, "B"] \arrow[d, dotted, "\odot"'] & & E & \\
X(\mu)\cdot Y(\mu) \arrow[rr, dotted, "M_h"] & & L^p(\mu) \arrow[u, dotted, "T"'] \arrow[r, hook] & L^1(\nu). \arrow[ul, "I_\nu"']
\end{tikzcd}$$

The reader can find this result in [23, Theorem 4.14]. Note that by the proof given there, the result is still true if this is not the case, that is, if T is not μ-determined. It is well-known that the space $L^1(\nu)$ is a Banach function space over a Rybakov measure η for

ν, and $\eta \ll \mu$ because of the continuity of T; we can change then the inclusion by the identification of classes $[f]_\mu \mapsto [f]_\eta$, what is sometimes called an inclusion/quotient map, and the factorization is still preserved. Summing up all these comments, we get that

$$B(f,g) = \int_\Omega f(t)\, g(t)\, h(t)\, d\nu(t)$$

for all $f \in X(\mu)$ and $g \in Y(\mu)$.

(iii) $\Rightarrow$ (i) A direct computation just using the formula gives this implication. Indeed, if $f_1,...,f_n \in X(\mu)$ and $g_1,...,g_n \in Y(\mu)$,

$$\begin{aligned}\Big(\sum_{i=1}^n \|B(f_i,g_i)\|_E^p\Big)^{1/p} &\leqslant \Big(\sum_{i=1}^n \|f_i\, g_i\, h\|_{L^1(\nu)}^p\Big)^{1/p} \leqslant \Big(\sum_{i=1}^n \|f_i\, g_i\, h\|_{L^p(\mu)}^p\Big)^{1/p} \\ &= \Big\|h\Big(\sum_{i=1}^n |f_i \cdot g_i|^p\Big)^{1/p}\Big\|_{L^p(\mu)} \leqslant \Big\|\Big(\sum_{i=1}^n |f_i \cdot g_i|^p\Big)^{1/p}\Big\|_{X(\mu)\cdot Y(\mu)}. \qquad \square\end{aligned}$$

An integral with respect to a vector measure is still a rather abstract representation for the bilinear operator B. However, using the same result for $p = 1$ we can still improve the representation for getting a kernel-type operator whenever the range space E has the Radon-Nikodym property. Although we will show a special representation for the specific case of classical bilinear integral operators, we can improve the integral formula given above for the case of 0-product preserving bilinear operators factoring through a 1-concave linear map. As in Theorem 4.3, we suppose without loss of generality that the constant appearing in the inequality in (i) equals to 1, that is, no specific constant appears.

Corollary 4.15 Let μ be a finite measure. Consider a compatible couple of Banach function spaces $X(\mu)$ and $Y(\mu)$ with order continuous norms. Suppose that E is a Banach space with the Radon-Nikodym property. For a continuous bilinear operator $B : X(\mu) \times Y(\mu) \to E$, the following statements are equivalent.

(i) For $f_1,...,f_n \in X(\mu)$ and $g_1,...,g_n \in Y(\mu)$,

$$\sum_{i=1}^n \|B(f_i,g_i)\| \leqslant \Big\|\sum_{i=1}^n |f_i \cdot g_i|\Big\|_{X(\mu)\cdot Y(\mu)}. \tag{4.7}$$

(ii) There is a (norm one) E-valued vector measure Bochner integrable function $\Phi \in L^\infty(\mu, E)$ such that

$$B(f,g) = \int_{\Omega} f(t)\, g(t)\, h(t)\, \Phi(t)\, d\mu(t), \tag{4.8}$$

where $h \in \Big(X(\mu) \cdot Y(\mu) \Big)'$.

Proof. (i) $\Rightarrow$ (ii) Applying the assumption (i) and Theorem 4.3 we conclude that the variation of the vector measure is finite. Indeed, it is seen from (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) in Theorem 4.3, we get that there is a vector measure ν defined by $\nu(A) = T(\chi_A)$ for all $A \in \Sigma$. It follows that $\|\nu(A)\| \leqslant \|T\|\mu(A)$ for all $A \in \Sigma$. Thus,

$$|\nu|(\Omega) = \sup \sum_{i=1}^{n} \|\nu(A_i)\| \leqslant \sup \sum_{i=1}^{n} \|T\|\mu(A_i) = \|T\|\mu(A)$$

for all partitions $\Omega = \bigcup_{i=1}^{n} A_i$ of Ω. Since μ is finite, it implies $|\nu|(\Omega) < \infty$. Then, this result is obtained as an application of Theorem 4.3. By hypothesis the vector measure ν provided by this theorem defines an operator $w \mapsto \int_{\Omega} w\, d\nu : L^1(\mu) \mapsto E$ that closes a factorization diagram as in the theorem. Using the Radon-Nikodym property of E we get that there is an integrable vector-valued density Φ for the vector measure, in such a way that $d\nu = \Phi d\mu$ and so $w(t) \mapsto \int_{\Omega} \Phi(t) w(t)\, d\mu(t)$. This gives (ii).

(ii) $\Rightarrow$ (i) Taking into account that $\Sigma \ni A \mapsto \int_A \Phi d\nu \in E$ defines a vector measure, (iii) $\Rightarrow$ (i) in Theorem 4.3 for the case $p = 1$ gives (i). $\square$

4.1.4 Applications: Representation Formulas for Integral Bilinear Maps

Here, we apply the results given in the previous sections to some particular classes of bilinear operators that are defined by integral formulas. In order to do that, we will have to enlarge the notion of product preserving map by including some measurable transformations. We are interested in considering classical operators as the Hilbert transform, but the class we will deal with is broader than this. Let us start with a simple example.

Application 4.1 For $1 \leqslant p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, let us consider the bilinear operator $B : L^p([0,1]) \times L^q([0,1]) \to L^2([0,1])$ defined by

$$B(f,g)(x) = \sum_{i=1}^{n} \Big(\int_0^1 r_n(y) f(y) g(y) dy \Big) g_n(x), \tag{4.9}$$

where r_n denotes the n^{th} Rademacher function (see page 9), $g_n(x) = 2^{-(n+1)/2} \chi_{[2^{-n}, 2^{(-n+1)}]}(x)$

for $x \in [0,1]$ and $n = 1,2,\ldots$. It is clear that this bilinear operator is zero product preserving since $B(f,g)(x) = 0$ if $f(y)g(y) = 0$ dy-a.e. for all $y \in [0,1]$. Therefore, by Theorem 4.1, it can be written as a linear integral operator such that $T(f \odot g) = T(h) = \sum_{i=1}^{n} \left(\int_0^1 r_n(y)h(y)dy \right) g_n(x)$, where $h \in L^2([0,1]) = L^p([0,1]) \odot L^q([0,1])$.

Application 4.2 The Hilbert transform of a function $f(x)$ is given by

$$H(f)(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} f(x-t)\frac{dt}{t} \tag{4.10}$$

where $p.v.$ denotes the Cauchy principal value. This transform can be considered as the convolution of $f(x)$ with the tempered distribution $p.v.\frac{1}{\pi t}$. The bilinear Hilbert transform was introduced by Calderón as the following

$$H_{\alpha_1,\alpha_2}(f,g)(x) = p.v. \int_{\mathbb{R}} f(x-\alpha_1 t)g(x-\alpha_2 t)\frac{dt}{t}. \tag{4.11}$$

In [64], Grafakos and Li have obtained a uniform bound for the bilinear Hilbert transform $H_{1,\alpha} : L^{p_1} \times L^{p_2} \to L^p$ for the real parameter α and $1 > \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{2}$. If we suppose $\alpha = 1$, then it is obviously seen that $H_{1,1}(f,g)(x) = (f \odot g) * (p.v.\frac{1}{t}) = H(f \odot g)(x)$. Since $H_{1,1}(f,g)(x) = 0$ if $f \odot g = 0$, the operator $H_{1,1}$ is a zero product preserving map. Then it has a linear factorization that is the Hilbert transform defined on $L^{p_1} \odot L^{p_2} = L^p$ into L^p.

We can give a more general result for the bilinear Hilbert transform when it is considered as acting on products of Lorentz spaces. Villarroya defined the generalized bilinear Hilbert transform by using an arbitrary distribution instead of the tempered distribution in [65]. The generalized bilinear Hilbert transform is given by

$$H_{u,\alpha}(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x-\alpha t)u(t)dt \tag{4.12}$$

where u is a distribution, $\alpha \in \mathbb{R}$ and f, g are elements of the function space $C_0^{\infty}(\mathbb{R})$ of the smooth functions with compact support in $\mathbb{R}$. A generalized bilinear Hilbert transform $H_{u,\alpha}$ is said to be $(p_i, q_i)_{i=1,2,3}$ bounded if it is possible to extend it to a bounded operator from $L^{p_1,q_1} \times L^{p_2,q_2}$ to L^{p_3,q_3}, where $0 < p_i < \infty, 0 < q_i < \infty$ for $i = 1,2,3$ and $L^{p,q}$ denotes the Lorentz function space that consists of measurable functions f endowed with the norm

$$\|f\|_{p,q} = \begin{cases} \left\{ \frac{q}{p} \int_0^{\infty} t^{q/p} (\inf \lambda > 0 : m(\{x \in \mathbb{R} : |f(x)| > \lambda\}))^q \frac{dt}{t} \right\}^{1/q}, & 0 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p} (\inf \lambda > 0 : m(\{x \in \mathbb{R} : |f(x)| > \lambda\})), & 0 < p \leqslant \infty, q = \infty, \end{cases}$$

we refer to [66] for Lorentz spaces. Now, consider a $(p_i, q_i)_{i=1,2,3}$ bounded generalized Hilbert transform with the parameter $\alpha = 1$. Then, it is seen that the transform $H_{u,1}$ is a zero product preserving operator and $H_{u,1}(f,g)(x) = (f \odot g) * u$. Since simple functions are dense in a Lorentz space $L^{p,q}$ for $0 < p,q < \infty$, by Corollary 4.5 we get a factorization such that $T : [(L^{p_1,q_1})^{1/2}(L^{p_2,q_2})^{1/2}]^{(1/2)} \to L^{p_3,q_3}$ defined by $H_{u,1} = T(f \odot g) = (f \odot g) * u$, where the Calderón space $[(L^{p_1,q_1})^{1/2}(L^{p_2,q_2})^{1/2}]^{(1/2)}$ is the product space of the compatible couple L^{p_1,q_1} and L^{p_2,q_2}.

Although the pointwise product of functions appears explicitly in many of the classical examples of integral operators, most of them are not strictly 0-product preserving. For example, consider operators defined by the formula of the bilinear Hilbert transform given above but with compact support,

$$H(f,g) := \int_K f(x-t)g(x-\alpha t)\frac{dt}{t}, \quad f,g \in L^2(\mu), \tag{4.13}$$

where (K, Σ, dt) is Lebesgue space on a compact set of the real line K, are not product preserving in general except that $\alpha = 1$. In this section we show that it is also possible to find a weak version of our representation theorem in this case. In order to do that, let us recall and introduce some concepts and notations.

Let I be $[0,1]$ or $[0,\infty)$ and m is the Lebesgue measure over I. A Banach function space is called *rearrangement invariant (r.i. shortly)* or *symmetric* if $f \in X$ and g is equimeasurable with f, that is f and g have the same distribution functions $d_f = d_g$, where $d_f(\alpha) = m(\{x \in I : |f(x)| > \alpha\})$, $\alpha \geqslant 0$, then $g \in X$ and $\|f\|_X = \|g\|_X$; see [66, Section 2.4].

Let (Ω, Σ, μ) be a σ-finite measure space and $X(\mu)$ is a Banach function space over μ. Let $\phi : \Omega \to \Omega$ be a bimeasurable (measurable in both directions) bijection. We define $X_\phi(\mu)$ as the space of (classes of μ-a.e. equal) functions

$$X_\phi(\mu) := \left\{f \in L^0(\mu) : f \circ \phi^{-1} \in X(\mu)\right\}$$

endowed with the norm

$$\|f\|_{X_\phi(\mu)} := \|f(\phi^{-1}(\cdot))\|_{X(\mu)}, \quad f \in X_\phi(\mu).$$

Note that such a ϕ defines an isometry, that is, the transformation

$$\Delta_\phi : X(\mu) \to X_\phi(\mu) \text{ given by } h \mapsto \Delta_\phi(h) = h \circ \phi \in X_\phi(\mu),$$

that is clearly defined for all $h \in X(\mu)$, is an isometric isomorphism. The functions ϕ we are thinking about are typically simple transformations as, for the case of Lebesgue measure space $([0,1], \mathfrak{B}([0,1]), dx)$, $\phi_{1/2}(x) = x + 1/2 \mod 1$, $x \in [0,1]$. If we take a rearrangement invariant space, for example if $X(\mu) = L^p([0,1])$ for $1 \leqslant p \leqslant \infty$, we have that $L^p_{\phi_{1/2}}([0,1]) = L^p([0,1])$ isometrically.

We will consider couples of parametric families $\{\phi^1_x\}_{x \in \Omega}$ and $\{\phi^2_x\}_{x \in \Omega}$ of such bimeasurable bijections satisfying the requirement that $X_{\phi^1_x}(\mu)$ and $Y_{\phi^2_x}(\mu)$ are compatible for each $x \in \Omega$. Our idea is to recover using these tools a similar definition that the one that gives for example the bilinear Hilbert transform. Note that the simplest example of such a parametric family is when $\phi^1_x = \phi^1$ and $\phi^2_x = \phi^2$ for fixed functions ϕ^1 and ϕ^2; we use it just below.

We are now ready to define *a general class of integral-type bilinear operators.* Let $Z(\mu)$ be a Banach function space over μ. Let $X(\mu)$ and $Y(\mu)$ be compatible Banach function spaces on μ. In this context, we will say in what follows that a bilinear operator $B: X(\mu) \times Y(\mu) \to Z(\mu)$ is an integral bilinear operator if it is defined by a formula as

$$B(f,g)(x) := \int_\Omega f(\phi^1_x(t))\, g(\phi^2_x(t))\, K(x,t)\, dt, \quad x \in \Omega, \quad f \in X(\mu),\ g \in Y(\mu), \tag{4.14}$$

where $K: \Omega \times \Omega \to \mathbb{R}$ is an integrable kernel such that the expression inside the integral is well-defined for each x, t, f and g, and integrable, in such a way that $B(f,g)(\cdot) \in Z(\mu)$.

Independently of the case of pointwise type bounds depending on x that we will explain later, we can get direct results when the functions ϕ^1 and ϕ^2 are fixed from the general framework constructed along the paper. So, let us assume for the next result that ϕ^1_x and ϕ^2_x do not depend on x. As a consequence of Lemma 3.1 we obtain the following general result. Note that it is not restricted to the case of integral bilinear operators, although it can be applied to this concrete context by its definition. It can be easily checked that the requirements in the following result are fulfilled in some simple —but meaningful— cases. For example, using for ϕ^1 and ϕ^2 the transformation $\phi_{1/2}$ explained above, we have that clearly the formula $f \circledast g := f \circ \phi^1 \cdot g \circ \phi^2$ defines an n.p. product.

Corollary 4.16 With the same notation and in the setting explained above, suppose that $X_{\phi^1}(\mu)$ and $Y_{\phi^2}(\mu)$ define a compatible couple. Assume also that the map given by $X(\mu) \times$

$Y(\mu) \ni (f,g) \mapsto f \circ \phi^1 \cdot g \circ \phi^2$, is an n.p. product. Then the following assertions are equivalent.

(i) There is a constant $k > 0$ such that for every $f_1,...,f_n \in X(\mu)$ and $g_1,...,g_n \in Y(\mu)$,

$$\Big\| \sum_{i=1}^{n} B(f_i, g_i) \Big\| \leqslant k \Big\| \sum_{i=1}^{n} f_i \circ \phi^1 \cdot g_i \circ \phi^2 \Big\|_{X_{\phi^1}(\mu) \cdot Y_{\phi^2}(\mu)}. \tag{4.15}$$

(ii) B is an integral bilinear map that factors through $X_{\phi^1}(\mu) \cdot Y_{\phi^2}(\mu)$ as

$$B(f,g) = T(f \circ \phi^1 \cdot g \circ \phi^2), \quad f \in X(\mu),\ g \in Y(\mu), \tag{4.16}$$

where $T : X_{\phi^1}(\mu) \cdot Y_{\phi^2}(\mu) \to Z(\mu)$ is a linear continuous operator.

Inspired in part by the example of the general Hilbert transform with compact support explained above, we start now to give a more accurate analysis of the problem of representing integral bilinear operators.

Lemma 4.1 Let $X(\mu)$ and $Y(\mu)$ be order continuous Banach function spaces over the measure space (Ω, Σ, μ). Consider an integral bilinear operator $B : X(\mu) \times Y(\mu) \to Z(\mu)$. Fix $x \in \Omega$, and let ϕ_x^1 and ϕ_x^2 be two measurable bijections defining isometries $X(\mu) \to X_{\phi_x^1}(\mu)$ and $Y(\mu) \to Y_{\phi_x^2}(\mu)$, respectively. Assume also that $X_{\phi_x^1}(\mu)$ and $Y_{\phi_x^2}(\mu)$ define a compatible couple.

Then there is a factorization through the product space $X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu)$ of the bilinear functional $B_x : X(\mu) \times Y(\mu) \to \mathbb{R}$ defined by

$$B_x(f,g) := B(f,g)(x) = \int_{\Omega} f(\phi_x^1(t))\, g(\phi_x^2(t))\, K(x,t)\, dt, \quad f \in X(\mu),\ g \in Y(\mu). \tag{4.17}$$

Moreover, the functional $\varphi_x \in (X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu))^*$ that closes the factorization diagram is $\varphi_x(h(t)) := \int_{\Omega} h(t) K(x,t) dt \in \mathbb{R}$, and so we have that

$$B(f,g)(x) = \big\langle f(\phi_x^1(\cdot)) \cdot g(\phi_x^2(\cdot)), \varphi_x(\cdot) \big\rangle, \quad f \in X(\mu),\ g \in Y(\mu). \tag{4.18}$$

Proof. It is worth noting that clearly order continuity of the spaces $X(\mu)$ and $Y(\mu)$ is automatically transferred to the spaces $X_{\phi_x^1}(\mu)$ and $Y_{\phi_x^2}(\mu)$. So, the lemma is just a consequence of the factorization theorem for zero product preserving operators and the construction. Indeed, taking into account that $\Delta_{\phi_x^1}$ and $\Delta_{\phi_x^2}$ are isometries, we can define a bilinear map

$B'_x : X_{\phi_x^1}(\mu) \times Y_{\phi_x^2}(\mu) \to \mathbb{R}$

by $B'_x = B_x \circ (\Delta_{\phi_x^1}^{-1} \times \Delta_{\phi_x^2}^{-1})$. So, we have a factorization as

$$\begin{tikzcd} X(\mu) \times Y(\mu) \arrow[r, "B_x"] \arrow[d, dotted, "\Delta_{\phi_x^1} \times \Delta_{\phi_x^2}"'] & \mathbb{R}, \\ X_{\phi_x^1}(\mu) \times Y_{\phi_x^2}(\mu) \arrow[ur, "B'_x"'] & \end{tikzcd}$$

where B'_x is a bilinear integral and symmetric operator. Therefore, by Theorem 4.1, and taking into account that the spaces are order continuous —and so simple functions are dense—, it can be also factored as

$$\begin{tikzcd} X_{\phi_x^1}(\mu) \times Y_{\phi_x^2}(\mu) \arrow[r, "B'_x"] \arrow[d, dotted, "\odot"'] & \mathbb{R}. \\ X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu) \arrow[ur, "T_x"'] & \end{tikzcd}$$

Once the existence of the factorization through the product space $X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu)$ has been established, it is clear that T_x has to be the linear and continuous functional

$$\varphi_x(h) := h \mapsto \int_\Omega h(t)K(x,t)dt \in \mathbb{R},$$

that belongs to the dual space $\big(X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu)\big)^*$. Thus, we get that the pointwise evaluation of $B(f,g)$ at x can be written as

$$B_x(f,g) = B(f,g)(x) = \big\langle f(\phi_x^1(\cdot)) \cdot g(\phi_x^2(\cdot)), \varphi_x(\cdot)\big\rangle. \qquad \square$$

Let $(\Omega_1, \Sigma_1, \mu)$ and $(\Omega_2, \Sigma_2, \nu)$ be σ-finite measure spaces and h is an integrable function on $\Omega_1 \times \Omega_2$, then the well known Fubini's Theorem indicates that

$$\int_{\Omega_1 \times \Omega_2} h(x,y)d(\mu \times \nu) = \int_{\Omega_1} \Big(\int_{\Omega_2} h(x,y)d\mu\Big)d\nu = \int_{\Omega_2} \Big(\int_{\Omega_1} h(x,y)d\nu\Big)d\mu.$$

This theorem gives directly the next result.

Lemma 4.2 Let $Z(\mu)$ be order continuous. In the same setting and with the same notation that in Lemma 4.1 and assuming also that $K(x,t)$ (and so φ_x) depends only on t, —that is $\varphi_x = \varphi$ for all x for a certain functional φ—, we have that for every element $\psi \in Z(\mu)'$,

$$\Big\langle B(f,g), \psi \Big\rangle = \Big\langle \int_\Omega (f \circ \phi_x^1)(t) \cdot (g \circ \phi_x^2)(t)\psi(x)d\mu(x), \varphi(t) \Big\rangle \tag{4.19}$$

for every pair $f \in X(\mu)$ and $g \in Y(\mu)$.

The previous results suggest that our general class of integral bilinear operators can be analyzed in the product factorization framework constructed in the present paper whenever some requirements on the pointwise domination are assumed. Thus, next theorem gives a characterization of general bilinear operators by means of inequalities involving bimeasurable bijections. As it is seen, the arguments used in the proof are adaptations of the ones we have used in the rest of the study and belong to the same cycle of ideas. We use the notation of Theorems 4.1 and 4.2. Note that, as we explained before, the requirement on the equality of the products is natural if we are working with the class of rearrangement invariant Banach function spaces.

Theorem 4.4 Let $X(\mu)$ and $Y(\mu)$ be a compatible couple of order continuous Banach function spaces such that its product is also order continuous, and consider a bilinear continuous operator $B : X(\mu) \times Y(\mu) \to Z(\mu)$. Consider a couple of parametric families $\{\phi_x^1\}_{x\in\Omega}$ and $\{\phi_x^2\}_{x\in\Omega}$ of bimeasurable bijections satisfying that $X_{\phi_x^1}(\mu) \cdot Y_{\phi_x^2}(\mu) = X(\mu) \cdot Y(\mu)$ isometrically for each $x \in \Omega$. The following assertions are equivalent.

(i) There is a constant $k > 0$ such that for every $f_1, ..., f_n \in X(\mu)$, $g_1, ..., g_n \in Y(\mu)$ and $x_1, ..., x_n \in \Omega$,

$$\sum_{i=1}^{n} B(f_i, g_i)(x_i) \leqslant k \Big\| \sum_{i=1}^{n} f_i \circ \phi_{x_i}^1 \cdot g_i \circ \phi_{x_i}^2 \Big\|_{X(\mu)\cdot Y(\mu)}. \tag{4.20}$$

(ii) There is a constant $k > 0$ and a function h_0 such that the bilinear continuous map B is an integral map that can be written as

$$B(f,g)(x) = k \int_\Omega (f \circ \phi_x^1)(t) \cdot (g \circ \phi_x^2)(t)\, h_0(t)\, d\mu(t), \tag{4.21}$$

where $f \in X(\mu)$, $g \in Y(\mu)$, $x \in \Omega$, and $h_0 \in B_{(X(\mu)\cdot Y(\mu))'}$.

Proof. For (i) $\Rightarrow$ (ii) we use a standard separation argument in a Maurey-Rosenthal fashion, as in the previous section. Consider all the functions $\Phi : B_{(X(\mu)\cdot Y(\mu))'} \to \mathbb{R}$ defined as

$$\Phi(h) := \sum_{i=1}^{n} B(f_i, g_i)(x_i) - k \sum_{i=1}^{n} \int_\Omega f_i \circ \phi_{x_i}^1 \cdot g_i \circ \phi_{x_i}^2\, h\, d\mu \tag{4.22}$$

for given $f_i \in X(\mu)$, $g_i \in Y(\mu)$ and $x_i \in \Omega$. Each of these functions is convex with respect to h and continuous with respect to weak*-topology. Besides, the whole family is concave.

For each fixed function, by the Hahn-Banach Theorem and the inequality in (i), there is an element $h_\Phi \in B_{(X(\mu)\cdot Y(\mu))'}$ such that $\Phi(h_\Phi) \leqslant 0$ (alternatively, such an element can found by applying Ky Fan's lemma for concave family of continuous functions; see [15, Section 9.10]). Then we get a function $h_0 \in B_{(X(\mu)\cdot Y(\mu))'}$ such that

$$B(f,g)(x) \leqslant k \int_\Omega f \circ \phi_x^1 \cdot g \circ \phi_x^2 h_0(t)\, d\mu(t) \tag{4.23}$$

for all functions f, g and all $x \in \Omega$. Since this must happen for all functions f and g we can change the signus in the inequality above just by changing for example f by $-f$. Thus, we obtain for all f, g and x that

$$B(f,g)(x) = k \int_\Omega f \circ \phi_x^1 \cdot g \circ \phi_x^2 h_0(t)\, d\mu(t). \tag{4.24}$$

(ii) $\Rightarrow$ (i) is given by a direct calculation. □

4.2 $\odot_{P\times Q}$-Factorable Bilinear Operators acting in Sequence Spaces

In this section, we will concern with the pointwise product acting on sequence spaces and we will give a factorization theorem for zero product preserving bilinear maps defined on a Cartesian product of Banach sequence spaces. Using the isomorphisms, we will introduce a general notion for pointwise product.

Remark 4.8 The pointwise product $\odot : \ell^p \times \ell^q \to \ell^r$, $((x_k),(y_k)) \rightsquigarrow x_k \cdot y_k$ is an n.p. product, where $1/p + 1/q = 1/r$ and $1 \leqslant r < p, q < \infty$ (see [51, Example 1] and references therein). In particular, it is a norming product.

Proof. Indeed, for $(x_k) \in \ell^p$ and $(y_k) \in \ell^q$ we get $(x_k \cdot y_k) \in \ell^r$ and $\|(x_k \cdot y_k)\|_r \leqslant \|(x_k)\|_p \|(y_k)\|_q$ by the Hölder-Rogers inequality (see page 13 or [24, Lemma 1]). To establish the converse, let us assume that $(z_k) \in \ell^r$. Define the (x_k) and (y_k) by setting $x_k := |z_k|^{r/p} sgn z_k$ and $y_k := |z_k|^{r/q}$ for all $k \in \mathbb{N}$, where $sgn z_k$ is the sign function of z_k. We obtain that $(z_k) = (x_k)\cdot(y_k)$ and $(x_k) \in \ell^p$, $(y_k) \in \ell^q$ with the norms $\|(x_k)\|_p = \left\|(|z_k|^{r/p} sgn z_k)\right\|_p = \left\|(z_k)\right\|_r^{r/p}$ and $\|(y_k)\|_q = \left\|(|z_k|^{r/q})\right\|_q = \left\|(z_k)\right\|_r^{r/q}$. So, $\left\|(z_k)\right\|_r = \left\|(z_k)\right\|_r^{r/p} \left\|(z_k)\right\|_r^{r/q} = \|(x_k)\|_p \|(y_k)\|_q$. Therefore, $\|(x_k \cdot y_k)\|_r = \inf\{\|(x'_k)\|_p \|(y'_k)\|_q : (x'_k) \in \ell^p, (y'_k) \in \ell^q, (x_k \cdot y_k) = (x'_k \cdot y'_k)\}$. The inclusion is seen easily if we choose the $\|(z_k)\|_r < 1$. Consequently, $\odot$ is an n.p. product on $\ell_p \times \ell_q$. □

Theorem 4.5 Let $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leqslant r < p, q < \infty$. Consider the Banach space-valued bilinear operator $B : \ell^p \times \ell^q \to Y$. The following assertions are equivalent.

(1) The operator B is $\odot$-factorable. That is, there is a linear and continuous operator $T : \ell^r \to Y$ such that $B = T \circ \odot$.

(2) There is a positive real number K such that, for all $a_1, ..., a_n \in \ell^p$ and $b_1, \dots, b_n \in \ell^q$, the following inequality holds

$$\Big\| \sum_{i=1}^{n} B(a_i, b_i) \Big\|_Y \leqslant K \Big\| \sum_{i=1}^{n} a_i \odot b_i \Big\|_{\ell^r}. \tag{4.25}$$

(3) The operator B is zero product preserving, that is, $B(x,y) = 0$ whenever $x \odot y = 0$.

Namely, the following diagram commutes whenever one of the above conditions holds.

$$\begin{tikzcd} \ell^p \times \ell^q \arrow[r, "B"] \arrow[dr, dotted, "\odot"'] & Y \\ & \ell^r . \arrow[u, dotted, "T"'] \end{tikzcd}$$

Proof. (1)$\Leftrightarrow$(2) is proved in Lemma 3.1 for an arbitrary product and $(2) \Rightarrow (3)$ is obvious. It only remains to show that (3) implies (1). So let us show that there is a linear continuous operator T such that $B := T \circ \odot$ whenever the operator B is a zero $\odot$-preserving. Indeed, define the map $T_n : \ell^p \odot \ell^q \to Y$, $T_n(z) := B(z \odot \chi_{\{1,2,\dots,n\}}, \chi_{\{1,2,\dots,n\}})$ for all $n \in \mathbb{N}$, where $z \in \ell^p \odot \ell^q$. Note that $z \odot \chi_{\{1,2,\dots,n\}} \in \ell^p$, and $\chi_{\{1,2,\dots,n\}} \in \ell^q$, and so T_n is well defined for each $n \in \mathbb{N}$. The linearity of T_n is a consequence of the linearity of the bilinear operator B in the first variable. To show the boundedness of the map T_n, we give an equivalent formula for this operator. Since $\chi_{\{1,2,\dots,n\}} = \sum_{i=1}^{n} \chi_{\{i\}}$, we have

$$T_n(a \odot b) = B(a \odot b \odot \chi_{\{1,2,\dots,n\}}, \chi_{\{1,2,\dots,n\}}) = \sum_{i=1}^{n} B(a \odot b \odot \chi_{\{1,2,\dots,n\}}, \chi_{\{i\}}).$$

The pointwise product of $a = (\alpha_k)_{k=1}^{\infty} \in \ell^p$ and $b = (\beta_k)_{k=1}^{\infty} \in \ell^q$ is $a \odot b = (\alpha_k \beta_k)_{k=1}^{\infty} = \sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}}$. By the continuity of B, the image of the couple $(a,b) \in \ell^p \times \ell^q$ under the bilinear operator B is

$$\begin{aligned} B(a,b) &= B\Big(\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}, \sum_{m=1}^{\infty} \beta_m \chi_{\{m\}} \Big) \\ &= \sum_{k=1}^{\infty} \alpha_k \sum_{m=1}^{\infty} \beta_m B(\chi_{\{k\}}, \chi_{\{m\}}). \end{aligned}$$

Since $\chi_{\{k\}} \odot \chi_{\{m\}} = 0 \, (k \neq m)$ and by the zero $\odot$-preserving property of the operator B, we have $B(a,b) = \sum_{k=1}^{\infty} \alpha_k \beta_k B(\chi_{\{k\}}, \chi_{\{k\}})$. Thus,

$$
\begin{aligned}
T_n(a \odot b) &= \sum_{i=1}^{n} B(a \odot b \odot \chi_{\{1,2,\dots,n\}}, \chi_{\{i\}}) \\
&= \sum_{i=1}^{n} B\Big(\sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}} \odot \chi_{\{1,2,\dots,n\}}, \chi_{\{i\}}\Big) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_k \beta_k B(\chi_{\{k\}}, \chi_{\{i\}}).
\end{aligned}
$$

Using the zero $\odot$-preserving property once again, we obtain

$$
\begin{aligned}
T_n(a \odot b) &= \sum_{i=1}^{n} \alpha_i \beta_i B(\chi_{\{i\}}, \chi_{\{i\}}) \\
&= B\Big(\sum_{i=1}^{n} \alpha_i \beta_i \chi_{\{i\}}, \sum_{i=1}^{n} \chi_{\{i\}}\Big) \\
&= B\Big(\sum_{i=1}^{n} \alpha_i \chi_{\{i\}}, \sum_{i=1}^{n} \beta_i \chi_{\{i\}}\Big).
\end{aligned}
$$

By the boundedness of the bilinear operator B, it follows that

$$
\begin{aligned}
\sup_{z \in U_{\ell^r}} \|T_n z\|_Y &= \sup_{\substack{(a,b) \in U_{\ell^p} \times U_{\ell^q} \\ z = a \odot b}} \Big\| B\Big(\sum_{i=1}^{n} \alpha_i \chi_{\{i\}}, \sum_{i=1}^{n} \beta_i \chi_{\{i\}}\Big) \Big\|_Y \\
&\leqslant \sup_{\substack{(a,b) \in U_{\ell^p} \times U_{\ell^q} \\ z = a \odot b}} \sum_{i=1}^{n} |\alpha_i \beta_i| \|B(\chi_{\{i\}}, \chi_{\{i\}})\|_Y < \infty.
\end{aligned}
$$

This shows that T_n is (uniformly) bounded, $n \in \mathbb{N}$, and therefore $(T_n)_{n=1}^{\infty}$ is a bounded sequence of linear operators acting on ℓ^r, since $\ell^r = \ell^p \odot \ell^q$. Indeed, note that since $\odot$ is an n.p. product, we have that it is surjective and preserves the norm, and so for every $x \in \ell^r$ we find adequate $a \in \ell^p$ and $b \in \ell^q$ such that $x = a \odot b$.

The sequence $(T_n(a \odot b))_{n=1}^{\infty}$ is a Cauchy sequence for every $a \in \ell^p$ and $b \in \ell^q$ and it is convergent by completeness of the Banach space Y. Indeed, since $a \odot b \in \ell^r$, then for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$
\Big\| \sum_{i=n}^{\infty} \alpha_i \chi_{\{i\}} \Big\|_{\ell^p} \Big\| \sum_{i=n}^{\infty} \beta_i \chi_{\{i\}} \Big\|_{\ell^q} < \frac{\varepsilon}{\|B\|} \quad (\forall n > N).
$$

Using again that $B(\chi_{\{i\}}, \chi_{\{j\}}) = 0$ if $i \neq j$, we obtain

$$\begin{aligned}\|T_m(a\odot b)-T_n(a\odot b)\|_Y &= \Big\|B\Big(\sum_{i=n+1}^{m}\alpha_i\beta_i\chi_{\{i\}},\sum_{i=n+1}^{m}\chi_{\{i\}}\Big)\Big\|_Y\\ &\leqslant \|B\|\Big\|\sum_{i=n+1}^{m}\alpha_i\chi_{\{i\}}\Big\|_{\ell^p}\Big\|\sum_{i=n+1}^{m}\beta_i\chi_{\{i\}}\Big\|_{\ell^q}\\ &< \varepsilon \quad (\forall m>n>N).\end{aligned}$$

Let us define now the limit operator $T:\ell_r\to Y$ of the operator sequence $(T_n)_{n=1}^{\infty}$, that is $T(a\odot b)=\lim_{n\to\infty}T_n(a\odot b)$. It is easily seen that T is well defined and linear. This allows us to define the operator T_n in all ℓ^r. Since $(T_n(a\odot b))$ converges for every $a\odot b\in\ell^r$, then it is bounded for every $a\odot b$. By the Uniform Boundedness Theorem, it follows that T is continuous. Therefore, we obtain

$$\begin{aligned}B(a,b) &= \lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i\beta_i B(\chi_{\{i\}},\chi_{\{i\}})\\ &= \lim_{n\to\infty}T_n(a\odot b)=T(a\odot b).\end{aligned}$$

Besides, the image of an element is independent from its representation. Indeed, for the element $x=a_1\odot b_1=a_2\odot b_2$, we obtain

$$\begin{aligned}T(a_1\odot b_1) &= \lim_{n\to\infty}B(a_1\odot b_1\odot\chi_{\{1,2,\dots,n\}},\chi_{\{1,2,\dots,n\}})\\ &= \lim_{n\to\infty}B(a_2\odot b_2\odot\chi_{\{1,2,\dots,n\}},\chi_{\{1,2,\dots,n\}})=T(a_2\odot b_2).\end{aligned}$$

Hence we obtain the factorization of the bilinear operator B through the pointwise product as $B=T\circ\odot$. This finishes the proof. □

Now we will give a general version of the theorem above. Consider two Banach spaces E and F that are isomorphic —as Banach spaces— to ℓ^p and ℓ^q, respectively, and the isomorphisms are given by the operators $P:E\to\ell^p$ and $Q:F\to\ell^q$. We define the product $\odot_{P\times Q}:E\times F\to\ell^r$ by

$$\odot_{P\times Q}(x,y)=P(x)\odot Q(y),\quad x\in E,\ y\in F.$$

Let us illustrate this definition by the following diagram;

$$\begin{tikzcd} E \times F \arrow[r, "\odot_{P\times Q}"] \arrow[d, dotted, "P\times Q"'] & \ell^r. \\ \ell^p \times \ell^q \arrow[ur, dotted, "\odot"'] & \end{tikzcd}$$

In this situation, a bilinear map $B : E \times F \to Y$ will be called *zero $\odot_{P\times Q}$-preserving* if

$$\odot_{P\times Q}(x,y) = 0 \quad \text{implies} \quad B(x,y) = 0$$

for all $x \in E$ and $y \in F$.

Corollary 4.17 Let $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leqslant r < p,q < \infty$. Let the Banach spaces E and F be isomorphic to ℓ^p and ℓ^q by means of the isomorphisms P and Q, respectively. Consider a Banach space valued bilinear operator $B : E \times F \to Y$. The following assertions imply each other.

(1) The operator B is $\odot_{P\times Q}$-factorable. That is, there exists a linear continuous operator $T : \ell^r \to Y$ such that $B = T \circ \odot_{P\times Q}$, and the following diagram commutes;

$$\begin{tikzcd} E \times F \arrow[r, "B"] \arrow[d, dotted, "P\times Q"'] & Y. \\ \ell^p \times \ell^q \arrow[r, dotted, "\odot"] & \ell^r \arrow[u, dotted, "T"'] \end{tikzcd}$$

(2) There is a positive real number K such that, for every finite set of elements $\{x_i\}_{i=1}^n \in E$ and $\{y_i\}_{i=1}^n \in F$, the following inequality holds

$$\Big\|\sum_{i=1}^n B(x_i,y_i)\Big\|_Y \leqslant K \Big\|\sum_{i=1}^n P(x_i) \odot Q(y_i)\Big\|_{\ell^r}. \tag{4.26}$$

(3) The operator B is zero $\odot_{P\times Q}$-preserving, that is, $x \odot_{P\times Q} y = 0$ implies $B(x,y) = 0$.

Proof. Let us prove that (3) implies (1). Under the conditions of the theorem, consider the bilinear map $\overline{B} = B \circ (P^{-1} \times Q^{-1}) : \ell^p \times \ell^q \to Y$. We have that for all $x \in E$ and $y \in F$, $x \odot_{P\times Q} y = P(x) \odot Q(y) = 0$ implies that $0 = B(x,y) = \overline{B}(P(x),Q(y)) = 0$. That is, since P and Q are isomorphisms, we have that for all $a \in \ell^p$ and $b \in \ell^q$, $a \odot b = 0$ implies that $\overline{B}(a,b) = 0$.

We are in situation of using Theorem 4.5 for $\overline{B}$. So we have that there is a linear operator $T : \ell^r \to Y$ such that $\overline{B} = T \circ \odot$. By the definition of $\overline{B}$, we obtain $B = \overline{B} \circ (P \times Q) =$

$T \circ \odot \circ (P \times Q)$, the required factorization.

The equivalences among the three statements of the corollary follow directly using Lemma 3.1 and this factorization. $\square$

It is easily seen that any $\odot$-factorable bilinear map $B : \ell^p \times \ell^p \to Y$ that factors through ℓ^r for $2r = p$ is symmetric in the sense that $B(a,b) = B(b,a)$ for all $a, b \in \ell^p$, since $B((a_n),(b_n) = T((a_n \odot b_n)) = T((b_n \odot a_n)) = B((b_n),(a_n)$ holds for all $(a_n), (b_n) \in \ell^p$ by the commutativity of the pointwise product.

Corollary 4.18 Let the Banach space X be isomorphic to ℓ^p for $p \geqslant 2$ by means of the isomorphism P. Then any zero $\odot_{P\times P}$-preserving bilinear map $B : X \times X \to Y$ satisfies the symmetry condition, that is $B(x,y) = B(y,x)$ for all $x, y \in X$.

Proof. Since the map B is zero $\odot_{P\times P}$-preserving, it is $\odot_{P\times P}$-factorable. Then, for $r = p/2$ there is a linear continuous map $T : \ell^r \to Y$ defined by $B(x,y) = T \circ \odot \circ (P \times P)(x,y) = T(P(x) \odot P(y))$. By the commutativity of the pointwise product we get the symmetry;

$$B(x,y) = T(P(x) \odot P(y)) = T(P(y) \odot P(x)) = B(y,x).$$

$\square$

Remark 4.9 The extension of the result given in Theorem 4.5 from the case of $\odot$ to the case of $\odot_{P\times Q}$ products implicitly shows a fundamental fact about factorization through the pointwise product. The requirement "$a \odot b = 0$ implies $B(a,b) = 0$" can be understood as a lattice-type property: indeed, note that for sequences a and b in the corresponding spaces, $a \odot b = 0$ if and only if a and b are disjoint, and so we can rewrite the requirement of being zero $\odot$-preserving as "if $|a| \wedge |b| = 0$, then $B(a,b) = 0$". Since P and Q are just (Banach space) isomorphisms, we have shown that the property is primarily related to the pointwise product, and not to the lattice properties. The result is particularly meaningful if we consider P and Q to be the isomorphisms associated to changes of unconditional basis of ℓ^p and ℓ^q whose elements are not in general disjoint.

Remark 4.10 Consider the bilinear map $B : E \times E^* \to Y$, where E is isomorphic to ℓ^p and * denotes the topological dual of E. This bilinear map can only be $\odot_{P\times Q}$-factorable through the sequence space ℓ^1. Indeed, let P denote the isomorphism between E and ℓ^p $(p \geqslant 1)$. Since the duals of isomorphic spaces are isomorphic, it follows that E^* is isomorphic to $(\ell^p)^* = \ell^{p^*}$ for $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$ by the isomorphism P^* that is adjoint map of

P. Therefore B can only be $\odot_{P\times P*}$-factorable and in this case it is factored through ℓ^1.

4.2.1 Compactness and Summability Properties

Corollary 4.17 provides a fundamental tool to obtain the main properties including the compactness and summability properties of zero $\odot_{P\times Q}$-preserving bilinear maps. It is already clear that (weak) compactness of the factorization map T is necessary and sufficient condition for the (weak) compactness of the zero $\odot_{P\times Q}$-preserving map B by the definition of the norm preserving product. Indeed,

$$\begin{aligned}
\text{zpp map } B \text{ is (weakly) compact} &\Longleftrightarrow B(U_X\times U_Y) \text{ is relatively (weakly) compact}\\
&\Longleftrightarrow B(P^{-1}(U_{\ell^p})\times Q^{-1}(U_{\ell^q})) \text{ is relatively (weakly) compact}\\
&\Longleftrightarrow \overline{B}(U_{\ell^p}\times U_{\ell^q}) \text{ is relatively (weakly) compact}\\
&\Longleftrightarrow T\circ\odot(U_{\ell^p}\times U_{\ell^q}) \text{ is relatively (weakly) compact}\\
&\Longleftrightarrow T(U_{\ell^r}) \text{ is relatively (weakly) compact}\\
&\Longleftrightarrow T \text{ is (weakly) compact.}
\end{aligned}$$

Now, we will give more specific situations. Note that the norming property of the pointwise product $\odot$ can be expanded to closed unit ball as $B_{\ell^r}\subseteq B_{\ell^p}\odot B_{\ell^r}$ by choosing $\|(z_k)\|_r\leqslant 1$ in Remark 4.8, where $1/p+1/q=1/r$ and $1\leqslant r<p,q<\infty$.

Proposition 4.1 Let $\dfrac{1}{p}+\dfrac{1}{q}=\dfrac{1}{r}$ for $1\leqslant r<p,q<\infty$. Suppose that there are isomorphisms $P:E\to\ell^p$ and $Q:F\to\ell^q$ such that the bilinear operator $B:E\times F\to Y$ is zero $\odot_{P\times Q}$-preserving. Then

(i) $B(E\times F)$ is a linear space.

(ii) If P and Q are isometries, then $B(B_E\times B_F)$ is convex.

(iii) If $r=1$ and Y is reflexive, then $B(B_E,y)$ is a relatively compact set for every $y\in F$ as well as $B(x,B_F)$ is relatively compact for every $x\in E$.

(iv) If $r>1$, then $B(B_E\times B_F)$ is relatively weakly compact.

(v) If $1\leqslant s<r<\infty$ and $Y=\ell^s$, then $B(B_E\times B_F)$ is relatively compact.

Proof. Consider the factorization for B given by $B=T\circ(P\odot Q)$.

(i) Since $\odot$ is a norming product and B factors through it by Theorem 4.5, we have that

$B(E\times F)=T(\ell^p\odot\ell^q)=T(\ell^r)$, that is, the range of a linear map. So it is a linear space.

(ii) Clearly, $A=P\odot Q(B_E\times B_F)=B_{\ell^p}\odot B_{\ell^q}=B_{\ell^r}$ is a convex set, and so $T(A)$ is also convex.

(iii) Note that there is a sequence $b=Q(y)$ such that $A=P\odot Q(B_E,y)$ is equivalent to $B_{\ell^p}\odot b\subset\ell^1$. Recall that $1<p,q<\infty$. Note also that $T:\ell^1\to Y$ is weakly compact by the reflexivity of the range space Y. Since the closed unit ball of a reflexive space is weakly compact, this gives rise to weak compactness of the set A in ℓ^1. Thus, we have that $T(A)$ is relatively compact by the Dunford–Pettis property of ℓ^1.

(iv) Since $B(B_E\times B_F)=T(P(B_E)\odot Q(B_F))$, and $P(B_E)\odot Q(B_F)$ is equivalent to the unit ball of the reflexive space ℓ^r, we get the result.

(v) Recall that by Pitt's Theorem (see page 8), every bounded linear operator from ℓ^r into ℓ^s is compact whenever $1\leqslant s<r<\infty$. The factorization gives directly the result. □

As in Section 4.1.2.2, our first summability property for zero preserving bilinear maps is a direct consequence of Grothendieck's Theorem. It also provides an integral domination for B. The second corollary is obtained as a result of the Schur's property of ℓ^1 and it is again an application of the compactness properties of the bounded subsets of ℓ^1.

Corollary 4.19 Let H_1,H_2 and H_3 be separable Hilbert spaces. Let $B:H_1\times H_2\to H_3$ be a zero $\odot_{P\times Q}$-preserving bilinear operator. Then

(i) for every $x_1,\ldots,x_n\in H_1$, $y_1,\ldots,y_n\in H_2$ there is a constant $K>0$ such that

$$\sum_{i=1}^{n}\Big\|B(x_i,y_i)\Big\|\leqslant K\sup_{z'\in B_{\ell^\infty}}\sum_{i=1}^{n}\Big|\Big\langle P(x_i)\odot Q(y_i),z'\Big\rangle\Big|, \tag{4.27}$$

(ii) and there is a regular Borel measure η over B_{ℓ^∞} such that

$$\|B(x,y)\|\leqslant K\int_{B_{\ell^\infty}}|\langle P(x)\odot Q(y),z'\rangle|\,d\eta(z'),\quad x\in H_1,\ y\in H_2. \tag{4.28}$$

Proof. Let us consider the $\odot_{P\times Q}$-preserving bilinear map $B:H_1\times H_2\to H_3$. Since any separable Hilbert space is isomorphic to the sequence space ℓ^2, we can define a bilinear map $\overline{B}=B(P^{-1}\times Q^{-1}):\ell^2\times\ell^2\to H_3$. The $\odot_{P\times Q}$-preserving property of B implies the $\odot$-preserving property of the map $\overline{B}$. Therefore, by Corollary 4.17 we have the factorization

$\overline{B} := T \circ \odot$, where $T : \ell^1 \to H_3$. One of the result of Grothendieck's Theorem states that every linear operator from ℓ^1 to a Hilbert space is 1-summing. It follows that, for every $x_1,...,x_n \in H_1$, $y_1,...,y_n \in H_2$ there is a constant $K > 0$ such that

$$\sum_{i=1}^{n} \left\| B(x_i,y_i) \right\| = \sum_{i=1}^{n} \left\| \overline{B}(P(x_i),Q(y_i)) \right\| \leqslant K \sup_{z' \in B_{\ell^\infty}} \sum_{i=1}^{n} \left| \left\langle P(x_i) \odot Q(y_i), z' \right\rangle \right|.$$

The second inequality of the corollary is clearly seen by Pietsch domination theorem (see page 9 or [15, Theorem 2.12]). This theorem states that every 1-summable operator has such a regular Borel measure. Thus, we get a regular Borel measure η over B_{ℓ^∞} satisfying

$$\|B(x,y)\| = \|\overline{B}(P(x),Q(y))\| \leqslant K \int_{B_{\ell^\infty}} |\langle P(x) \odot Q(y), z' \rangle| \, d\eta(z'), \quad x \in H_1, \, y \in H_2.$$

□

The following corollary is obtained as a result of the Schur's property of ℓ^1 and it is again an application of the compactness properties of the bounded subsets of ℓ^1:

Corollary 4.20 Let H_1, H_2 and H_3 be separable Hilbert spaces. Let $B : H_1 \times H_2 \to H_3$ be a zero $\odot_{P \times Q}$-preserving bilinear operator. Then:

(i) For every couple of sequences $(x_i)_{i=1}^{\infty}$ in H_1 and $(y_i)_{i=1}^{\infty}$ in H_2 such that $(P(x_i) \odot Q(y_i))_{i=1}^{\infty}$ is weakly convergent, we have that $(B(x_i,y_i))_{i=1}^{\infty}$ converges in the norm.

(ii) For $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ such that $P(S_1) \odot Q(S_2) \subseteq \ell^1$ is relatively weakly compact, i.e. $S_1 \times S_2$ is $\odot_{P \times P}$-relatively weakly compact, we have that $B(S_1 \times S_2)$ is relatively compact.

We can obtain some summability results if we consider the range space Y with some cotype-related properties. It is known that a Banach space has the Orlicz property, if it is of cotype 2 (see page 9 or [18, §8.9]). It follows that for any zero $\odot_{P \times Q}$-preserving bilinear map $B : E \times F \to Y$ whose range space Y has the Orlicz property, we get a domination as follows: there exists $k > 0$ such that for $x_1,...,x_n \in E$ and $y_1,...,y_n \in F$,

$$\left(\sum_{i=1}^{n} \|B(x_i,y_i)\|_Y^2 \right)^{1/2} \leqslant k \sup_{\varepsilon_i \in \{-1,1\}} \| \sum_{i=1}^{n} \varepsilon_i \, (P(x_i) \odot Q(y_i)) \|_{\ell^r}.$$

4.2.2 Applications of Zero $\odot_{P \times Q}$-Preserving Bilinear Maps

Finally, let us give some applications for the zero $\odot_{P \times Q}$-preserving bilinear maps.

Application 4.3 Consider a bilinear continuous operator $B : \ell^2 \times \ell^2 \to \ell^1$. It is known that the pointwise product $\odot$ from $\ell^2 \times \ell^2$ to ℓ^1 is a norming product. Let $(a,b) = (\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}, \sum_{m=1}^{\infty} \beta_m \chi_{\{m\}}) \in \ell^2 \times \ell^2$. Then the image of this element under pointwise product is

$$a \odot b = \Big(\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}\Big) \odot \Big(\sum_{m=1}^{\infty} \beta_m \chi_{\{m\}}\Big) = \sum_{k=1}^{\infty} \alpha_k \sum_{m=1}^{\infty} \beta_m (\chi_{\{k\}} \odot \chi_{\{m\}}) = \sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}}.$$

Thus, for the finite sets of sequences $a_1, ...a_n$, $b_1, ..., b_n$ we have

$$\sum_{i=1}^{n} a_i \odot b_i = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \alpha_{ik} \beta_{ik} \chi_{\{k\}} = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \alpha_{ik} \beta_{ik}\Big) \chi_{\{k\}}.$$

This is a sequence in absolutely summable sequence space ℓ^1 such that its general term is $z_k = \sum_{i=1}^{n} \alpha_k^i \beta_k^i$ for every $k \in \mathbb{N}$.

The ℓ^1 norm of this sequence is $\|z_k\|_{\ell^1} = \sum_{k=1}^{\infty} |\sum_{i=1}^{n} \alpha_{ik} \beta_{ik}|$. By Lemma 3.1, we obtain that the bilinear operator B is factorable by the pointwise product if and only if there is a constant K for all finite sequences $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n \in \ell^2$ such that

$$\Big\|\sum_{i=1}^{n} B(x_i, y_i)\Big\|_1 \leqslant K \sum_{k=1}^{\infty} \Big|\sum_{i=1}^{n} \alpha_{ik} \beta_{ik}\Big|.$$

Let us consider now a more specific bilinear operator $B : \ell^2 \times \ell^2 \to \ell^1$: a diagonal multilinear operator. Recall that a bilinear operator $B \in \mathcal{B}(\ell^2 \times \ell^2, \ell^1)$ is called bilinear diagonal if there is a bounded sequence $\xi = (\xi_k)_k$ such that $B(a,b) = \sum_{k=1}^{\infty} \xi_k \alpha_k \beta_k \chi_{\{k\}}$. By Hölder inequality, it is easily seen that $B \in \mathcal{B}(\ell^2 \times \ell^2, \ell^1)$ if and only if $\xi \in \ell^\infty$. For arbitrary finite sequences $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n \subset \ell^2$, we obtain

$$\begin{aligned}\Big\|\sum_{i=1}^{n} B(x_i, y_i)\Big\|_1 &= \Big\|\sum_{i=1}^{n} \sum_{k=1}^{\infty} \xi_k \alpha_{ik} \beta_{ik} \chi_{\{k\}}\Big\|_1 \\ &\leqslant \|\xi_k\|_\infty \sum_{k=1}^{\infty} \Big|\sum_{i=1}^{n} \alpha_{ik} \beta_{ik}\Big| = K \sum_{k=1}^{\infty} \Big|\sum_{i=1}^{n} \alpha_{ik} \beta_{ik}\Big|.\end{aligned}$$

Therefore, we obtain that every bilinear diagonal operator is factorable through $\odot$. Remark that a bilinear diagonal operator satisfies that $B(x,y) = 0$ whenever $x \odot y = 0$. Namely, every diagonal operator is zero product preserving.

Application 4.4 Take into account the bilinear continuous operator $B : \ell^2 \times \ell^2 \to \ell^1$ given by $B(x,y) = a * (x \odot y)$ where $a \in \ell^1$ and $*$ is convolution defined on ℓ^1 by the rule $(a * b)_n = \sum_{m=-\infty}^{\infty} a_m b_{n-m}$ for $a, b \in \ell^1$ (it is known that $\ell^1(\mathbb{Z})$ is a unital Banach algebra under convolution and satisfies $\ell^1(\mathbb{Z}) * \ell^1(\mathbb{Z}) = \ell^1(\mathbb{Z})$). It is clear that if $x \odot y = 0$ then $B(x,y) = 0$. Then by Theorem 4.5, we obtain a factorization operator $T : \ell^1 \to \ell^1$ for B such that $T(z) = T(x \odot y) = B(x,y)$ defined by $T(z) = a * z$. It is known that an operator T on $\ell^1(\mathbb{Z})$ is of the form $a * z$ if and only if it is translation invariant, i.e. $TA = AT$, where $A(z)(n) := z(n+1)$ ([67, pp. 63]). Thus, we conclude that a zero product preserving bilinear operator $B : \ell^2 \times \ell^2 \to \ell^1$ factors through a translation invariant linear operator if and only if there is an $a \in \ell^1$ such that $B(x,y) = a * (x \odot y)$ for all $x, y \in \ell^2$.

Application 4.5 Consider any bilinear map $B : L^2[0,2\pi] \times L^2[0,2\pi] \to Y$ such that $B(f,g) = 0$ whenever $f \odot_{\widehat{}\times\widehat{}} g = \widehat{f} \odot \widehat{g} = 0$ for $f, g \in L^2[0,2\pi]$, where $\widehat{}$ denotes the Fourier transform. Plancherel's theorem states that the Banach space $L^2[0,2\pi]$ is isometrically isomorphic to ℓ^2 by the Fourier transform (see page 16). Therefore, by Corollary 4.17 we get a factorization for the $\odot_{\widehat{}\times\widehat{}}$-factorable bilinear map B such that $B = T \circ \odot \circ (\widehat{} \times \widehat{})$ and the bilinear map B is symmetric by Corollary 4.18. The class of these bilinear maps was investigated by Erdoğan et al in [68] and the results of this study will be given in the first section of the next chapter.

Now, we will give a more specific example. $\mathscr{H}$ and H^2 stand for the holomorphic functions on the unit disc $\mathbb{D}$ and the Hardy space of functions, respectively. The Hardy space H^2, that is a closed subspace of $L^2[0,2\pi]$, consists of the functions whose all Fourier coefficients with negative index are zero. It is possible to represent any holomorphic function $f \in \mathscr{H}$ as a Taylor polynomial $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Moreover, this representation is given by the Fourier coefficients for the elements of H^2 and H^2 is isomorphically isomorphic to the sequence space ℓ^2 by means of Fourier transform.

Arregui and Blasco defined the u-convolution of the holomorphic functions f and g in $\mathscr{H}$ given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ as $f *_u g(z) = \sum_{n=0}^{\infty} u(a_n, b_n) z^n$, where $u : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a bilinear continuous map (see [69, Definition 1.1.]). If we consider the bilinear map u defined as $u(a_n, b_n) = a_n \odot b_n$, then we get $f *_u g(z) = \sum_{n=0}^{\infty} (a_n \odot b_n) z^n$. Therefore, it is seen that u-convolution defined on $H^2 \times H^2$ to H^2 is zero $\odot_{\widehat{}\times\widehat{}}$-preserving, since $f \odot_{\widehat{}\times\widehat{}} g = \widehat{f}(n) \odot \widehat{g}(n) = 0$ implies $f *_u g = 0$ for all $f, g \in H^2$. By Corollary 4.17, it follows that there is a linear map $T : \ell^1 \to H^2$ such that $f *_u g = T(\widehat{f}(n) \odot \widehat{g}(n)) = \sum_{n=0}^{\infty} x_n z^n$, where (x_n) is the sequence in ℓ^1 obtained by the pointwise product $\widehat{f}(n) \odot \widehat{g}(n)$. Also, by Corollary 4.18 it is obtained that u-convolution is a symmetric map.

CHAPTER 5

CONVOLUTION FACTORABILITY OF BILINEAR OPERATORS

Throughout the chapter we will use the convolution operator defined on the Banach space $L^1(G)$ by the formula

$$f * g(x) = \int_G f(y^{-1}x)g(y)d\mu(y), \tag{5.1}$$

where G is a compact Abelian group with Haar measure μ.

In particular, we are interested in the compact Abelian group $\mathbb{T}$ that is the circle group –the real line mod 2π. It is known that the convolution operation $*$ on $L^1(\mathbb{T})$ has commutativity and associativity properties, that is, $f * g = g * f$ and $f * (g * h) = (f * g) * h$ for all $f, g, h \in L^1(\mathbb{T})$; see [26, Chapter 1].

The aim of this chapter is to obtain a class of bilinear maps acting on a product of Hilbert spaces of integrable functions, respectively, a product of Banach algebras of integrable functions that can be factored through the convolution operation. Moreover, we will see that the class of these bilinear maps what will be called $*$-factorable is equivalent to zero convolution product bilinear maps. We will investigate compactness and summability conditions of $*$-factorable bilinear maps under the assumptions of some classical properties. Finally, we present some applications of $*$-factorable bilinear maps and a representation for Hilbert-Schmidt operators.

Note that the results of Section 5.1 and Section 5.2 are preseted in the published papers [68] and [70], respectively.

5.1 $*$-Factorable Bilinear Maps on Hilbert Spaces of Integrable Functions

In Section 2.2, we give some preliminaries related with group algebras. Since we state that we will be interested in compact Abelian group $\mathbb{T}$, it is useful to recall these informations for the circle group $\mathbb{T}$. The complex space $\mathcal{W}(\mathbb{T})$ of functions defined on $\mathbb{T}$ spanned by all continuous positive-definite functions on $\mathbb{T}$ coincides with the set of functions which have absolutely convergent Fourier series. The space $\mathcal{W}(\mathbb{T})$ is a unital Banach algebra of functions under pointwise operations known as Wiener algebra ([29, Theorem 32.10]). It is isomorphic to the Banach algebra $\ell^1(\mathbb{Z})$ by the isomorphism given by Fourier transform and it is endowed with the norm $\|f\|_{\mathcal{W}} = \|\hat{f}\|_1$ for $f \in \mathcal{W}(\mathbb{T})$, where $\hat{f}$ denotes the Fourier transform of f.

Now we will show that the convolution operation $*$ is an n.p. product from $L^2(G) \times L^2(G)$ to $\mathcal{W}(G)$ for an arbitrary compact Abelian group G, where $\mathcal{W}(G)$ is the unital Banach algebra of the functions with absolutely convergent Fourier series.

Remark 5.1 Let G be a compact Abelian group. Convolution map acting in $L^2(G) \times L^2(G)$ to $\mathcal{W}(G)$ is an n.p. product. In particularly, it is a norming product.

Proof. Let us consider $h \in U_{\mathcal{W}(G)}$. By using Theorem 34.15 in [29] it is seen that there exist the functions $f, g \in L^2(G)$ such that $h = f * g$ and these functions can be chosen in such way that $1 > \|h\|_{\mathcal{W}} = \|f\|_2^2 = \|g\|_2^2$, that is $(f, g) \in U_{L^2(G)} \times U_{L^2(G)}$. Thus, we get $U_{\mathcal{W}} \subseteq *(U_{L^2(G)} \times U_{L^2(G)})$.

Now, let us show $\|f * g\|_{\mathcal{W}} = \inf\{\|f'\|_2\|g'\|_2 : f', g' \in L^2(G), f' * g' = f * g\}$ for every $f, g \in L^2(G)$. By Theorem 34.14 given in [29], we have that $\|f * g\|_{\mathcal{W}} \leqslant \|f\|_2\|g\|_2$. Since the inequality is obtained for all couples (f', g') satisfying $f * g = f' * g'$, it follows that $\|f * g\|_{\mathcal{W}} \leqslant \inf\{\|f'\|_2\|g'\|_2 : f', g' \in L^2(G), f' * g' = f * g\}$. For the converse inequality, consider arbitrary elements $f, g \in L^2(G)$. Then, $h = f * g \in \mathcal{W}(G)$ and satisfies $\|f\|_2^2 = \|g\|_2^2 = \|f * g\|_{\mathcal{W}} = \|h\|_{\mathcal{W}}$ (see [29, Theorem 34.14 and 34.15]). Therefore, $\|h\|_{\mathcal{W}} = \|f * g\|_{\mathcal{W}} = \|f\|_2\|g\|_2$ and $\|f * g\|_{\mathcal{W}} = \inf\{\|f'\|_2\|g'\|_2 : f', g' \in L^2(G), f' * g' = f * g\}$ is obtained for every $f, g \in L^2(G)$. □

Hereafter, we deal with the bilinear maps $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$, so we will consider the n.p product $*$ defined on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ to the Wiener algebra $\mathcal{W}(\mathbb{T})$.

Theorem 5.1 Let $\mathbb{T}$ be the real line mod 2π and let Y be an arbitrary Banach space. For a bilinear continuous operator $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$, the following statements are equal;

i) The bilinear map B is zero product preserving, that is, $B(f,g) = 0$ if $f * g = 0$,

ii) There exists a linear and continuous map $T : \mathcal{W}(\mathbb{T}) \to Y$ such that $B = T \circ *$, i.e. the operator B is $*$-factorable.

iii) There is a constant K such that for all $f_1, f_2, \dots, f_n, g_1, g_2, \dots, g_n \in L^2(\mathbb{T})$, we have

$$\Big\| \sum_{i=1}^{n} B(f_i, g_i) \Big\|_Y \leqslant K \Big\| \sum_{i=1}^{n} x_i * y_i \Big\|_{\mathcal{W}}. \tag{5.2}$$

In this case, the following triangular diagram commutes;

$$\begin{tikzcd} L^2(\mathbb{T}) \times L^2(\mathbb{T}) \arrow[r, "B"] \arrow[dr, dotted, "*"'] & Y \\ & \mathcal{W}(\mathbb{T}). \arrow[u, "T"'] \end{tikzcd}$$

Proof. Assume first B is zero product preserving. By Plancherel's theorem it is known that the linear map $\check{}: \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ is an isometric isomorphism, then we can define the bilinear operator

$$\widetilde{B}\big((a_n),(b_n)\big) := B\big((a_n)^{\vee},(b_n)^{\vee}\big), \quad (a_n),(b_n) \in \ell^2(\mathbb{Z}), \tag{5.3}$$

which clearly provides $\widetilde{B}(\widehat{f}, \widehat{g}) = B(f,g)$ and the commutativity of the diagram

$$\begin{tikzcd} L^2(\mathbb{T}) \times L^2(\mathbb{T}) \arrow[r, "B"] \arrow[d, "\hat{}\times\hat{}"'] & Y. \\ \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[ur, "\widetilde{B}"'] & \end{tikzcd} \tag{5.4}$$

Now, we prove that there is a bounded linear map $\widetilde{T} : \ell^1(\mathbb{Z}) \to Y$ such that the following diagram commutes

$$\begin{tikzcd} \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \arrow[r, "\widetilde{B}"] \arrow[d, "\odot"'] & Y, \\ \ell^1(\mathbb{Z}) \arrow[ur, "\widetilde{T}"'] & \end{tikzcd} \tag{5.5}$$

where $\odot$ is the pointwise product of sequences.

For each $N \in \mathbb{N}$ we define the linear map $\widetilde{T}_N : \ell^1(\mathbb{Z}) \to Y$ by

$$\widetilde{T}_N\big((a_n)\big) := \widetilde{B}\big((a_n), \chi_{[-N,N]\cap\mathbb{Z}}\big), \quad (a_n) \in \ell^1(\mathbb{Z}). \tag{5.6}$$

We *claim* that, since B is zero product preserving, we have that $\widetilde{B}(\chi_{\{i\}},\chi_{\{j\}}) = 0$ when $i \neq j$. Indeed, since

$$0 = \chi_{\{i\}} \cdot \chi_{\{j\}} = \widehat{\widetilde{\chi_{\{i\}}}} \cdot \widehat{\widetilde{\chi_{\{j\}}}} = \left(\widetilde{\chi_{\{i\}}} * \widetilde{\chi_{\{j\}}}\right)\widehat{},$$

the zero product preserving property of B together with (5.4) gives $\widetilde{B}\left(\chi_{\{i\}},\chi_{\{j\}}\right) = B\left(\widetilde{\chi_{\{i\}}},\widetilde{\chi_{\{j\}}}\right) = 0$. Using this remark it is easy to see that

$$\widetilde{T}_N\left((a_n)\right) = \sum_{|j|\leqslant N} a_j \widetilde{B}\left(\chi_{\{j\}},\chi_{\{j\}}\right). \tag{5.7}$$

Therefore

$$\begin{aligned}
\|\widetilde{T}_N\left((a_n)\right)\|_Y &\leqslant \sum_{|j|\leqslant N} \left\|a_j \widetilde{B}(\chi_{\{j\}},\chi_{\{j\}})\right\|_Y \\
&= \sum_{|j|\leqslant N} \left\|\widetilde{B}(a_j\chi_{\{j\}},\chi_{\{j\}})\right\|_Y \\
&\leqslant \|\widetilde{B}\| \sum_{|j|\leqslant N} \|a_j\chi_{\{j\}}\|_{\ell^2(\mathbb{Z})} \|\chi_{\{j\}}\|_{\ell^2(\mathbb{Z})} \\
&= \|\widetilde{B}\| \sum_{|j|\leqslant N} |a_j| \leqslant \|\widetilde{B}\| \|(a_n)\|_{\ell^1(\mathbb{Z})},
\end{aligned}$$

and so $\widetilde{T}_N$ is continuous; in fact the family $\{\widetilde{T}_N : N \in \mathbb{N}\}$ is uniformly bounded, since $\|\widetilde{T}_N\| \leqslant \|\widetilde{B}\|$ for all $N \in \mathbb{N}$. Moreover, for each fixed $(a_n) \in \ell^1(\mathbb{Z})$, $\left(\widetilde{T}_N\left((a_n)\right)\right)$ is a Cauchy sequence in the Banach space Y, and so it is convergent. Indeed, for a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\sum_{|j|>N} |a_j| < \varepsilon/\|\widetilde{B}\|$ for all $N \geqslant k$. By using (5.7) we have that for all $M > N \geqslant k$,

$$\begin{aligned}
\|\widetilde{T}_M\left((a_n)\right) - \widetilde{T}_N\left((a_n)\right)\|_Y &= \|\sum_{|j|\leqslant M} a_j \widetilde{B}(\chi_{\{j\}},\chi_{\{j\}}) - \sum_{|j|\leqslant N} a_j \widetilde{B}(\chi_{\{j\}},\chi_{\{j\}})\|_Y \\
&\leqslant \sum_{N<|j|\leqslant M} \|\widetilde{B}(a_j\chi_{\{j\}},\chi_{\{j\}})\|_Y \\
&\leqslant \|\widetilde{B}\| \sum_{|j|>N} |a_j| < \varepsilon.
\end{aligned}$$

Therefore by using Banach-Steinhaus theorem —the Uniform Boundedness Principle— the map $\widetilde{T} : \ell^1(\mathbb{Z}) \to Y$ given by

$$\widetilde{T}\left((a_n)\right) := \lim_{N\to\infty} \widetilde{T}_N\left((a_n)\right), \tag{5.8}$$

is linear and bounded. Finally, by the continuity of $\widetilde{B}$ and again (5.7) we have

$$
\begin{aligned}
\widetilde{B}\big((a_n),(b_n)\big) &= \sum_{j=-\infty}^{\infty} a_j b_j \widetilde{B}(\chi_{\{j\}},\chi_{\{j\}}) \\
&= \lim_{N\to\infty} \sum_{|j|\leqslant N} a_j b_j \widetilde{B}(\chi_{\{j\}},\chi_{\{j\}}) \\
&= \lim_{N\to\infty} \widetilde{T}_N\big((a_n b_n)\big) = \widetilde{T}\circ\odot\big((a_n),(b_n)\big),
\end{aligned}
$$

and so the commutativity of (5.5) follows.

On the other hand, given a linear map $\widetilde{T}:\ell^1(\mathbb{Z})\to Y$ we can use that the Wiener algebra $\mathcal{W}(\mathbb{T})$ is isometrically isomorphic to $\ell^1(\mathbb{Z})$ by the Fourier transform to define an operator $T:\mathcal{W}(\mathbb{T})\to Y$ by

$$
T(f) := \widetilde{T}(\widehat{f}), \quad f\in\mathcal{W}(\mathbb{T}). \tag{5.9}
$$

This gives the factorization

$$
\begin{tikzcd}
\ell^1(\mathbb{Z}) \arrow[r, "\widetilde{T}"] \arrow[d, "\check{}"'] & Y. \\
\mathcal{W}(\mathbb{T}) \arrow[ur, "T"'] &
\end{tikzcd} \tag{5.10}
$$

Finally, the classical identity $\widehat{f*g} = \widehat{f}\cdot\widehat{g}$ that works in general for $f,g\in L^1(\mathbb{T})$ allows to write $* = \check{}\circ\odot\circ(\hat{}\times\hat{})$. Hence, we obtain the commutativity of the diagram

$$
\begin{tikzcd}
L^2(\mathbb{T})\times L^2(\mathbb{T}) \arrow[r, "\hat{}\times\hat{}"] \arrow[rrr, bend right, "*"'] & \ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z}) \arrow[r, "\odot"] & \ell^1(\mathbb{Z}) \arrow[r, "\check{}"] & \mathcal{W}(\mathbb{T}),
\end{tikzcd} \tag{5.11}
$$

and (ii) holds.

The equivalery of the statements (ii) and (iii) have already been seen in Lemma 3.1. The zero product preserving property of the B is obvious by both the statements (ii) and (iii). The proof is completed. $\square$

Actually, putting together the commutativity of diagrams (5.4), (5.5), (5.10) and (5.11) we have proved that:

Corollary 5.1 Let $\mathbb{T}$ be the real line mod 2π and let Y be an arbitrary Banach space. For the bilinear continuous operator $B: L^2(\mathbb{T})\times L^2(\mathbb{T})\to Y$ to be zero product preserving, it is a necessary and sufficient condition that there exist linear and bilinear continuous operators such that the following diagram commutes

$$\begin{tikzcd}
L^2(\mathbb{T})\times L^2(\mathbb{T}) \arrow[r, "B"] \arrow[d, "\widehat{\ }\times\widehat{\ }"'] \arrow[ddd, bend right, "*"'] & Y \\
\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z}) \arrow[ur, "\widetilde{B}"'] \arrow[d, "\odot"'] & \\
\ell^1(\mathbb{Z}) \arrow[uur, "\widetilde{T}"'] \arrow[d, "\check{\ }"'] & \\
\mathcal{W}(\mathbb{T}) \arrow[uuur, "T"'] &
\end{tikzcd}$$

where $\widehat{\ }$ and $\check{\ }$ stand for the Fourier and the Inverse Fourier transforms, respectively, and $\odot$ is the pointwise product of sequences.

Remark 5.2 Seeing the proof of our result we can give an explicit formula for the operator T in terms of the classical Dirichlet kernel. We *claim* that the map T of the theorem is given by

$$T(f) = \lim_{N\to\infty} B\big(f, D_N\big),$$

where D_N stands for the *Dirichlet kernel* which is given by the formula

$$D_N(x) = \sum_{|j|\leqslant N} e^{ijx}.$$

Indeed just observe that

$$\big(\chi_{[-N,N]\cap\mathbb{Z}}\big)^{\vee} = \sum_{|j|\leqslant\infty} \chi_{[-N,N]\cap\mathbb{Z}}(j)e^{ijx} = D_N(x),$$

and use (5.9), (5.8), (5.6) and (5.3) to obtain

$$\begin{aligned}
T(f) &= \widetilde{T}\big(\widehat{f}\big) = \lim_{N\to\infty} \widetilde{T}_N\big(\widehat{f}\big) = \lim_{N\to\infty} \widetilde{B}\big(\widehat{f}, \chi_{[-N,N]\cap\mathbb{Z}}\big) \\
&= \lim_{N\to\infty} B\big(f, (\chi_{[-N,N]\cap\mathbb{Z}})^{\vee}\big) = \lim_{N\to\infty} B\big(f, D_N\big).
\end{aligned}$$

Corollary 5.2 A zero product preserving bilinear map $B : L^2(\mathbb{T})\times L^2(\mathbb{T}) \to Y$ is symmetric, that is $B(f,g) = B(g,f)$ for all $f,\, g \in L^2(\mathbb{T})$.

Proof. Since the map B is zero product preserving, then there is a linear continuous map $T : \mathcal{W}(\mathbb{T}) \to Y$ defined by $B(f,g) = T\circ *(f,g)$. By the commutativity of the convolution product we get $B(f,g) = T(f*g) = T(g*f) = B(g,f)$. □

5.1.1 Properties of $*$-Factorable Maps on Hilbert Spaces

Some direct consequences on the properties of zero product preserving bilinear maps defined on Hilbert spaces of integrable functions can be fixed by using some classical properties. We will analyze separately the main two cases that are reasonable to consider in our context: when Y is a reflexive space, and when Y is a Banach space with the Schur property. In the first one, —that regards topological properties— we will provide some information in the case that B is weakly compact. In the second one it will be shown that zero product preserving operators have good summability properties in case Y has some suitable geometric properties. We will finish the section by showing an application of our results to what is called generalized convolution.

5.1.1.1 Compactness Properties As in the pointwise product case, it is easily seen that a zero product preserving map B is (weakly) compact if and only if the linear operator T appearing in its factorization is (weakly) compact, due to the definition of product. Indeed,

$$\begin{aligned}
\text{the zpp map } B \text{ is (weakly) compact} &\iff B(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})}) \text{ is relatively (weakly) compact} \\
&\iff T \circ *(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})}) \text{ is relatively (weakly) compact} \\
&\iff T(U_{\mathcal{W}(\mathbb{T})}) \text{ is relatively (weakly) compact} \\
&\iff T \text{ is (weakly) compact.}
\end{aligned}$$

Now, we assume that the bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ is weakly compact and we will get some particular results. The following result shows that zero product preserving operators satisfy a certain kind of Dunford–Pettis property.

Corollary 5.3 Let $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ be a zero product preserving weakly compact bilinear map, and let A be an $*$-relatively weakly compact set (see Definition 3.6). Then $B(A)$ is relatively compact.

Moreover, if there is a $*$-relatively weakly compact bilinear operator B for Y, then Y is finite dimensional.

Proof. This is just a consequence of the fact that ℓ^1 has the Dunford–Pettis property.

Since we have the factorization though ℓ^1 of B, and such factorization satisfies that

$$*(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})}) = U_{W(\mathbb{T})},$$

we have that $\widehat{\ }(*(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})})) = U_{\ell^1}$. Therefore, $T \circ \check{\ } : \ell^1 \to Y$ is weakly compact. Take an $*$-relatively weakly compact set A. We have that $\widehat{\ }(*A)$ is then a relatively weakly compact of ℓ^1. The Dunford–Pettis property of ℓ^1 gives then that $T \circ \check{\ } \circ \widehat{\ } \circ *(A) = T \circ *(A) = B(A)$ is relatively compact. The last statement is then clear. $\square$

This theorem can be improved for the case that Y has the Schur property. We will prove that zero product preserving bilinear maps give a characterization for the space ℓ^1 under a bit more restrictive requirements on B.

Corollary 5.4 Let $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ be a zero product preserving weakly compact bilinear map. Let Y be a Banach lattice with the Schur property. Then $B\big(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})}\big)$ is a relatively compact set in Y.

Consequently, if B is norming product for Y, then Y is finite dimensional.

Proof. We use Theorem 2 in [71], that establishes that a Banach space Y has the Schur property if and only if every weakly compact operator from ℓ^1 to Y is compact. As we shown in the proof of Corollary 5.3 we have that $\widehat{\ }(*(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})})) = U_{\ell^1}$. Therefore, $T \circ \check{\ } : \ell^1 \to Y$ is weakly compact, and so the Schur property for Y gives that it is also compact. Then obviously norming property of B implies that Y has finite dimension. $\square$

Corollary 5.5 A Banach space Y is isomorphic to ℓ^1 if and only if it admits an equivalently zero product preserving bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$ such that B is a norm preserving product from $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ to Y (see the Definition 3.7 for equivalently zero product preserving bilinear map).

Proof. The direct implication is obvious: if $R : \ell^1 \to Y$ is an isomorphism, just take $B = R \circ \widehat{\ } \circ *$. For the converse implication, suppose that B satisfies the requirements. Now take into account that we have a factorization of B as $B = T \circ *$ as a consequence of Theorem 5.1. Since we also have that $B(f,g) = 0$ implies $f * g = 0$, we have that T (and so the operator $\widetilde{T}$ defined in the proof of Theorem 5.1) is injective. But B is a norm preserving product, and so we have that

$$\widetilde{T}(U_{\ell^1}) \subseteq kU_Y \subseteq kB(U_{L^2(\mathbb{T})} \times U_{L^2(\mathbb{T})}) = k\widetilde{T}(U_{\ell^1}).$$

This gives the result. $\square$

5.1.1.2 Summability Properties For a zero product preserving bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Y$, the factorization given by Theorem 5.1 implies that for all $f_1, \ldots, f_n$, $g_1, \ldots, g_n \in L^2(\mathbb{T})$,

$$\sum_{i=1}^{n} \|B(f_i, g_i)\|_Y \leqslant k \sum_{i=1}^{n} \|\widehat{f_i} \cdot \widehat{g_i}\|_{\ell^1} = k \sum_{i=1}^{n} \sum_{j=1}^{\infty} |a^i_j b^i_j|,$$

where (a^i_j) and (b^i_j) are the sequences of Fourier coefficients of f_i and g_i, respectively.

If we assign some conditions to the range space Y, we obtain improved summability results. Firstly, we will consider Hilbert valued bilinear maps and this assumption will give us an integral domination.

Theorem 5.2 If H is a Hilbert space and $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to H$ is zero product preserving, then there is a constant $k > 0$ such that the following equivalent assertions hold.

(i) For $f_1, \ldots, f_n$, $g_1, \ldots, g_n \in L^2(\mathbb{T})$,

$$\sum_{i=1}^{n} \|B(f_i, g_i)\|_H \leqslant k \sup_{\varphi \in B_{\ell^\infty}} \sum_{i=1}^{n} \left|\left\langle \widehat{f_i} \cdot \widehat{g_i}, \varphi \right\rangle\right| = k \sup_{(\varphi_j) \in B_{\ell^\infty}} \sum_{i=1}^{n} \Big| \sum_{j=1}^{\infty} a^i_j b^i_j \varphi_j \Big|, \tag{5.12}$$

where (a^i_j) and (b^i_j) are the sequences of Fourier coefficients of f_i and g_i, respectively.

(ii) For $f, g \in L^2(\mathbb{T})$,

$$\|B(f, g)\|_H \leqslant k \int_{B_{\ell^\infty}} \left|\left\langle \widehat{f} \cdot \widehat{g}, \varphi \right\rangle\right| d\eta(\varphi) = k \int_{B_{\ell^\infty}} \Big| \sum_{j=1}^{\infty} a_j b_j \varphi_j \Big| d\eta(\varphi), \tag{5.13}$$

where η is a regular probability measure on the unit ball of ℓ^∞ given by the Pietsch Domination Theorem, and (a_j) and (b_j) are the sequences of Fourier coefficients of f and g, respectively.

Proof. Since B is a zero product preserving map, by Corollary 5.1 it factors through the linear operators T and $\tilde{T}$ as $B = T \circ *$ and $B = \tilde{T} \circ \odot \circ \widehat{\ } \times \widehat{\ }$. The linear operator $\tilde{T} : \ell^1 \to H$ is a summing operator as a consequence of Grothendieck's Theorem $\mathcal{L}(\ell^1, H) = \Pi_1(\ell^1, H)$. The summability of the operator $\tilde{T}$ implies the summability of the operator T, by the isometry between the spaces ℓ^1 and $\mathcal{W}(\mathbb{T})$. The first assertion is seen directly by being 1-summing of the operator $\tilde{T}$ and the definition of the summable operator. Second statement is a result of Pietsch's Domination Theorem. $\square$

Secondly we will consider the space Y with a cotype-related property and as an ℓ^p-space, respectively. If we assume Y is of cotype 2, then this implies Y to have Orlicz property (see page 9 or [18, §8.9]). Thus, we get a domination for any zero product preserving bilinear map with a range space that has Orlicz property as follows: there exists $k > 0$ such that for $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^2(\mathbb{T})$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_Y^2\Big)^{1/2} \leqslant k \sup_{\varepsilon_i \in \{-1,1\}} \|\sum_{i=1}^{n} \varepsilon_i \widehat{f_i} \cdot \widehat{g_i}\|_{\ell^1}.$$

Corollary 5.6 Let $1 \leqslant p \leqslant \infty$, and take r satisfying $1/r = 1 - |1/p - 1/2|$. For a zero product preserving bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \ell^p$, there exists a constant $k > 0$ such that for $f_1, \ldots, f_n, g_1, \ldots, g_n \in L^2(\mathbb{T})$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_{\ell^p}^r\Big)^{1/r} \leqslant k \sup_{\varepsilon_i \in \{-1,1\}} \|\sum_{i=1}^{n} \varepsilon_i \widehat{f_i} \cdot \widehat{g_i}\|_{\ell^1} = k \sup_{(\varphi_j) \in B_{\ell^\infty}} \sum_{i=1}^{n} |\sum_{j=1}^{\infty} a_j^i b_j^i \varphi_j|,$$

where (a_j^i) and (b_j^i) are the sequences of Fourier coefficients of f_i and g_i, respectively.

Proof. It is easily seen by the factorization of the zero product preserving operator B through the space ℓ^1 and the following result that can be found found in [18, §34.11]

$$\mathcal{L}(\ell^1, \ell^p) = \Pi_{r,1}(\ell^1, \ell^p) \text{ for } 1 \leqslant p \leqslant \infty \text{ and } r \text{ such that } 1/r = 1 - |1/p - 1/2|.$$

$\square$

This corollary provides the same result given in Theorem 5.2 for the case $p = 2$.

Finally, we will give a result that is a consequence of the classical Littlewood inequality, that can be written as $\mathcal{L}(\ell^1, \ell^{4/3}) = \Pi_{4/3,1}(\ell^1, \ell^{4/3})$ (see [18, §34.12]): for a $\ell^{4/3}$-valued zero product preserving map B, we obtain that

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_{\ell^{4/3}}^{4/3}\Big)^{3/4} \leqslant k \sup_{\varphi \in B_{\ell^\infty}} \sum_{i=1}^{n} |\langle \widehat{f_i} \cdot \widehat{g_i}, \varphi \rangle|.$$

5.1.2 Applications of $*$-Factorable Maps Acting in Hilbert Spaces

We show some examples of zero product preserving bilinear operators concerning recent developments in bilinear Fourier analysis.

Application 5.1 The first example is constructed by using translation invariant linear operators, —sometimes called convolution operators or multipliers—: they are the operators that commute with translations. The bibliography on this topic is deep and wide, we mention here only the classical paper by Cowling and Fournier [72]. Thus, consider an operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ that satisfies that $Tf * g = T(f * g)$. For example, we can take a convolution operator $T_k : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with convolution kernel $k \in L^2(\mathbb{T})$, that is $T_k f := k * f$. Consider now the bilinear map $B_k : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by convolution of $B_k(\cdot,\cdot) := T_k(\cdot) * (\cdot)$. Then we have

$$B_k(f,g) = T_k(f) * g = (k * f) * g = k * (f * g) = T_k(f * g),$$

for $f, g \in L^2(\mathbb{T})$. It is easily seen that the bilinear map B_k is zero prouct preserving since $B_k(f,g) = 0$ if $f * g = 0$. Therefore it has a linear factorization by Theorem 5.1, besides the factorization map is T_k.

Although we note that for the convolution map defined in products of spaces of continuous functions, these arguments for translation invariant bilinear maps can already be found in the paper by Edwards [73]; see the proof of Proposition 1 in this paper. If $P(D)$ is a linear partial differential operator, bilinear maps as $P(D)(f * g)$ are also usual in applications of the harmonic analysis.

Application 5.2 Consider a Banach space Z and a Bochner 2-integrable function $\Phi : [0, 2\pi] \to Z$ and the vector-valued-kernel bilinear operator $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to Z$ defined by

$$B(f,g) := \int_0^{2\pi} \int_0^{2\pi} \Phi(x) f(x-y) g(y) dy dx. \tag{5.14}$$

This bilinear map can be written as $(f,g) \mapsto \int_0^{2\pi} \Phi(x)(f * g)(x)\,dx$, and then it is 0-valued when $f * g = 0$. Therefore, by the factorization theorem of bilinear maps on Hilbert spaces of integrable functions (Theorem 5.1), we get that B factors through a linear map $T : \mathcal{W}(\mathbb{T}) \to Z$ such that $B(f,g) = T(f * g) = T(h) = \int_0^{2\pi} \Phi(x) h(x)\,dx$, where $h = f * g$.

Application 5.3 Let us explain some relations of our class with a genuine bilinear version of convolution, that is given by the so-called translation invariant bilinear operators. A considerable effort has been made recently for understanding this class of maps in the setting of the multilinear harmonic analysis; we refer to [74] and the references therein for information on the topic. They are given —in the case we consider $\mathbb{R}$ as measurable space and the operator is defined by a non-negative regular Borel measure μ— by the formula

$$B_\mu(f,g) := \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(x-z)d\mu(y,z), \quad f,g \in L^2(\mathbb{R}), \tag{5.15}$$

(see [74] and the references therein). We consider the "compact group version" of this definition with a slight modification. Take $\mu = k(z)dydz$ for $k \in L^2[0,2\pi]$ and consider the map

$$\begin{aligned} B_k(f,g) &:= \int_0^{2\pi}\int_0^{2\pi} f(y-x)g(x-z)k(z)\,dydz \\ &= \int_0^{2\pi} f(y-x)(k*g)(x)dy, \quad f,g \in L^2[0,2\pi]. \end{aligned}$$

Using this, and if Ψ is a Z-valued Bochner 2-integrable function —Z is a Banach space—, we can define the Z-valued kernel bilinear map by

$$B_{\Psi,k}(f,g) = \int_0^{2\pi} \Psi(y)\big(\int_0^{2\pi} f(y-x)(k*g)(x)dx\big)dy, \quad f,g \in L^2[0,2\pi].$$

Clearly,

$$B_{\Psi,k}(f,g) = \int_0^{2\pi} \Psi(y)\,(k*(f*g))(y)\,dy,$$

and this is 0 if $f*g=0$. Thus, by the factorization theorem, $B_{\Psi,k}$ can be written as $B_{\Psi,k} = T_{\Psi,k}\circ *$, where $T_{\Psi,k}$ is a linear continuous map defined by $T_{\Psi,k}(h) = \int_0^{2\pi}\Psi(y)\,(k*h)(y)\,dy$ for all $h \in \mathcal{W}([0,2\pi])$.

Application 5.4 Let $1<p<\infty$ and consider the continuous bilinear map $u: \ell^p \times \ell^{p'} \to \ell^1$ given by the pointwise product $u((a_i),(b_i)) := (a_i)\odot(b_i) = (a_ib_i) \in \ell^1$. We will use for this example the u-convolution for spaces of Bochner integrable functions defined by Blasco in [75] (see also [76, 77]). Following [75] and the notation in this paper, the u-convolution can be defined as a bilinear map $*_u : L^1(\mathbb{T},\ell^p)\times L^1(\mathbb{T},\ell^{p'}) \to L^1(\mathbb{T},\ell^1)$ by the formula

$$\phi *_u \psi(t) = \int_0^{2\pi} u(\phi(e^{is}),\psi(e^{i(t-s)}))\frac{ds}{2\pi} \in L^1(\mathbb{T},\ell^1), \tag{5.16}$$

for $\phi \in L^1(\mathbb{T},\ell^p)$, $\psi \in L^1(\mathbb{T},\ell^{p'})$. Consider now to sequences of integrable functions

(k_i) and (v_i) and assume that the linear maps $T_1 : L^2(\mathbb{T}) \to L^1(\mathbb{T}, \ell^p)$ and $T_2 : L^2(\mathbb{T}) \to L^1(\mathbb{T}, \ell^{p'})$ given by

$$T_1(f)(w) := \sum_{i=1}^{\infty} (k_i * f)(w) e_i \in \ell^p, \ \ T_2(g)(w) := \sum_{i=1}^{\infty} (v_i * g)(w) e_i \in \ell^{p'},$$

are well-defined for all $f, g \in L^2(\mathbb{T})$ and continuous. We consider the bilinear map $B := *_u \circ (T_1, T_2) : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^1(\mathbb{T}, Z)$. Let us show that it is zero product preserving. Indeed, for a fixed couple of functions $f, g \in L^2(\mathbb{T})$, we have

$$\begin{aligned} B(f,g)(t) &= \int_0^{2\pi} \Big(\sum_{i=1}^{\infty} (k_i * f)(e^{is}) \, (v_i * g)(e^{i(t-s)}) \, e_i \Big) \frac{ds}{2\pi} \\ &= \sum_{i=1}^{\infty} \Big(\int_0^{2\pi} (k_i * f)(e^{is}) \, (v_i * g)(e^{i(t-s)}) \Big) \frac{ds}{2\pi} \Big) \ e_i \in \ell^1. \end{aligned}$$

Thus, $B(f,g) = \sum_{i=1}^{\infty} (k_i * v_i) * (f * g) \, e_i$, and so it is zero product preserving. As the result of Theorem 5.1, B can be factor through the convolution and the linear map $T(h) = \sum_{i=1}^{\infty} ((k_i * v_i) * h) \, e_i$.

Application 5.5 Generalized Convolution Let us finish the section by showing a remark on a new construction that has shown to be useful for applications. It concerns to what is called generalized convolution; the reader can find information about in [78, Definition 2.3] (see also the references in this paper for the original definitions). Let U_1, U_2 and U_3 be linear spaces (may be different) on the same field of scalars and let V be a commutative algebra. Suppose that $K_1 \in \mathcal{L}(U_1, V)$, $K_2 \in \mathcal{L}(U_2, V)$ and $K_3 \in \mathcal{L}(U_3, V)$ are the linear operators from U_1, U_2 and U_3 to V respectively.

Definition 5.1 (Definition 2.3 in [78]) A bilinear map $* : U_1 \times U_2 \to U_3$ is called the convolution with weight-element δ —an element of the algebra V— for K_3, K_1, K_2 (in that order) if the following identity holds:

$$K_3(*(f,g)) = \delta K_1(f) K_2(g), \tag{5.17}$$

for any $f \in U_1$ and $g \in U_2$. The equality above is called the factorization identity of the convolution.

Fix $U_1 = U_2 = L^2(\mathbb{T})$, $U_3 = V = \mathcal{W}(\mathbb{T})$ and $K_3 = id$ and consider $*$ as the usual convolution bilinear map. Let us write now a characterization of when a bilinear map defined as a product in the algebra of two linear operators define a generalized convolution associated to $*$. Indeed, as a consequence of Theorem 5.1 we directly obtain the following

Corollary 5.7 Consider two operators $S_1, S_2 : L^2(\mathbb{T}) \to \mathcal{W}(\mathbb{T})$ and $\delta \in \mathcal{W}(\mathbb{T})$, and consider the bilinear map $B : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to \mathcal{W}(\mathbb{T})$ given by $B(\cdot,\cdot) = \delta\, S_1(\cdot)\, S_2(\cdot)$. Then the following assertions are equivalent.

(i) B is zero product preserving.

(ii) There is an operator $T : \mathcal{W}(\mathbb{T}) \to \mathcal{W}(\mathbb{T})$ such that $*$ is a convolution with weight δ for T, S_1, S_2.

In this case, the factorization identity is $T \circ * = B = \delta\, S_1\, S_2$.

5.2 Factorization for Bilinear Maps Defined on Banach Modules

In Section 2.2 we stated that the subalgebras $L^p(\mathbb{T})\,(1 \leqslant p < \infty)$, $C(\mathbb{T})$, $\mathcal{W}(\mathbb{T})$ of the algebra $L^1(\mathbb{T})$ are left Banach $L^1(\mathbb{T})$-modules with respect to convolution for the circle group $\mathbb{T}$ such that the space of trigonometric polynomials $\mathcal{T}(\mathbb{T})$ is a dense subspace in these algebras. Thus implies that $L^1(\mathbb{T}) * M(\mathbb{T}) = M(\mathbb{T})$, where $M(\mathbb{T}) \in \left\{L^p(\mathbb{T})\,(1 \leqslant p < \infty), C(\mathbb{T}), \mathcal{W}(\mathbb{T})\right\}$. Moreover, a left bounded approximate identity of $L^1(\mathbb{T})$ is also a left bounded approximate identity for them, i.e. $\lim_\alpha \|h_\alpha * g - g\| = 0$, where (h_α) is a left bounded approximate identity of $L^1(\mathbb{T})$. Therefore we obtain the following remark.

Remark 5.3 As a consequence, we conclude that $*(U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})}) = U_{M(\mathbb{T})}$, where $M(\mathbb{T}) \in \left\{L^p(\mathbb{T})\,(1 \leqslant p < \infty), C(\mathbb{T}), \mathcal{W}(\mathbb{T})\right\}$, that is, convolution is a norming product on $L^1(\mathbb{T}) \times M(\mathbb{T})$.

Now we will give a factorization theorem through the convolution product for the zero product preserving bilinear maps acting on a Cartesian product of the above subalgebras. Since we consider the nets called *approximate identities* that appear as substitute for the unit element of a Banach algebra, we will need the generalization of Banach-Steinhaus Theorem to nets.

Theorem 5.3 [79, pp. 141] **Banach-Steinhaus Theorem for Nets** Let E be a barrelled space and F is locally convex. Let $(A_\alpha)_{\alpha \in I}$ be a net in $\mathcal{L}(E,F)$ such that for every $x \in E$ the net $(A_\alpha x)_{\alpha \in I}$ is bounded in F and converges to an element $A_0 x \in F$. Then $A_0 \in \mathcal{L}(E,F)$.

Theorem 5.4 For the bilinear operator $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$, where $M(\mathbb{T}) \in \left\{L^p(\mathbb{T})\,(1 \leqslant p < \infty), C(\mathbb{T}), \mathcal{W}(\mathbb{T})\right\}$, the following statements are equivalent.

i) $B(f,g)=0$ whenever $f*g=0$, i.e. B is zero product preserving.

ii) There is a linear continuous operator $T: M(\mathbb{T}) \to Y$ such that $B := T \circ *$, that is, B is $*-$factorable.

iii) There is a constant K such that for all $f_1, f_2, ..., f_n \in L^1(\mathbb{T})$ and $g_1, g_2, ..., g_n \in M(\mathbb{T})$, we have

$$\Big\| \sum_{i=1}^{n} B(f_i,g_i) \Big\|_Y \leqslant K \Big\| \sum_{i=1}^{n} f_i * g_i \Big\|_{M(\mathbb{T})}. \tag{5.18}$$

As a consequence of this theorem, we have a factorization for B which satisfies the following scheme whenever any of the statements above holds;

$$\begin{tikzcd} L^1(\mathbb{T}) \times M(\mathbb{T}) \arrow[r, "B"] \arrow[d, "*"'] & Y. \\ M(\mathbb{T}) \arrow[ur, "T"'] & \end{tikzcd}$$

Proof. Lemma 3.1 shows that the factorization gives the inequality written in (iii), and it is obvious that (iii) implies (i). Then, we will only prove that every zero product preserving bilinear operator has a factorization. Assume that the continuous bilinear operator B maps zero the couples of functions whose convolution is equal to zero. Since $\mathbb{T}$ is a compact topological group, the set of trigonometric polynomials $\mathfrak{T}(\mathbb{T})$ is dense in both $L^1(\mathbb{T})$ and $M(\mathbb{T})$. A trigonometric polynomial on $\mathbb{T}$ is an expression of the form $\sum_{n=-N}^{N} a_n e^{int}$, where (a_n) is a finite sequence of scalars. It is known that $L^1(\mathbb{T})$ has a bounded 1 approximate identity such that its elements are positive definite trigonometric polynomials and it is also an approximate identity for the subalgebra $M(\mathbb{T})$. Let us denote it by $(h_\alpha)_{\alpha\in I}$. Since h_α is a trigonometric polynomial for each α, it can be written by the expression $h_\alpha = \sum_{j=-N_\alpha}^{N_\alpha} h(\alpha)_j e^{ijt}$, where $h(\alpha)_j$ is the jth Fourier coefficiant of h_α.

Assume that f and g are trigonometric polynomials. So, they can be written with the forms $f = \sum_{k=-N_f}^{N_f} f_k e^{ikt}$ and $g = \sum_{l=-N_g}^{N_g} g_l e^{ilt}$. It is seen by the definition of the convolution operation that $e^{ikx} * e^{ilx} = 0$ if $k \neq l$. Indeed,

$$\begin{aligned} e^{ikx} * e^{ilx} &= \int_0^{2\pi} e^{ik(x-t)} e^{ilt} dt \\ &= e^{ikx} \int_0^{2\pi} e^{i(l-k)t} dt \\ &= e^{ikx} \int_0^{2\pi} [\cos(l-k)t + i\sin(l-k)t dt] = 0 \end{aligned}$$

It follows that

$$h_\alpha * f = \sum_{k=-N_1}^{N_1} h(\alpha)_k f_k e^{ikt} \text{ and } h_\alpha * g = \textstyle\sum_{k=-N_2}^{N_2} h(\alpha)_k g_k e^{ikt}$$

where $N_1 = min\{N_f, N_\alpha\}$ and $N_2 = min\{N_g, N_\alpha\}$, respectively.

For a fixed α, let us define the bilinear operator $B_\alpha : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$, $B_\alpha(f,g) = B(h_\alpha * f, h_\alpha * g)$. It is easily seen that $(B_\alpha)_{\alpha \in I}$ is a net of well-defined, continuous bilinear operators. Using the zero product preserving property of the operator B, for the trigonometric polynomials f and g, we get that

$$\begin{aligned}
B_\alpha(f,g) &= B(h_\alpha * f, h_\alpha * g) \\
&= B\Big(\sum_{n=-N_1}^{N_1} h(\alpha)_n f_n e^{int}, \sum_{k=-N_2}^{N_2} h(\alpha)_k g_k e^{ikt}\Big) \\
&= \sum_{n=-N}^{N} (h(\alpha)_n)^2 f_n g_n B(e^{int}, e^{int}) \quad (N = min\{N_1, N_2\}) \\
&= B\Big(\sum_{n=-N}^{N} h(\alpha)_n e^{int}, \sum_{n=-N}^{N} h(\alpha)_n f_n g_n e^{int}\Big) \\
&= B(h_\alpha, h_\alpha * f * g).
\end{aligned}$$

Now, we show the same equality for elements that are not trigonometric polynomials. Due to density, for each element $f \in L^1(\mathbb{T})$ and $g \in M(\mathbb{T})$, there are sequences $(s_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ of trigonometric polynomials such that $f = \lim_{n\to\infty} s_n$ and $g = \lim_{n\to\infty} r_n$. Then we obtain that

$$\begin{aligned}
B_\alpha(f,g) &= B_\alpha(\lim_{n\to\infty} s_n, \lim_{n\to\infty} r_n) = \lim_{n\to\infty} B_\alpha(s_n, r_n) \\
&= \lim_{n\to\infty} B(h_\alpha * s_n, h_\alpha * r_n) \\
&= \lim_{n\to\infty} B(h_\alpha, h_\alpha * s_n * r_n).
\end{aligned}$$

It is known that, continuous multilinear operators are separately continuous. By using the separately continuity of the bilinear operators B and $*$ defined on $L^1(\mathbb{T}) \times M(\mathbb{T})$, and commutativity of the convolution we obtain

$$
\begin{aligned}
B_\alpha(f,g) &= \lim_{n\to\infty} B(h_\alpha, h_\alpha * s_n * r_n) \\
&= B(h_\alpha, \lim_{n\to\infty} h_\alpha * s_n * r_n) \\
&= B(h_\alpha, \lim_{n\to\infty} s_n * r_n * h_\alpha) \\
&= B(h_\alpha, \lim_{n\to\infty} s_n * \lim_{n\to\infty} r_n * h_\alpha) \\
&= B(h_\alpha, f * g * h_\alpha) = B(h_\alpha, h_\alpha * f * g).
\end{aligned}
$$

Therefore, the values of the bilinear operator B_α defined above can be written as $B_\alpha(f,g) = B(h_\alpha, h_\alpha * f * g)$ for every $f \in L^1(\mathbb{T})$ and $g \in M(\mathbb{T})$.

Now, define the map $T_\alpha : M(\mathbb{T}) \to Y$ as $T_\alpha(v) = T_\alpha(f * g) = B_\alpha(f,g)$ for each $\alpha \in I$. The map T_α is well-defined, linear, continuous operator. Indeed, for $f_1 * g_1 = f_2 * g_2$,

$$
T_\alpha(f_1 * g_1) = B_\alpha(f_1, g_1) = B(h_\alpha, h_\alpha * f_1 * g_1) = B(h_\alpha, h_\alpha * f_2 * g_2) = T_\alpha(f_2 * g_2).
$$

Linearity of the map T is seen by the bilinearity of the operator B and the convolution product. Last we will show the continuity of T. Using the continuity of the operator B_α, the following holds

$$
\sup_{f*g \in U_{M(\mathbb{T})}} \|T_\alpha(f * g)\|_Y = \sup_{(f,g)\in U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})}} \|B_\alpha(f,g)\|_Y < \infty.
$$

Consequently, we obtain a net of continuous linear operators $(T_\alpha)_{\alpha\in I}$. Besides,

$$
\|T_\alpha(f * g)\|_Y = \|B(h_\alpha, h_\alpha * f * g)\|_Y \leqslant \|B\| \|h_\alpha\|_{L^1(\mathbb{T})} \|h_\alpha * f * g\|_{M(\mathbb{T})}.
$$

Let us say $\|h_\alpha * f * g\|_{M(\mathbb{T})} = c_\alpha$, then it is seen that $(T_\alpha(f * g))_{\alpha\in I}$ is a bounded net for each $f * g$ such that $\|T_\alpha(f * g)\|_Y \leqslant c_\alpha$. Define the pointwise limit operator $T(f * g) := \lim_\alpha T_\alpha(f * g)$. It is clearly well-defined and linear. Also, for $f * g \in M(\mathbb{T})$,

$$
\begin{aligned}
T(f * g) &= \lim_\alpha T_\alpha(f * g) = \lim_\alpha B_\alpha(f,g) \\
&= \lim_\alpha B(h_\alpha * f, h_\alpha * g) \\
&= B(\lim_\alpha h_\alpha * f, \lim_\alpha h_\alpha * g) = B(f,g).
\end{aligned}
$$

It follows that, the net $(T_\alpha(f * g))_{\alpha\in I}$ converges to $T(f * g) = B(f,g) \in Y$. Since normed spaces are locally convex and Banach spaces are barreled, the requirements of the Banach-

Steinhaus Theorem for nets are satisfied Thus, it is obtained that the limit operator T is continuous. Therefore, a factorization for the bilinear operator B is found as $B(f,g) = T \circ *(f,g) = T(f*g)$. This completes the proof. $\square$

Remark 5.4 Since there is not a unit element in $L^1(\mathbb{T})$, the concept of approximate identity comes into prominence. There are some well-known important examples of approximate identities in $L^1(\mathbb{T})$ such as Fejér Kernel and Poisson Kernel defined by $F_n(x) = \sum_{|j|\leqslant n}\left(1-\frac{|j|}{n}\right)e^{ijx}$ and $P(r,x) = \sum r^{|j|}e^{ijx}$, respectively. These are also positive summability kernels (see [26, Def. 2.2.]) that give rise to an approximate identity. $F*f$ and $p*f$ give Cesaro and Abel means of Fourier series of a function f and they are related with some important partial differential equations. One of these important kernels can be used in our factorization theorem. A more detailed account on summability kernels may be found in [26].

Corollary 5.8 Any zero product preserving bilinear map $B : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to Y$ satisfies the symmetry condition, that is $B(f,g) = B(g,f)$ for all $f,g \in L^1(\mathbb{T})$.

Proof. Since the map B is zero product preserving, then there is a linear continuous map $T : L^1(\mathbb{T}) \to Y$ defined by $B(f,g) = T \circ *(f,g)$. By the commutativity of the convolution product we get the symmetry; $B(f,g) = T(f*g) = T(g*f) = B(g,f)$. $\square$

5.2.1 Properties of $*$-Factorable Operators on Banach Modules

In this section, we investigate some compactness and summability properties for the $*-$factorable operators using some classical properties and results, such as Grothendieck's theorem or cotype-related properties.

5.2.1.1 Summability Properties Now we analyze some direct summability properties of $*-$factorable bilinear operators in two particular cases: when Y is a Hilbert space and when Y has some cotype-related properties.

Corollary 5.9 Let Y be a Hilbert space H and let $B : L^1(\mathbb{T}) \times \mathcal{W}(\mathbb{T}) \to Y$ be a zero product preserving operator, where $\mathcal{W}(\mathbb{T})$ denotes the Wiener algebra. Then the operator B factors through an absolutely summing operator $\tilde{T}$ as $B = \tilde{T} \circ \hat{} \circ *$, where $\hat{}$ denotes the Fourier transform. As a consequence, in this case there is a constant $c > 0$ such that the following statements hold.

(i) For $f_1,\dots,f_n\in L^1(\mathbb{T})$, $g_1,\dots,g_n\in \mathcal{W}(\mathbb{T})$,

$$\sum_{i=1}^{n}\|B(f_i,g_i)\|_H \leqslant c \sup_{\varphi\in B_{\ell^\infty}} \sum_{i=1}^{n}\big|\big\langle \widehat{f_i * g_i},\varphi\big\rangle\big| = c \sup_{(\varphi_j)\in B_{\ell^\infty}} \sum_{i=1}^{n}\big|\sum_{j=1}^{\infty} a^i_j\,\varphi_j\big|, \tag{5.19}$$

where (a^i_j) is the sequence of Fourier coefficients of the convolution product $f_i * g_i$ for $i=1,2,\dots,n$.

(ii) For $f\in L^1(\mathbb{T})$ and $g\in\mathcal{W}(\mathbb{T})$,

$$\|B(f,g)\|_H \leqslant c\int_{B_{\ell^\infty}} \big|\big\langle \widehat{f*g},\varphi\big\rangle\big|\,d\eta(\varphi) = c\int_{B_{\ell^\infty}} \big|\sum_{j=1}^{\infty} a_j\,\varphi_j\big|\,d\eta(\varphi), \tag{5.20}$$

where η is a regular probability measure on the unit ball of ℓ^∞ given by the Pietsch's Domination Theorem, and (a_j) is the sequence of Fourier coefficients of the convolution product $f*g$.

Proof. Assume that B is zero product preserving. By Theorem 5.4 we know that B has a factorization $B = T\circ *$. Using the fact that $\mathcal{W}(\mathbb{T})$ is isometrically isomorphic to the sequence space $\ell^1(\mathbb{Z})$ by the Fourier transform, we can define the linear continuous operator

$\tilde{T}((a_n)) := T(\widecheck{(a_n)})$, $(a_n)\in\ell^1(\mathbb{Z})$,

which satisfies $\tilde{T}(\widehat{f*g}) = \tilde{T}\circ\widehat{\ }(f*g) = T(f*g) = B(f,g)$ and the commutativity of the diagram

$$\begin{tikzcd}
L^1(\mathbb{T})\times\mathcal{W}(\mathbb{T}) \arrow[r, "B"] \arrow[d, "*"'] & H. \\
\mathcal{W}(\mathbb{T}) \arrow[ur, "T"] \arrow[d, "\widehat{\ }"'] & \\
\ell^1(\mathbb{Z}) \arrow[uur, "\tilde{T}"'] &
\end{tikzcd}$$

One of the well-known instances of Grothendieck's Theorem states that

$$\mathcal{L}(\ell^1(\mathbb{Z}),H) = \Pi_1(\ell^1(\mathbb{Z}),H).$$

Using this, we obtain that $\tilde{T}$ is a summing operator. The statements (i) and (ii) can be easily seen as a consequence of the definition of summing operator and the Pietsch's Domination Theorem. □

Corollary 5.10 Let $M(\mathbb{T})$ be $L^1(\mathbb{T})$ or $C(\mathbb{T})$, and let Y be a Hilbert space H. Consider the zero product preserving bilinear continuous map $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to H$. Then the operator B factors through an absolutely 2-summing operator. As a consequence, there is a constant $c > 0$ such that the following statements hold.

(i) For $f_1, \ldots, f_n \in L^1(\mathbb{T})$, $g_1, \ldots, g_n \in M(\mathbb{T})$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_H^2\Big)^{1/2} \leqslant c \sup_{\varphi \in B_{M(\mathbb{T})^*}} \Big(\sum_{i=1}^{n} |\langle \varphi, f_i * g_i \rangle|^2\Big)^{1/2}. \tag{5.21}$$

(ii) For $f \in L^1(\mathbb{T})$ and $g \in M(\mathbb{T})$,

$$\|B(f,g)\|_H \leqslant c \Big(\int_{B_{M(\mathbb{T})^*}} |\langle f * g, \varphi \rangle|^2 \, d\eta(\varphi) \Big)^{1/2}, \tag{5.22}$$

where η is a regular probability measure on the unit ball of $M(\mathbb{T})^*$ given by Pietsch's Domination Theorem.

Proof. By applying Grothendieck's Theorem it can be easily seen that $\mathcal{L}(L^1(\mathbb{T}), H) = \Pi_2(L^1(\mathbb{T}), H)$ and $\mathcal{L}(C(\mathbb{T}), H) = \Pi_2(C(\mathbb{T}), H)$. Therefore, we get the desired result from definition of summable operator and Pietsch's Domination Theorem. $\square$

Finally, consider a Banach space Y which is of cotype 2. Thus, it has the Orlicz property (see page 9 or [18, §8.9]). It follows that for any $*-$factorable bilinear map $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$, a domination is obtained by the assumption Y is of cotype 2:

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_Y^2\Big)^{1/2} \leqslant c \sup_{\varepsilon_i = \mp 1} \Big\| \sum_{i=1}^{n} \varepsilon_i (f_i * g_i) \Big\|_{M(\mathbb{T})}$$

for $f_1, \ldots, f_n \in L^1(\mathbb{T})$, $g_1, \ldots, g_n \in M(\mathbb{T})$.

Corollary 5.11 Let us assume that $B : L^1(\mathbb{T}) \times C(\mathbb{T}) \to Y$ is zero product preserving and the range space Y is of cotype 2. Then for some $1 \leqslant p < \infty$ there exists a probability measure $\mu \in C(\mathbb{T})^*$ with the property that for a given $\varepsilon > 0$ we can find an $N(\varepsilon) > 0$ such that for all $(f, g) \in L^1(\mathbb{T}) \times C(\mathbb{T})$

$$\|B(f,g)\|_Y \leqslant N(\varepsilon) \Big(\int_{\mathbb{T}} |f * g|^p d\mu \Big)^{1/p} + \varepsilon \|f * g\|_{C(\mathbb{T})}.$$

Proof. By Theorem 5.4, the operator B has a factorization $B := T \circ *$. It is known that any linear map from $C(K)$ –K is compact Hausdorff space– to a Banach space being of cotype 2 is 2-summing; see [15, Theorem 11.14]. Since the Banach space Y is of cotype 2 and $\mathbb{T}$ is compact, the linear operator $T : C(\mathbb{T}) \to Y$ is 2-summing. Therefore, it is weakly compact. Theorem 15.2 in [15] gives a characterization of weakly compact operators and states that a linear operator $T : C(\mathbb{T}) \to Y$ is weakly compact if and only if there exists a probability measure $\mu \in C(\mathbb{T})^*$ with the property that for a given $\varepsilon > 0$ we can find an $N(\varepsilon) > 0$ such that for all $h \in C(\mathbb{T})$

$$\|T(h)\|_Y \leqslant N(\varepsilon)\Big(\int_{\mathbb{T}} |h|^p d\mu\Big)^{1/p} + \varepsilon\|h\|_{C(\mathbb{T})}.$$

Since $h = f * g$ for $(f,g) \in L^1(\mathbb{T}) \times C(\mathbb{T})$, we get

$$\|B(f,g)\|_Y \leqslant N(\varepsilon)\Big(\int_{\mathbb{T}} |f * g|^p d\mu\Big)^{1/p} + \varepsilon\|f * g\|_{C(\mathbb{T})}. \qquad \square$$

In the next corollary we use the definition of equivalently zero product preserving map, see Definition 3.7 in page 30.

Corollary 5.12 A Banach space Y is isomorphic to the Banach space $M(\mathbb{T})$, where $M(\mathbb{T}) \in \Big\{L^p(\mathbb{T})\, (1 \leqslant p < \infty), C(\mathbb{T}), \mathcal{W}(\mathbb{T})\Big\}$ if and only if there exists an equivalently zero product preserving norming bilinear map $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$.

Proof. If Y is isomorphic to the Banach space $M(\mathbb{T})$ by the isomorphism $S : M(\mathbb{T}) \to Y$, then we obtain an equivalently zero product preserving norming bilinear map by $B = S \circ *$. For the converse, let us consider the equivalently zero product preserving norming bilinear map B. By Theorem 5.4, we have a factorization such that the linear operator T is injective since $f * g = 0$ whenever $B(f,g) = 0$. Using the norming property of B, we get

$$T(U_{M(\mathbb{T})}) \subseteq kU_Y \subseteq kB(U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})}) = kT(U_{M(\mathbb{T})}). \qquad \square$$

5.2.1.2 Compactness Properties Similarly to the pointwise product case, it is easily seen that a zero product preserving map B is (weakly) compact if and only if the linear operator T appearing in its factorization is (weakly) compact, due to the definition of product. Indeed,

the zpp map B is (weakly) compact $\iff B(U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})})$ is relatively (weakly) compact

$\iff T \circ *(U_{L^2(\mathbb{T})} \times U_{M(\mathbb{T})})$ is relatively (weakly) compact

$\iff T(U_{M(\mathbb{T})})$ is relatively (weakly) compact

$\iff T$ is (weakly) compact.

Theorem 5.5 A zero product preserving bilinear operator $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$ satisfy the following statements.

(i) The range set $B(L^1(\mathbb{T}) \times M(\mathbb{T}))$ is a linear space.

(ii) $B(U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})})$ is a convex set.

(iii) $B(U_{L^1(\mathbb{T})} \times U_{L^2(\mathbb{T})})$ is relatively weakly compact, that is, B is weakly compact operator.

(iv) Let $Y = \ell^s$, where $1 \leqslant s < 2$, $B(U_{L^1(\mathbb{T})} \times U_{L^2(\mathbb{T})})$ is relatively compact.

Proof. Since the given bilinear operator is zero product preserving, by Theorem 5.4, it is $*-$factorable by the linear operator $T : M(\mathbb{T}) \to Y$.

(i) By the $*-$factorability of B and norming property of the convolution product, $B(L^1(\mathbb{T}) \times M(\mathbb{T})) = T(L^1(\mathbb{T}) * M(\mathbb{T})) = T(M(\mathbb{T}))$. That shows that the range of B is a range of a linear map, therefore it is a linear space.

(ii) Since $U_{L^1(\mathbb{T})} * U_{M(\mathbb{T})} = U_{M(\mathbb{T})}$ and $U_{M(\mathbb{T})}$ is a convex set, the image $B(U_{L^1(\mathbb{T})} \times U_{M(\mathbb{T})}) = T(U_{M(\mathbb{T})})$ is also convex.

(iii) Since $L^2(\mathbb{T})$ is a reflexive space, the linear operator $T : L^2(\mathbb{T}) \to Y$ is weakly compact. By the factorization, it follows that $B(U_{L^1(\mathbb{T})} \times U_{L^2(\mathbb{T})}) = T(U_{L^1(\mathbb{T})} * U_{L^2(\mathbb{T})}) = T(U_{L^2(\mathbb{T})})$ is relatively weakly compact.

(iv) By Plancherel's Theorem it is well-known that the Fourier transform $\widehat{}$ is a linear isometry of $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$, and so the inverse Fourier transform $\check{}$ is a linear isometry of $\ell^2(\mathbb{Z})$ onto $L^2(\mathbb{T})$. Since we have a factorization of B through $L^2(\mathbb{T})$ by the convolution product, we get that $T \circ \check{} : \ell^2 \to \ell^s$ is a linear continuous operator. Thus this linear operator is compact by Pitt's Theorem (see page 8). Therefore, $T \circ \check{} \circ \widehat{} \circ *(U_{L^1(\mathbb{T})} \times U_{L^2(\mathbb{T})}) = T \circ \check{} \circ \widehat{} U_{L^2(\mathbb{T})} = T \circ \check{} \circ U_{\ell^2(\mathbb{T})} = B(U_{L^1(\mathbb{T})} \times U_{L^2(\mathbb{T})})$ is relatively compact. $\square$

Corollary 5.13 Let $M(\mathbb{T})$ be either the whole algebra $L^1(\mathbb{T})$ or the subalgebra $C(\mathbb{T})$ and B is a zero product preserving bilinear operator. Let the set $A \subset L^1(\mathbb{T}) \times M(\mathbb{T})$ be $*$- weakly compact (see Definition 3.6), then $B(A)$ is norm compact.

Proof. We obtain this as a consequence of the Dunford–Pettis property of the spaces $L^1(\mathbb{T})$ and $C(\mathbb{T})$. The zero product preserving bilinear operator B is $*-$factorable. Since $*(A)$ is weakly compact, $B(A) = T \circ *(A)$ is weakly compact. Using Dunford–Pettis property, we get compactness of the set $B(A)$. $\square$

5.2.2 Integral Representation for $*$-Factorable Maps

Since there is an identification between functions on the circle group $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, the measure on $\mathbb{T}$ is defined due to this identification. The interval $[0,2\pi)$ is considered as a model of $\mathbb{T}$ and the Lebesgue measure on $\mathbb{T}$ is the restriction of the Lebesgue measure on $\mathbb{R}$ to $[0,2\pi)$ (see [26, Chapter 1]). Moreover, the circle group $\mathbb{T}$ is Hausdorff.

From this point of view, we can establish integral representations for $*$-factorable bilinear maps defined on Banach modules. We will need some notions and results related with vector measures, so we refer to the reader Section 4.1.3.

Theorem 5.6 Consider a Banach space-valued bilinear continuous map $B : L^1(\mathbb{T}) \times L^p(\mathbb{T}) \to E$, where $1 \leqslant p < \infty$. Then the following statements imply each other.

(i) For any finite subsets $(f_i)_{i=1}^n \subset L^1(\mathbb{T})$ and $(g_i)_{i=1}^n \subset L^p(\mathbb{T})$,

$$\Big(\sum_{i=1}^n \|B(f_i,g_i)\|^p\Big)^{1/p} \leqslant \Big\|\big(\sum_{i=1}^n |f_i * g_i|^p\big)^{1/p}\Big\|_{L^p(\mathbb{T})}. \tag{5.23}$$

(ii) There exist a norm one multiplication operator $M_h : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ and a linear operator $S : L^p(\mathbb{T}) \to E$ such that B factors as $B = S \circ M_h \circ *$, that is,

$$\begin{tikzcd} L^1(\mathbb{T}) \times L^p(\mathbb{T}) \arrow[r, "B"] \arrow[d, dotted, "*"'] & E. \\ L^p(\mathbb{T}) \arrow[r, dotted, "M_h"] & L^p(\mathbb{T}) \arrow[u, dotted, "S"'] \end{tikzcd}$$

(iii) There is an E-valued vector measure ν such that $L^p(\mathbb{T}) \hookrightarrow L^1(\nu)$, and

$$B(f,g) = \int_{\mathbb{T}} (f(t) * g(t))\, h(t)\, d\nu(t), \tag{5.24}$$

where h defines a multiplication operator $M_h : L^p(\mathbb{T}) \to L^p(\mathbb{T})$.

Proof. Firstly, let us show (i) $\Rightarrow$ (ii). The inequality given in (i) implies the $*$-factorability of B. Thus, by Theorem 5.4, B factors through the linear map $T : L^p(\mathbb{T}) \to E$. This linear map satisfies the following by the hypothesis for $(f_i)_{i=1}^n \subset L^1(\mathbb{T})$ and $(g_i)_{i=1}^n \subset L^p(\mathbb{T})$;

$$\Big(\sum_{i=1}^n \|T(f_i * g_i)\|^p\Big)^{1/p} \leqslant \Big\|\Big(\sum_{i=1}^n |f_i * g_i|^p\Big)^{1/p}\Big\|_{L^p(\mathbb{T})}.$$

This gives the p-concavity of the linear map T. The Corollary 5 given in [63] states, as an application of Maurey-Rosenthal theorem, that for a p-concave linear operator with a p-convex order continuous domain there is a norm one multiplication operator M_h such that the linear operator factors through this multiplication operator. Since $L^p(\mathbb{T})$ is p-convex order continuous and T is p-concave, then there is a multiplication operator M_h : $L^1(\mathbb{T}) * L^p(\mathbb{T})(= L^p(\mathbb{T})) \to L^p(\mathbb{T})$ and a linear continuous map $S : L^p(\mathbb{T}) \to E$ such that $T := S \circ M_h$. Consequently, we get the factorization $B = S \circ M_h \circ *$.

(ii)$\Rightarrow$(iii) Since $L^p(\mathbb{T})$ is order continuous, the linear map $S : L^p(\mathbb{T}) \to E$ defines a countably additive vector measure $\nu(A) = S(\chi_A)$, where $A \in \mathfrak{B}(\mathbb{T})$. We have that $L^p(\mathbb{T}) \hookrightarrow L^1(\nu)$ by the optimality of the space $L^1(\nu)$. Since the optimal extension of S is the integral operator

$$I_\nu(s) = \int_{\mathbb{T}} s d\nu,$$

for $s \in L^1(\nu)$, the following commutative diagram is obtained:

$$\begin{tikzcd} L^1(\mathbb{T}) \times L^p(\mathbb{T}) \arrow[rr, "B"] \arrow[d, dotted, "*"'] & & E & \\ L^p(\mathbb{T}) \arrow[rr, dotted, "M_h"] & & L^p(\mathbb{T}) \arrow[u, dotted, "S"'] \arrow[r, hook] & L^1(\nu). \arrow[ul, "I_\nu"'] \end{tikzcd}$$

This result can be found in [23, Th.4.14]. It is well-known that the space $L^1(\nu)$ is a Banach function space over a Rybakov measure η for ν, and $\eta \ll dt$ because of the continuity of T; we can change then the inclusion by the identification of classes $[f]_{dt} \mapsto [f]_\eta$, what is sometimes called an inclusion/quotient map, and the factorization is still preserved. Composing all these, we obtain

$$B(f,g) = \int_{\mathbb{T}} (f(t) * g(t))\, h(t)\, d\nu(t),$$

for all $(f,g) \in L^1(\mathbb{T}) \times L^p(\mathbb{T})$.

(iii) $\Rightarrow$ (i) by a direct computation, we get for all $(f_i)_{i=1}^n \subset L^1(\mathbb{T})$ and $(g_i)_{i=1}^n \subset L^p(\mathbb{T})$,

$$\begin{aligned}\Big(\sum_{i=1}^n \|B(f_i,g_i)\|_E^p\Big)^{1/p} &\leqslant \Big(\sum_{i=1}^n \|(f_i * g_i)h\|_{L^1(\nu)}^p\Big)^{1/p} \leqslant \Big(\sum_{i=1}^n \|(f_i * g_i)h\|_{L^p(\mathbb{T})}^p\Big)^{1/p} \\ &= \Big\|h\Big(\sum_{i=1}^n |f_i * g_i|^p\Big)^{1/p}\Big\|_{L^p(\mathbb{T})} \leqslant \Big\|\Big(\sum_{i=1}^n |f_i * g_i|^p\Big)^{1/p}\Big\|_{L^p(\mathbb{T})}. \qquad \square\end{aligned}$$

As a result, it is easily seen that any $*$-factorable bilinear map $B : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to E$ has an integral representation $B(f,g) = \int_{\mathbb{T}} (f * g)h\,dt$, where $h \in L^\infty(dt,E)$, if E has the Radon-Nikodym property. Such a representation can be given for a $*$-factorable bilinear map acting in $L^1(\mathbb{T}) \times L^1(\mathbb{T})$ with an arbitrary range space under the assumption of weak compactness.

Theorem 5.7 Any weakly compact $*$-factorable bilinear continuous map $B : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to E$ has an integral representation $B(f,g) = \int_{\mathbb{T}} (f * g)h\,dt$ where $h \in L^\infty(dt,E)$ for all $f, g \in L^1(\mathbb{T})$.

Proof. Since B is $*$-factorable, it has linear continuous factorization operator $T : L^1(\mathbb{T}) \to E$ defined as $B(f,g) = T(f * g)$. T is weakly compact since B is weakly compact. By the strong version of Dunford–Pettis Theorem, the weakly compact linear operator T has an integral representation such that $T(f * g) = \int_{\mathbb{T}} (f * g)h\,dt$ ([18, Appendix C]). This gives the desired representation. $\square$

We will finish the section by providing an integral representation for weakly compact operators acting in $L^1(\mathbb{T}) \times C(\mathbb{T})$.

Theorem 5.8 A Banach space-valued $*$-factorable bilinear operator $B : L^1(\mathbb{T}) \times C(\mathbb{T}) \to E$ is weakly compact if and only if B has the integral representation $B(f,g) = \int_{\mathbb{T}} (f * g)d\nu$ for $f \in L^1(\mathbb{T})$ and $g \in C(\mathbb{T})$, where ν is a countably additive E-valued vector measure on the Borel sets in $\mathbb{T}$.

Proof. By the $*$-factorability of the map B, it has a linear factorization through the space $C(\mathbb{T})$. By the norming property of $*$, B is weakly compact if and only if T is so. Since the Banach-valued bounded linear operator T is weakly compact if and only if there can be found a countably additive E-valued vector measure ν on the Borel sets in $\mathbb{T}$ such that $B(f,g) = T(f * g) = \int_{\mathbb{T}} (f * g)d\nu$ for all $f * g = h \in C(\mathbb{T})$ (see [21, Section VI.2]). $\square$

5.2.3 Applications of $*$-Factorable Operators

We finish the section by giving some applications for integral transforms and Hilbert-Schmidt integral operators.

Application 5.6 $*-$Factorable Integral Operator Our first application is related to the integral transformations that have an important role in many fields such as optics and signal processing, due to their usefulness to solve problems in linear differential equations. Fourier, Mellin and Laplace transforms are some of these well known integral transformations. Let us give a general definition for them. Consider function spaces $X(\mu)$ and $Y(\nu)$ —where μ and ν are σ-finite measure spaces—, and let I denote an integral transformation. The integral transform of a function $f(t)$ with respect to the kernel $K(x,\alpha)$ is given by the equation

$$I(f) = \int_{\Omega} f(x)K(x,\alpha)dx \tag{5.25}$$

(for more information see [80, Section 1]). All integral transforms satisfy the *convolution property,* that states that the integral transform of convolution of two functions are equal to the product of their transforms. Namely, integral transforms satisfy the following equality;

$$I(f*g) = I(f)I(g)$$

for all functions f,g in the corresponding domain.

Consider an integral transformation $I : L^1(\mathbb{T}) \to Y$ and let $B : L^1(\mathbb{T}) \times M(\mathbb{T}) \to Y$ given by $B(f,g) = I(f)I(g)$, where $M(\mathbb{T}) \in \left\{L^p(\mathbb{T})\,(1 \leqslant p < \infty), C(\mathbb{T}), \mathcal{W}(\mathbb{T})\right\}$. Then, the map B is a zero product preserving operator if and only if B satisfies a factorization through a linear operator $T : M(\mathbb{T}) \to Y$ such that $T(f*g) = I(f*g)$ for all f,g in the corresponding domains. That is, B factors through an integral transform.

Application 5.7 Representations For Hilbert-Schmidt Operators By Factorable Maps

Let us show now some applications of the representations for Hilbert-Schmidt operators.

Let $T \in \mathcal{L}(H)$. Recall that T is called a Hilbert-Schmidt operator if for some complete orthonormal system $(e_i)_{i\in I} \subset H$ the sum $\sum_{i\in I} \|Te_i\|^2$ is convergent. The space of all Hilbert-Schmidt operators defined on H is denoted by $HS(H)$; see [81, Definition 1.b.14].

Theorem 5.9 [81, Proposition 1.b.15] Let (Ω,μ) be a finite measure space and $H = L^2(\Omega,\mu)$, and consider a linear operator $T : H \to H$. Then $T \in HS(H)$ if and only if there is a kernel $k \in L^2(\Omega^2,\mu^2)$ such that

$$Tf(x) = \int_{\Omega} k(x,y)f(y)d\mu(y), \quad f \in H,\ x\ \mu - a.e. \tag{5.26}$$

Theorem 5.10 (Mercer theorem) [81, Theorem 3.a.1] Let (Ω,μ) be a finite measure space and $k \in L^{\infty}(\Omega^2,\mu^2)$ be a kernel such that $T_k : L^2(\Omega,\mu) \to L^2(\Omega,\mu)$ is positive, that is, $\langle T_k(f), f\rangle_2 \geqslant 0$ for all f. Then the eigenvalues $(\lambda_n(T_k))$ of T_k are absolutely summable. Besides the eigenfunctions $f_n \in L^2(\Omega,\mu)$ of T_k, associated with those n such that $\lambda_n(T_k) \neq 0$, and normalized by $\|f_n\|_2 = 1$, actually belong to $L^{\infty}(\Omega,\mu)$ with $sup_n \|f_n\|_{\infty} < \infty$

$$k(x,y) = \sum_{n\in\mathbb{N}} \lambda_n(T_k)\overline{f_n(y)}f_n(x)$$

holds μ^2-a.e., where the series converges absolutely and uniformly μ^2-a.e.

By Remark 5.3, we know that every function $f \in L^2(\mathbb{T})$ can be written as a convolution product of the functions h and g such that $h \in L^1(\mathbb{T})$ and $g \in L^2(\mathbb{T})$. By using this, we obtain the following.

Corollary 5.14 Let $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$. $T \in HS(L^2(\mathbb{T}))$ if and only if there is a kernel $k \in L^2(\mathbb{T}^2)$ such that

$$\begin{aligned} T(h*g)(x) &= \int_{\mathbb{T}}\int_{\mathbb{T}} k(x,y)h(y-z)g(z)d\mu(z)d\mu(y) \\ &= \int_{\mathbb{T}} k(x,y)(h*g)(y)d\mu(y) \end{aligned}$$

for all $h \in L^1(\mathbb{T})$ and $g \in L^2(\mathbb{T})$.

Corollary 5.15 Consider the elements appearing in Mercer Theorem and assume that its requirements are satisfied. Let $(f_n)_{n=1}^{\infty}$ be the sequence of the eigenvectors of T_k, and suppose that each of them can be written as the convolution product $f_n = h_n * g_n$, $n \in \mathbb{N}$. Then the series expansion of the kernel can be written by using the convolution product as follow.

$$k(x,y) = \sum_{n\in\mathbb{N}} \lambda_n(T_k)\overline{(h_n*g_n)(y)}(h_n*g_n)(x).$$

Recall that a bounded linear map $T : H_1 \to H_2$ between Hilbert spaces is called *nonnega-*

tive if $\langle Tf, f\rangle_2 \geqslant 0$ for all $f \in H_1$ [82, pp. 24]. It is well-known that nonnegative operators are self adjoint, i.e. $T^* = T$ for a nonnegtive T.

Corollary 5.16 Consider a compact zero product preserving bilinear operator $B : L^1(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ such that $\langle B(f,g), f * g\rangle_2 \geqslant 0$ for all $f \in L^1(\mathbb{T})$, $g \in L^2(\mathbb{T})$. Then B has a uniformly convergent series representation.

Proof. By applying Theorem 5.4 we obtain a factorization operator T for B such that $T(f * g) := B(f,g)$. Due to the norming property of the convolution it is seen that T is compact if and only if the operator B is compact. Since the condition $\langle B(f,g), f * g\rangle_2 \geqslant 0$ is given, it follows that $\langle T(f * g), f * g\rangle_2 \geqslant 0$. This shows that T is a nonnegative self-adjoint operator. By the Spectral Theorem (see [82, pp. 24]), it follows that a linear, self-adjoint, compact operator admits a uniformly convergent representation such that $T = \sum_{n\in\mathbb{N}} \lambda_n(T)\langle\, .\, , \phi_n\rangle\phi_n$, where λ_n's and ϕ_n's are eigenvalues and eigenvectors of T, respectively. By using $L^1(\mathbb{T}) * L^2(\mathbb{T}) = L^2(\mathbb{T})$, it is seen that every eigenvector ϕ_n allows a factorization such that $\phi_n = a_n * b_n$, where $a_n \in L^1(\mathbb{T})$, $b_n \in L^2(\mathbb{T})$. Then we obtain that $B(f,g) = \sum_{n\in\mathbb{N}} \lambda_n(T)\langle f * g, a_n * b_n\rangle(a_n * b_n)$ for all $(f,g) \in L^1(\mathbb{T}) \times L^2(\mathbb{T})$. □

It is clear that every linear operator $T : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ gives rise to a continuous bilinear operator $B : L^1(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T})$ defined by $T(f) = B(h,g)$, where $f = h * g$. Moreover, if this linear operator is a Hilbert-Schmidt operator, then B is an integral operator.

CHAPTER 6

RESULTS AND DISCUSSION

We obtained that several developments in functional analysis follow the same idea: a factorization through a canonical product. We find this with the convolution product, the pointwise product for function spaces, the Banach algebra product.

The characterizations in all these cases are similar. We have an algebraic condition as being a zero preserving map, that is equivalent to an summability inequality, and this can be written as a factorization theorem through the product and a linear map.

The summability characterization allows to relate bilinear maps with p-summing operators, what give a lot of information about the bilinear maps. We show some applications in this direction.

Factorization allows to obtain a lot of topological information by means of the results that we know for the linear factor. Compactness and weak compactness are easily obtained.

Summing up all what we got, we found a complete description of the class of product factorable equations. Applications to integral and kernel bilinear operators are also given.

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APPENDIX-A

BASIC DEFINITIONS

A-1 Weak Topology

The *weak topology* or the topology $\sigma(X,X^*)$ of a normed space X is the induced topology by the topological dual space X^* of X. That is the smallest topology for the space provided that every functional in the dual space is continuous [14].

A base for the weak topology of the space X is the collection of the all sets of the form $\{v \in X : |f_i(u-v)| < \varepsilon, f_i \in X^*, i = 1,2,...,n\}$, where $u \in X$ and n is a positive finite integer.

If a topological property is satisfied with respect to the weak topology, it is said to be a *weak property* or to hold *weakly*. For example, we say a sequence $(x_i)_{i=1}^{\infty}$ is weakly convergent to $x \in X$ –denoted by $x_i \to^w x$– if it converges with respect to the topology $\sigma(X,X^*)$ and a set A in X is *weakly compact* if it is compact in the $\sigma(X,X^*)$.

A-2 Vector Lattices

Let $(A,\preccurlyeq)$ be an ordered set. A is called *lattice* if a least upper bound denoted by $x \vee y = \sup(x,y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x,y)$ exist for any $x,y \in A$ [83, Section 1.1].

An ordered set A is called *order bounded* if it is bounded both above and below [83, Section 1.1].

A real vector space A equipped with an order $\preccurlyeq$ is called *ordered vector space* if it satisfies translation invariance and positive homogeneity [83, Section 1.1].

In addition, if $(E,\preccurlyeq)$ is an ordered vector lattice that order structure is lattice, then E is called *Riesz space* [83, Section 1.1].

The positive cone E^+ of a Riesz space E is the set $\{x \in E : x \succcurlyeq 0\}$ [19, Section 1.a].

The elements $x, y \in E$ are said to be *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$, where $|x|$ denotes the absolute value of $x \in E$ defined by $|x| = x \vee (-x)$ [83, Section 1.1].

A Riesz space is said to be *Archimedean* if $x \preccurlyeq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above [83, Definition 1.1.7(i)]. Function spaces are important examples of Archimedean vector lattices.

A Riesz space is called *Dedekind complete* if every non-empty order bounded set has a supremum and an infimum [83, Definition 1.1.7(ii)].

Let E be a Riesz space. A sequence $(x_n)_{n=1}^{\infty} \in E$ converges u-uniformly to an element x in E if for given $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - x| \preccurlyeq \varepsilon u$ for all $n \succcurlyeq N$. It is said that the sequence $(x_n)_{n=1}^{\infty} \in E$ converges relatively uniformly to x if $(x_n)_{n=1}^{\infty}$ converges u-uniformly to x for some $u \in E^+$ [84, Theorem 16.2].

A subset A of the Riesz space E is called (relatively) uniformly closed if for every relatively uniformly convergent sequence in A, all relatively uniform limits of the sequence are in A. The empty set and E itself are uniformly closed, and finite unions and arbitrary intersections of uniformly closed sets are uniformly closed. Thus, the uniformly closed sets are exactly the closed sets of a certain topology in E, the *relatively uniform topology* [84, p.84].

For an e in E^+, the sequence $(x_n)_{n=1}^{\infty}$ in E is called an e-uniform Cauchy sequence whenever, for any $\varepsilon > 0$, $|x_m - x_n| \preccurlyeq \varepsilon e$ for every $m, n \succcurlyeq N$. The Riesz space E is called uniformly complete if, for every e in E^+, every e-uniform Cauchy sequence has an e-uniform limit [84, p.248].

Let E be a Riesz space with with the norm $\|.\|$. E is called a *Banach lattice* if the norm $\|.\|$ is a lattice norm, that is $|x| \preccurlyeq |y|$ implies $\|x\| \preccurlyeq \|y\|$, and E is complete with respect to this norm [83, Definition 1.1.5].

For a set $(x_i)_{i\in\Lambda}$ in the Banach lattice E, $\bigvee_{i\in\Lambda} x_i$ or *l.u.b.* $(x_i)_{i\in\Lambda}$ denote the unique element $x \in E$ provided

- $x \succcurlyeq x_i$ for every $i \in \Lambda$

- for a $y \in E$, $y \geqslant x$ whenever $y \geqslant x_i$ for all $i \in \Lambda$

A Banach lattice is said to be *order complete (σ-order complete)* whenever every order bounded set (sequence) has a least upper bound [19, Definition 1.a.3.]. A Banach lattice E is *order continuous (σ-order continuous)* if downward directed sets (sequences) converging to 0 converge also in the norm [19, Definition 1.a.6]. For a Banach lattice E, the following implications hold (This result can be found in [19, Proposition 1.a.8.]);

$$E \text{ is } \sigma\text{-complete and } \sigma\text{-o.c.} \Leftrightarrow \quad E \text{ is o.c.} \quad \Leftrightarrow \; E \text{ is complete and o.c.}$$

Any Köthe function space is a Banach lattice with the order defined by $f \geqslant 0$ if $f(x) \geqslant 0$ a.e. and this lattice is order complete [19, Section 1.b]. Thus, it is seen that a B.f.s. is o.c. if and only if it is σ-o.c.

Let E and F be Riesz spaces. A linear continuous map $T : E \to F$ is called *lattice homomorphism* if

$T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$

for all $x, y \in E$ [83, Definition 1.3.10].

A linear operator $T : X \to Y$ between Archimedean Riesz spaces is called *positive* if $x \geqslant 0$ implies $Tx \geqslant 0$ and is called *regular* if it can be written as a difference of two positive linear maps [83, Definition 1.3.1]. The operator T is called *increasing* if $x \geqslant y$ implies $Tx \geqslant Ty$, so T preserves the order.

A linear map defined on a vector lattice to a vector lattice is called *order bounded* if it maps order bounded set to an order bounded set.

Let X, Y, Z be Riesz spaces. A bilinear map $B : X \times Y \to Z$ is called *Riesz bimorphism (respectively, bipositive)* if the maps

for any $y \in Y^+$, $\quad x \longmapsto B(x,y)\,(x \in X)$

for any $x \in X^+$, $\quad y \longmapsto B(x,y)\,(y \in Y)$

are *homomorphism (respectively, positive)* [7]. Note that the term *bipositive* was used by Fremlin in the paper [3] and it is known that a bilinear map is bipositive if and only if it is positive in the sense that $B(x,y) \geqslant 0$ for all $x \in X^+$ and $y \in Y^+$ [7]. Analogously, the bilinear map $B : X \times Y \to Z$ is called *regular* if it can be written as a difference of two positive bilinear maps [7]. Finally, recall that a bilinear map $B : X \times Y \to Z$ is said to be *orthoregular* if it is difference of two positive orthosymmetric bilinear map [5].

A commutator $[\,,\,]_B$ of a bilinear map $B : X \times X \to Z$ is defined by $[x,y]_B = B(x,y) - B(y,x)$ for all $x, y \in X$.

A-3 Banach Algebras

A linear space A over the field $\mathbb{K}$ is called *algebra* with an *associative* multiplication $x, y \to xy$ of $A \times A$ to A, called *product*, such that it is *distibutive* and satisfies for all $x, y \in A$ and $\alpha \in \mathbb{K}$

$\alpha(xy) = x(\alpha y) = (\alpha x)y.$

Moreover, if the algebra A is a Banach space endowed with the norm $\|.\|$, then A is said to *Banach algebra* whenever it holds $\|xy\| \geqslant \|x\|\|y\|$ for all $x, y \in A$. For definitions, see Chapter 1 in [85].

Let A be a normed algebra with norm $\|.\|$ and L be a normed linear space with norm $|\|.\||$ over the same field $\mathbb{K}$. L is said to be a *normed left A-module* with the *bilinear* map $A \times L \to L$ defined by $(a,l) \to al$ provided the followings

i) $a_1(a_2 l) = (a_1 a_2)l \quad (a_1, a_2 \in A, l \in L);$

ii) there exists a positive constant K such that $|\|al\|| \leqslant K\|a\||\|l\|| \quad (a \in A, l \in L).$

The *normed right A-module* is defined similarly, and the space L is called *A-bimodule* if it is both right and left A-module. A normed left (right) A-module is called a *Banach left (right) A-module* if it is complete as a normed linear space. These definitions can be found in [85, §9].

A Banach algebra A is said to be *unital*, if there is a (unique) element $e \in A$, called *unit (or identity) element*, such that $a = ae = ea$ for every $a \in A$ [85, Definition 1]. Some of the non-unital Banach algebras have a net called *approximate identity* that deals as a substitute for a unit element. Namely, a net $(e_\lambda)_{\lambda \in I}$ in a non-unital Banach algebra A is called *left (right) approximate identity* if $(e_\lambda x)_{\lambda \in I}$ $((xe_\lambda)_{\lambda \in I})$ converges to x for every $x \in A$. The left (right) approximate identity $(e_\lambda)_{\lambda \in I}$ is said to be *bounded* –by bound k- if there is a positive constant k such that $\|e_\lambda\| \leqslant k$ [85, §11].

A linear map $T : E \to F$ between Banach algebras is called *algebra homomorphism* if

$T(xy) = T(x)T(y)$ holds for every $x, y \in E$ [85, Chapter 1].

Finally, note that a C^*-algebra is a closed algebra of bounded linear operators defined on a Hilbert space H such that the operation of taking adjoints of operators is closed [85, §12].

APPENDIX-B

TENSOR PRODUCT AND LINEARIZATION

In this appendix, we will explain tensor products and how they act as a ”linearizing space” for bilinear maps.

B-1 Tensor Product of Banach Spaces

The tensor product $X \otimes Y$ of the Banach spaces X and Y is a space of linear functionals on $B(X \times Y, \mathbb{K})$ that is constructed as the following way; for the elements $x \in X$ and $y \in Y$ let us define the map

$x \otimes y : B(X \times Y, \mathbb{K}) \to \mathbb{K}$

by

$(x \otimes y)\psi = \langle \psi, x \otimes y \rangle = \psi(x, y)$

for each bilinear functional ψ on $X \times Y$. The form $x \otimes y$ given by the evaluation at the point (x, y) is called an *elementary tensor* [86, Chapter 1].

The subspace of the dual $B(X \times Y, \mathbb{K})^*$ spanned by the elementary tensors $\{x \otimes y : x \in X, y \in Y\}$ is called the *tensor product of* X, Y and is denoted by $X \otimes Y$. The elements of the space $X \otimes Y$ is called *tensors* and a *typical* tensor $u \in X \otimes Y$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i,$$

where $n \in \mathbb{N}$, $(\lambda_i)_{i=1}^n \in \mathbb{K}$, $(x_i)_{i=1}^n \in X$, $(y_i)_{i=1}^n \in Y$ [86, Chapter 1]. Using the properties of elementary tensors, representation of a tensor u can be rewritten in the form

$$u = \sum_{i=1}^{n} x_i \otimes y_i.$$

For the detailed theory of tensor product we refer to [13] and [86].

A Linearization Tool for Bilinear Maps

Since it is sometimes difficult to deal with bilinear maps the answer of whether we can linearize the bilinear maps comes into prominence. This is possible by the tensor product space $X \otimes Y$ of the X, Y. The philosophy of tensor products is that: to exchange bilinear maps on a given space with *simpler* linear maps on a *more complicated* space.

For the Banach spaces X, Y we consider the special bilinear map

$(x, y) \in X \times Y \to x \otimes y \in X \otimes Y$

behaves like a *universal* bilinear map, that is, every other bilinear map on $X \times Y$ can be factored through this space via a linear mapping [86, Chapter 1].

The Proposition 1.4 in [86] states that for every bilinear mapping $B : X \times Y \to Z$ there is a unique linear map $B_L : X \otimes Y \to Z$ defined by

$B_L\Big(\sum_{i=1}^{n} x_i \otimes y_i\Big) = \sum_{i=1}^{n} B(x_i, y_i).$

The correspondence $B \longleftrightarrow B_L$ identifies an isomorphism between the vector spaces $B(X \times Y, Z)$ and $L(X \otimes Y, Z)$ and the linear map B is called the *linearization* of the bilinear map B.

The situation is illustrated by the following diagram;

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[dr] & Z \\ & X \otimes Y. \arrow[u, "B_L"'] \end{tikzcd}$$

B-2 The Projective Tensor Norm

It is important to investigate a norm on the tensor product $X \otimes Y$ in order to obtain a linearization for bounded bilinear operators.

The *natural* norm, known as *projective norm* on $X \otimes Y$ is defined as

$$\pi(u) = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\},$$

where the infimum is taken over all possible representations of the tensor u [86, Chapter 2].

This norm satisfies the equality $\pi(x \otimes y) = \|x\|\|y\|$ for all elementary tensors $x \otimes y$.

The tensor product $X \otimes Y$ endowed with the projective norm π is denoted by $X \otimes_\pi Y$ and the completion of $X \otimes_\pi Y$ by $X \hat{\otimes}_\pi Y$. An element $u \in X \hat{\otimes}_\pi Y$ has the representation $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ such that $\sum_{i=1}^{\infty} \|x_i\|\|y_i\| < \infty$, therefore the projective norm of u is

$$\pi(u) = \inf\left\{\sum_{i=1}^{\infty} \|x_i\|\|y_i\| : u = \sum_{i=1}^{\infty} x_i \otimes y_i\right\}$$

where the infimum is taken over all possible representations of the tensor u as above [86, Chapter 2].

Linearization of Continuous Bilinear Operators

For every continuous bilinear operator $B : X \times Y \to Z$ there exists a unique continuous linear map $B_L : X \hat{\otimes}_\pi Y \to Z$ defined by

$$B_L(x \otimes y) = B(x, y),$$

for every $x \in X$, $y \in Y$. The correspondence $B \longleftrightarrow B_L$ is an isometric isomorphism between the vector spaces $\mathcal{B}(X \times Y, Z)$ and $\mathcal{L}(X \hat{\otimes}_\pi Y, Z)$ [86].

B-3 Injective Tensor Norm

The *injective tensor norm* on $X \otimes Y$ is defined by

$$\varepsilon(u) = \sup\left\{\left|\sum_{i=1}^{n} \phi(x_i)\psi(y_i)\right| : \phi \in B_{X*},\ \psi \in B_{Y*}\right\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of u in tensor product $X \otimes Y$ [86, Section 3.1].

The tensor product $X \otimes Y$ with the injective norm ε is denoted by $X \otimes_\varepsilon Y$ and the completion of $X \otimes_\varepsilon Y$, called the *injective tensor product of* X, Y, by $X \hat{\otimes}_\varepsilon Y$ [86, Chapter 3].

Similar to the projective tensor norm, the injective tensor norm satisfies the equality $\varepsilon(x \otimes$

$y) = \|x\|\|y\|$ for all elementary tensors $x \otimes y$ and these norms hold the inequality $\varepsilon(u) \leqslant \pi(u)$ for every tensor u [86, Proposition 3.1.].

B-4 Reasonable Crossnorm

A. Grothendieck gave the general study of tensor norms and he provided the properties that a tensor norm has to possess.

A norm α on $X \otimes Y$ is called *reasonable crossnorm* if the following properties hold.

1. $\alpha(x \otimes y) \leqslant \|x\|\|y\|$ for every $x \in X$ and $y \in Y$,
2. For every $\phi \in X^*$ and $\psi \in Y^*$, the linear functional $\phi \otimes \psi$ on $X \otimes Y$, defined by $\phi \otimes \psi(u) = \sum_{i=1}^{n} \phi(x_i)\psi(y_i)$ for $u = \sum_{i=1}^{n} x_i \otimes y_i$, is continuous and $\|\phi \otimes \psi\| \leqslant \|\phi\|\|\psi\|$ (see [86, §6.1] or [18, §12.1]).

The projective and the injective tensor norms satisfy these conditions and they are the *greatest* and the *least* reasonable crossnorms, respectively.

Proposition 6.1 in [86] shows that a norm α on $X \otimes Y$ is a reasonable crossnorm if and only if the inequalities $\varepsilon(u) \leqslant \alpha(u) \leqslant \pi(u)$ hold for all $u \in X \otimes Y$ and moreover, for any reasonable crossnorm α the equality $\alpha(x \otimes y) = \|x\|\|y\|$ holds for every $x \in X$, $y \in Y$.

CURRICULUM VITAE

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EDUCATION

Degree	Department	University	Date of Graduation
Master	Mathematical Engineering	Yıldız Technical University	2013
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WORK EXPERIENCE

Year	Institute	Enrollment
2012-	Marmara University	Research Assistant

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2016-2017	Univ. Politècnica de València (Spain)	Researcher

PUBLISHMENTS

Papers

1. Erdoğan, E. and Karakaya, V., (2014). "On Spectral Properties of a New Operator over Sequence Spaces c and c_0", Acta Mathematica Scientia, 34(5): 1481-1494.
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Conference Papers

1. Erdoğan, E. and Karakaya V., (2012). "The Fine Spectra of a New Operator W on Sequence Space c_0", International Conference on Applied Analysis and Applications (ICAAA), 20-24 June 2012, Yıldız Technical University, Istanbul.

2. Erdoğan, E. and Karakaya V., (2012). "The Spectrum of Matrix Operators on a Difference Sequence Space of Weighted Means", International Eurasian Conference on Mathematical Sciences And Applications (IECMSA), 3-7 August 2012, Piristina.

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4. Erdoğan E., (2014). "Bazı Dizi Uzaylarındaki Spektral Analiz", I. Kadın Matematikçiler Derneği Çalıştayı, 2-4 May 2014, Kocaeli, (poster).

5. Erdoğan, E. and Karakaya V., (2014). "On the Spectrum of the Product Operator W on the Sequence Space b_v", International Conference on Recent Advances in Pure and Applied Mathematics (ICRAPAM), 6-9 November 2014, Antalya.

6. Kocabaş S., Erdoğan, E. and Dernek A.N., (2016). "Some Results on the Generalized Mellin Transforms and Applications", International Conference on Analysis and Applications (ICAA), 12-15 July 2016, Kırşehir.

7. Erdoğan, E. and Karakaya V., (2016). "Operator Ideal of s-Type Operators Using Weighted Mean Sequence Space", International Conference on Analysis and Applications (ICAA), 12-15 July 2016, Kırşehir.

8. Erdoğan, E., Calabuig J. M. and Sánchez Perez E. A., (2017). "Convolution-Continuous Bilinear Operators Acting on Hilbert Spaces of Integrable Functions", Banach Spaces And Operator Theory with Applications (BSOTA), 3-7 July 2017, Poznan.

9. Erdoğan E., (2017). " S-Sayıları ve Multilineer Operatörler", 3. Marmara Matematik Günleri, 28-29 September 2017, Istanbul.

10. Erdoğan, E., (2018). "Convolution Factorability of Zero Product Preserving Bilinear Maps", Workshop on Infinite Dimensional Analysis, 1-3 February 2018, Valencia, (poster).

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12. Erdoğan E. and Sánchez Pérez E. A., (2018). "Product Factorability of Bilinear Maps on Banach Function Spaces", The 12th International Conferences on Function Spaces, 9-13 July 2018, Krakow.

AWARDS AND SCHOLARSHIPS

1. Top scoring student in the department, Sakarya University, 2010.

2. The Scientific and Technological Research Council of Turkey (TÜBİTAK) National Scholarship for M.S. Students-2010.

3. The Scientific and Technological Research Council of Turkey (TÜBİTAK) National Scholarship for Ph.D. Students-2013.