REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

ON SOLVABILITY OF NONLINEAR COUPLED WAVE SYSTEMS ARISING IN DNA DYNAMICS

Meltem UZUN

DOCTOR OF PHILOSOPHY THESIS
Department of Mathematics
Mathematics Program

Advisor
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A thesis submitted by Meltem UZUN in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 07.05.2021 in Department of Mathematics, Mathematics Program.

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Dedicated to my family

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LIST OF SYMBOLS

A^*	adjoint of the operator A
$\partial\Omega$	boundary of Ω
$\langle v, v' \rangle$	bracket of duality between $v \in V$ and $v' \in V'$
$\overline{\Omega}$	closure of Ω
$H_0^m(\Omega)$	closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$
D^α	differential operator of order $\lvert\alpha\rvert$ with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, $D^\alpha = D_1^{\alpha_1} ... D_N^{\alpha_N}$
$X \times Y$	direct product space, $X \times Y = \{(x, y) : x \in X, y \in Y\}$
$X \bigoplus Y$	direct sum of two vector spaces X and Y
$D(A)$	domain of the operator A
V'	dual space of V
$H^{-m}(\Omega)$	dual space of $H_0^m(\Omega)$
$\lvert x\rvert$	Euclidean norm of x
$\nabla = \sum_{k=1}^N D_k$	gradient operator
$\mathscr{D}(\Omega)$	identical to $C_0^\infty(\Omega)$
$H^m(\Omega)$	identical to $W_2^m(\Omega)$
(x,y)	inner (scalar) product on a Hilbert space
$\Delta = \sum_{k=1}^N D_k^2$	Laplace operator
$L_p(0, T; X)$	Lebesgue space of functions $u : (0, T) \to X$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\lVert x\rVert$	norm of x in a Banach space
$D_k f$ or $\frac{\partial f}{\partial x_i}$	partial derivative of $f(x) : \mathbb{R} \to X$ with respect to the variable x_i
$B(V, W)$	set of bounded operators from V into W
$\mathbb{Z}$	set of integers

$\mathscr{L}(V,W)$	set of linear continuous operators from V into W
$\mathbb{N}$	set of Natural Numbers
$\mathbb{Z}_+$	set of Nonnegative Integers
$\mathbb{R}_+$	set of Nonnegative Real Numbers
$\mathbb{R}$	set of Real Numbers
$W_p^m(\Omega)$	Sobolev space of all measurable functions $u:\Omega\to\mathbb{R}$, where $m = 0,1,...$ and $1\leq p<\infty$
$C_b^k(\Omega)$	space of bounded functions on Ω which are k-times continuously differentiable, $k\in Z_+$
$\mathscr{D}'(\Omega)$	space of distributions on Ω, i.e. the set of continuous linear forms on $\mathscr{D}(\Omega)$, the dual of $\mathscr{D}(\Omega)$
$C_0^\infty(\Omega)$	space of functions of class C^∞ on Ω which have compact support in Ω
$C_0^k(\Omega)$	space of functions on Ω of class C^k which have compact support in Ω
$C^\infty(\Omega)$	space of functions on Ω which are infinitely continuously differentiable
$C^k(\Omega)$	space of functions on Ω which are k-times continuously differentiable, $k\in Z_+$
$C^k([0,T];X)$	space of k-times continuously differentiable functions $u:(0,T)\to X$
$L_\infty(\Omega)$	space of measurable functions on Ω which are bounded almost everywhere, i.e. essentially bounded
$L_p(\Omega)$	space of measurable functions on Ω whose p-th power is Lebesgue integrable
$\sup_{x\in B} f(x)$	supremum or upper bound of $f(x)$ in $\mathbb{R}$, where $f:B\to\mathbb{R}$
A^T	transpose of the operator A
Ω	usually denotes an open set in a topological space
$V\hookrightarrow H$	V is embedded into H

LIST OF FIGURES

ABSTRACT

On Solvability of Nonlinear Coupled Wave Systems Arising in DNA Dynamics

Meltem UZUN

Department of Mathematics
Doctor of Philosophy Thesis

Advisor: Assoc. Prof. Dr. Özgür YILDIRIM

In this thesis, we investigate the existence and uniqueness of weak solutions for the system of finite difference schemes for the coupled sine-Gordon equations. The novel first order of accuracy unconditionally stable difference scheme is considered. The variational method, which is also known as the energy method is employed to prove the unique weak solvability. We also present a novel unified approach for the numerical solution of the system by combining the difference scheme with a convenient adaptation of fixed point theory. Several test problems are considered and results of the numerical experiments are provided with error analysis in order to verify the accuracy of the proposed numerical method.

Keywords: Abstract evolution equations, existence-uniqueness, a priori estimates, weak solutions, finite difference methods

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ÖZET

DNA Dinamiğinde Ortaya Çıkan Nonlineer Bağlı Dalga Denklem Sistemlerinin Çözülebilirliği Üzerine

Meltem UZUN

Matematik Anabilim Dalı
Doktora Tezi

Danışman: Assoc. Prof. Dr. Özgür YILDIRIM

Bu tezde, bağlı sinüs-Gordon denklem sistemi için sonlu farklar sisteminin zayıf çözümünün varlığı ve tekliği ele alınmıştır. Bu sistemin tek türlü zayıf çözülebilirliğini kanıtlamak için birinci mertebeden koşulsuz kararlı fark şemasına, enerji metodu olarak da bilinen varyasyonel yöntem uygulanmıştır. Ayrıca yeni bir hibrit nümerik metod geliştirilerek ikili sinüs-Gordon denklem sistemi için başlangıç sınır değer probleminin yaklaşık çözümü elde edilmiştir. Teorik bulguları destekleyen çeşitli test problemleri ele alınarak hata analizleriyle birlikte sunulmuştur.

Anahtar Kelimeler: Soyut oluşum denklemleri, varlık-teklik, önsel kestirimler, zayıf çözümler, sonlu fark metodu

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Review

1.1.1 General Overview

Wave propagation problems are studied in several areas of engineering, physics, and applied mathematics including relativistic quantum mechanics, acoustics, biomedical engineering, and field theory problems (see, [1–12] and the references given therein). There have been extensive theoretical and numerical studies on nonlinear wave systems such as sine-Gordon, Klein-Gordon and coupled sine-Gordon equations in the literature (see, [13–17] and the references given therein). Such type of problems attracted much attention in the last decades due to the presence of soliton solutions. Solitons are nonlinear waves which occur in proteins, signal conduction between neurons and deoxyribonucleic acid (DNA) [18, 19].

Due to low regularity of coefficients and source functions, unique solvability in the weak sense have drawn remarkable interest for many problems occurring in real world phenomena, including coupled sine-Gordon equations. In the weak solvability, solutions of complicated nonlinear systems, and also linear or semi-linear problems which don't have a corresponding mild formulation, can be obtained even under less regularities of data. Solutions of these problems are obtained in the space of distributions by the energy method, also known as the variational method, which serves as a versatile equipment in theory of partial differential equations (PDE).

The weak solvability of nonlinear systems are widely investigated in the literature (see, [9, 11, 20–33] and the references given therein). In [20], endemic equilibrium for the PDE model of Zika virus, which leads to a major global public health emergency, is studied. In [23], approximate solution of coupled sine-Gordon equation with periodic boundary conditions is investigated. Also in [9], the global weak solvability of coupled damped sine-Gordon equations in abstract form is proved and the finite element method is used. The weak solutions for non-gradient coupled sine-Gordon equations is studied in [11]. Regularity criteria of the weak solutions for the 3-dimensional incompressible viscous magnetohydrodynamics equations are discussed in [30]. Several types of prey-predator models are investigated in [32]. Weak solvability of finite difference schemes for several coupled systems are studied in [28, 34–40].

Finite difference method is a powerful instrument for obtaining numerical solutions of PDE and it is applied in many types of problems occurring in real world phenomena (see, [2, 4, 12, 15, 41–49] and the related references given herein).In recent years, growing attention is paid on the study of unconditionally stable difference schemes due to the fact that they don't require any assumption with regard to grid steps on space and time variables (see, [41–46] and the references given therein). In this study, we use the unconditionally stable difference schemes since they provide good convergence and stability results.

The early investigations about the convergence of difference scheme for the hyperbolic PDEs are contributed by Courant, Friedrichs, Lewy, von Neumann, Lax, and Richtmeyer et al. In studying these problems, a necessary condition for convergence of a finite difference scheme is Courant-Friedrichs-Lewy condition, or in short, CFL conditions. In the present study, the employed difference scheme provides good convergence and stability results without the need of a CFL condition. In numerical analysis, a unified approach which combines the difference schemes and fixed point iteration with some error tolerances is used. Combining with fixed point iteration, the numerical experiments for solutions of these difference schemes gives accurate results.

1.1.2 DNA Modelling

DNA (deoxyribonucleic acid) is a biopolymer which plays a significant role in the storage of genetic information in prokaryotic and eukaryotic organisms. Therefore, an appropriate model must be created in the theoretical study of the dynamics of this complex structure. Genetic information from genes transferring to certain proteins and enzymes is based on local expansion (denaturation) of DNA chains [50]. This local opening of DNA is mathematically defined as a traveling wave that moves along the helix. Studies in this area mainly cover the presence of solitons and nonlinear motion of DNA polynucleotide chains. A model that expresses the dynamical change of DNA during the denaturation of the DNA double helices was proposed by M. Peyrard et al. [50]. In addition, with the model developed by S. W. Englander et al., the information that the DNA model was reduced to the sine-Gordon equation (SGE) was brought into the literature [51]. Nonlinear dynamics of DNA with the hypothesis of "solitons in DNA" attracted the attention of many researchers, especially theoretical physicists, and many models have been formulated [6, 52–59]. Other important articles in this area are Yomosa S., (1983) and Zhang, C.T. (1987), under the same title, published in the same journal [60, 61]. In the same period, Yakushevich et al. (1987) described the weak homogeneities in the simple DNA parts consisting of the given type and subsequently other types of uniform base sequences in terms of perturbed SGE [62]. According to the standard Watson-Crick double helix B - form DNA model, the two helix polynucleotide strands are held together by hydrogen H-bonds. Yomosa discussed the Watson-Crick model in [60, 63]. In this model, the zeroth level energies of DNA polynucleotides and the average of disrupted double and triple hydrogen bonds between A-T (Adenine-Thymine), G-C (Guanine-Cytosine) base pairs are discussed [64]. Yumosa stated that stacking energies, consisting of H-bonding and electrostatic exchange, charge transfer and induction and distribution interactions, are roughly proportional to the overlaps of molecular orbitals [63, 65, 66].

The open state of the DNA helix is expressed by the Hamiltonian equation below:

$$\begin{aligned} H = \sum_m \Big[& \frac{I}{2}\left(\dot{\omega}_m^2 + \dot{\omega}_m'^2\right) + \kappa\left[2 - \cos\left(\omega_{m+1} - \omega_m\right)\right. \\ & \left. - \cos\left(\omega'_{m+1} - \omega'_m\right)\right] - \eta\left[1 - \cos\left(\omega_m - \omega'_m\right)\right] \\ & + \lambda\left\{2q_0^2 - \left[\sin\left(\omega_{m+1} - \omega_m\right) - q_0\right]^2\right. \\ & \left. - \left[\sin\left(\omega'_{m+1} - \omega'_m\right) - q_0\right]^2\right\}\Big] \end{aligned} \tag{1.1}$$

Here, ω_i and ω'_i represent the rotational angles of the base pair, $I = 1/2A^2$ symbolizes the moment of inertia around the axis of rotation at mth point, A and κ are the uniaxial magnetocristal anisotropy along the axis and the exchange interaction of ferromagnetic spin, respectively. Also, λ specifies the elasticity constant related with twist deformation and η is a constant. In equation (1.1), the terms

$$\sum_m \frac{I}{2}\left(\dot{\omega}_m^2 + \dot{\omega}_m'^2\right),\ \sum_m \kappa\left[2 - \cos\left(\omega_{m+1} - \omega_m\right) - \cos\left(\omega'_{m+1} - \omega'_m\right)\right] \tag{1.2}$$

and

$$\sum_m \eta\left[1 - \cos\left(\omega_m - \omega'_m\right)\right] \tag{1.3}$$

refer to the kinetic energy resulting from the rotation of the bases, energy of the hydrogen bonds in complementary base pairs between chains, and the cluster energy between adjacent bases in the helix, respectively. These terms determine stability of the DNA double helix [64]. The movement of the DNA helix is identified by taking derivative of equation (1.1) according to the variable t, and the following equations are obtained:

$$
\begin{aligned}
I\omega_{mtt} &= [\kappa+2\lambda\cos(\omega_{m+1}-\omega_m)]\sin(\omega_{m+1}-\omega_m) \\
&\quad -[\kappa+2\lambda\cos(\omega_m-\omega_{m-1})]\sin(\omega_m-\omega_{m-1})+\eta\sin\left(\omega_m-\omega'_m\right) \\
&\quad -2\lambda q_0[\cos(\omega_{m+1}-\omega_m)-\cos(\omega_m-\omega_{m-1})],
\end{aligned} \tag{1.4}
$$

$$
\begin{aligned}
I\omega'_{mtt} &= \left[\kappa+2\lambda\cos\left(\omega'_{m+1}-\omega'_m\right)\right]\sin\left(\omega'_{m+1}-\omega'_m\right) \\
&\quad -\left[\kappa+2\lambda\cos\left(\omega'_m-\omega'_{m-1}\right)\right]\sin\left(\omega'_m-\omega'_{m-1}\right)+\eta\sin\left(\omega'_m-\omega_m\right) \\
&\quad -2\lambda q_0\left[\cos\left(\omega'_{m+1}-\omega'_m\right)-\cos\left(\omega'_m-\omega'_{m-1}\right)\right].
\end{aligned} \tag{1.5}
$$

These equations determine the discrete dynamics of DNA double helices when the helix nature of the molecule is represented as a twist-like deformation. Assuming that the difference in angular rotation of the bases relative to the adjacent bases along the two strips is too small, and after redefining time and parameter η, separate motion equations are obtained in the following form

$$
\omega_{tt} = \frac{(\kappa+2\lambda)}{I}\omega_{zz}+\eta\sin\left(\omega-\omega'\right), \tag{1.6}
$$

$$
\omega'_{tt} = \frac{(\kappa+2\lambda)}{I}\omega'_{zz}+\eta\sin\left(\omega'-\omega\right). \tag{1.7}
$$

Using the substitution $\Psi=\omega-\omega'$ and choosing $2\eta=-1$, the sine-Gordon equation

$$
\Psi_{tt}-\Psi_{zz}+\sin\Psi=0 \tag{1.8}
$$

is obtained. Using $t=\frac{x+y}{2}$, $z=\frac{x-y}{2}$ equation (1.8) is obtained in the form

$$
\Psi_{xy}=\sin\Psi \tag{1.9}
$$

In addition, the dynamics of the DNA twist molecule are expressed with coupled nonlinear equations, as follows:

$$
\begin{aligned}
I\omega_{mtt} &= [\kappa+2\lambda\cos(\omega_{m+1}-\omega_m)]\sin(\omega_{m+1}-\omega_m) \\
&\quad -[\kappa+2\lambda\cos(\omega_m-\omega_{m-1})]\sin(\omega_m-\omega_{m-1})+\eta\sin\left(\omega_m-\omega'_m\right) \\
&\quad -2\lambda q_0[\cos(\omega_{m+1}-\omega_m)-\cos(\omega_m-\omega_{m-1})],
\end{aligned} \tag{1.10}
$$

$$
\begin{aligned}
I\omega'_{mtt} &= \left[\kappa+2\lambda\cos\left(\omega'_{m+1}-\omega'_m\right)\right]\sin\left(\omega'_{m+1}-\omega'_m\right) \\
&\quad -\left[\kappa+2\lambda\cos\left(\omega'_m-\omega'_{m-1}\right)\right]\sin\left(\omega'_m-\omega'_{m-1}\right)+\eta\sin\left(\omega'_m-\omega_m\right) \\
&\quad -2\lambda q_0\left[\cos\left(\omega'_{m+1}-\omega'_m\right)-\cos\left(\omega'_m-\omega'_{m-1}\right)\right],
\end{aligned} \tag{1.11}
$$

$$
\begin{aligned}
I\nu_{mtt} &= [\kappa + 2\lambda\cos(\nu_{m+1} - \nu_m)]\sin(\nu_{m+1} - \nu_m) \\
&\quad -[\kappa + 2\lambda\cos(\nu_m - \nu_{m-1})]\sin(\nu_m - \nu_{m-1}) + \eta\nu_m\cos\left(\omega_m - \omega'_m\right) \\
&\quad -2\lambda q_0[\cos(\nu_{m+1} - \nu_m) - \cos(\nu_m - \nu_{m-1})], \qquad (1.12)
\end{aligned}
$$

$$
\begin{aligned}
I\nu'_{mtt} &= \left[\kappa + 2\lambda\cos\left(\nu'_{m+1} - \nu'_m\right)\right]\sin\left(\nu'_{m+1} - \nu'_m\right) \\
&\quad -\left[\kappa + 2\lambda\cos\left(\nu'_m - \nu'_{m-1}\right)\right]\sin\left(\nu'_m - \nu'_{m-1}\right) + \eta\nu'_m\cos\left(\omega'_m - \omega_m\right) \\
&\quad -2\lambda q_0\left[\cos\left(\nu'_{m+1} - \nu'_m\right) - \cos\left(\nu'_m - \nu'_{m-1}\right)\right]. \qquad (1.13)
\end{aligned}
$$

The equations (1.10)-(1.13) express the interaction of two pairs of DNA helices with each other in discrete twist deformation. With small angles approach under the continuity limit

$$
\omega_{i+1} - 2\omega_i + \omega_{i-1} \rightarrow \omega_{zz} \qquad (1.14)
$$

rescaling time and redefining parameter η yields

$$
\begin{aligned}
\omega_{tt} &= \frac{(\kappa + 2\lambda)}{I}\omega_{zz} + \eta\sin\left(\omega - \omega'\right), \qquad (1.15) \\
\omega'_{tt} &= \frac{(\kappa + 2\lambda)}{I}\omega'_{zz} + \eta\sin\left(\omega' - \omega\right), \qquad (1.16)
\end{aligned}
$$

$$
\begin{aligned}
\nu_{tt} &= \frac{(\kappa + 2\lambda)}{I}\nu_{zz} + \eta\sin\left(\nu - \nu'\right), \qquad (1.17) \\
\nu'_{tt} &= \frac{(\kappa + 2\lambda)}{I}\nu'_{zz} + \eta\sin\left(\nu' - \nu\right). \qquad (1.18)
\end{aligned}
$$

Applying of algebraic operations to equations (1.15)-(1.18) and rescaling parameter z finally gives the coupled sine-Gordon equation

$$
\begin{aligned}
\Phi_{tt} - \Phi_{zz} + \sin\Phi &= 0 \\
\Psi_{tt} - \Psi_{zz} + \sin\Psi &= 0. \qquad (1.19)
\end{aligned}
$$

Here, $\Phi = \omega - \omega'$, $\Psi = \nu - \nu'$ and $\eta = -1$. Because of its physical and biological importance, many scientists focused on obtaining analytical and approximate solution of (1.19) with the help of numerical and ansatz methods [8, 9, 13, 16–18, 48, 64, 67–69].

1.1.3 Solitons

Soliton is a nonlinear wave, which is solitary traveling wave pulse solution for a nonlinear PDE [70, 71]. The nonlinearity effect plays a major role in solitons. Solitary

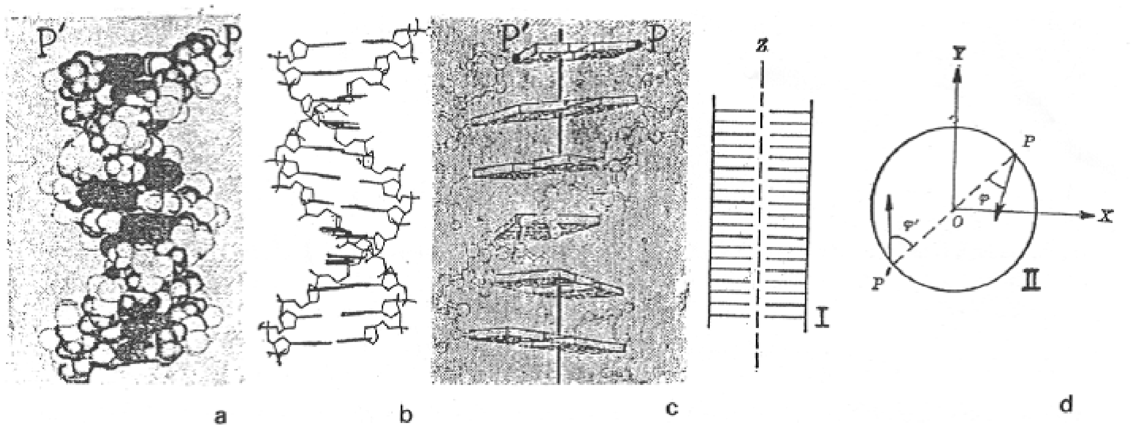

Figure 1.1 the solid structure of DNA double helices and its dynamical plane base-rotator model:**a.)** molecular structure **b.)**linear map **c.)**plane base-rotator model **d.)** outline graph of base-rotator: **I.)** side elevation, **II.)**plane figure [65]
.

waves scatter and lose energy due to the radiation, in dispersive evolution equations. However, solitons retains their identities with same speed and shape after a nonlinear interaction [72]. Therefore, it have substantial stability properties, which plays a significant role in soliton physics. The beginning of soliton physics dates back to the observation of John Scott Russell, called “great wave of translation”, on month of August 1834 [73]. It is well known as a special form of a surface water wave, by many scientists such as Stokes, Boussinesq, Korteweg, de Vries, and Rayleigh. In 1895, Korteweg and de Vries obtained the equation describing the propagation of wave on the surface of a channel.

Solitons may occur in many applications such as fibre optics, magnets and some biological process [54, 74–77]. "Solitons in DNA" hypotheses reveals that solitons are associated with the low-frequency collective motion in DNA and proteins [78]. A model in neuroscience proposed that signals are transmitted by neurons in the form of solitons [79, 80].

1.2 Objective of the Thesis

In this study, the nonlinear system of coupled sine Gordon equations

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} - \beta_1 \Delta u + \gamma_1 \sin(\delta_{11}u + \delta_{12}v) \\ +\rho_{11}u + \rho_{12}v = f \text{ in } R, \\ \frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} - \beta_2 \Delta v + \gamma_2 \sin(\delta_{21}u + \delta_{22}v) \\ +\rho_{21}u + \rho_{22}v = g \text{ in } R \end{cases} \tag{1.20}$$

with the boundary conditions

$$u = 0 \text{ and } v = 0 \text{ on } S, \tag{1.21}$$

and the initial conditions

$$u(0,x) = \xi_1(x) \text{ in } \Omega \text{ and } \frac{\partial u}{\partial t}(0,x) = \eta_1(x) \text{ in } \Omega, \tag{1.22}$$

$$v(0,x) = \xi_2(x) \text{ in } \Omega \text{ and } \frac{\partial v}{\partial t}(0,x) = \eta_2(x) \text{ in } \Omega \tag{1.23}$$

is considered. Here, $\Omega \subset \mathbb{R}^n$ is a bounded open set with piecewise smooth boundary $\Gamma = \partial\Omega$ and Δ is the Laplacian. The spaces R and S are defined as $R = (0,1) \times \Omega$ and $S = (0,1) \times \Gamma$, respectively. The given constants are

$$\alpha_{ij}, \beta_i, \gamma_i, \delta_{ij}, \rho_{ij}, \text{ which are bounded nonzero real numbers for } i,j = 1,2. \tag{1.24}$$

Let $A = -\Delta$ be defined as an unbounded self-adjoint and positive-definite operator in a Hilbert space H with the domain

$$D(A) = \{\varphi(x) \in V : A\varphi(x) \in H, \varphi(x) = 0 \text{ on } \Gamma\}. \tag{1.25}$$

Then problem (1.20)-(1.23) can be written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} + \beta_1 A u + \gamma_1 sin\left(\delta_{11}u + \delta_{12}v\right) \\ +\rho_{11}u + \rho_{12}v = f, 0 < t < 1, \\ \frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} + \beta_2 A v + \gamma_2 \sin\left(\delta_{21}u + \delta_{22}v\right) \\ +\rho_{21}u + \rho_{22}v = g, 0 < t < 1, \\ u(0) = u_0 \in V, \frac{du}{dt}(0) = u_0' \in H, \\ v(0) = v_0 \in V, \frac{dv}{dt}(0) = v_0' \in H. \end{cases} \tag{1.26}$$

Here, V is the Hilbert space satisfying the relation $V \subset H$. In the literature, a special case of the system in the form

$$\begin{cases} u_{tt} - u_{xx} = -\delta^2 \sin(u - v), \\ v_{tt} - v_{xx} = \sin(u - v) \end{cases} \tag{1.27}$$

which describes the open states of DNA double helices is studied by many researchers (see, [18, 19] and the references given therein). Note that some applications and numerical results of the present study, without proof, are presented in [48, 49].

1.3 Hypothesis

Unique solvability of problem (1.26) is presented as the limit of first order of accuracy unconditionally stable difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\alpha_{11}(2\tau)^{-1}(u_{k+1}-u_{k-1}) \\ +\alpha_{12}(2\tau)^{-1}(v_{k+1}-v_{k-1})+\beta_1 Au_{k+1} \\ +\gamma_1\sin(\delta_{11}u_k+\delta_{12}v_k)+\rho_{11}u_{k+1}+\rho_{12}v_{k+1}=f_k, \\ f_k=f(t_k), t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ \tau^{-2}(v_{k+1}-2v_k+v_{k-1})+\alpha_{21}(2\tau)^{-1}(u_{k+1}-u_{k-1}) \\ +\alpha_{22}(2\tau)^{-1}(v_{k+1}-v_{k-1})+\beta_2 Av_{k+1} \\ +\gamma_2\sin(\delta_{21}u_k+\delta_{22}v_k)+\rho_{21}u_{k+1}+\rho_{22}v_{k+1}=g_k, \\ g_k=g(t_k), t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ u_0=\varphi_1, u_0'=\frac{u_1-u_0}{\tau}=\psi_1, \\ v_0=\varphi_2, v_0'=\frac{v_1-v_0}{\tau}=\psi_2, \end{cases} \tag{1.28}$$

with the modification for nonlinear damped system. The set of a family of grid points

$$\Omega_h=[0,1]_\tau\times[0,\pi]_h=(t_k,x_n): t_k=k\tau,\ \ 0\le k\le N,$$

$$N\tau=1, x_n=nh, 0\le n\le M, Mh=\pi\} \tag{1.29}$$

with small parameters τ, h is considered for approximate solutions of (1.26). Here, f_k, g_k, φ_1, φ_2, ψ_1, and ψ_2 are given nonzero functions. Convergence and stability issues for the linear and undamped form of difference scheme (1.28) are presented in [2, 4].

In this thesis, the unique solvability of first order of accuracy unconditionally stable difference schemes for coupled sine-Gordon system, in the weak sense, is proved. Compared with other existing studies in the literature, the novelty of the present work is two fold: one is the generality of nonlinearity and damping effects in weak solvability via finite difference method, and the other is the unified numerical approach based on a first order of accuracy unconditionally stable finite difference schemes and the fixed point theory. Moreover, we can express the originality of this study as follows:

Obtaining a smooth and continuous solution of this nonlinear system under the appropriate initial and limit values is not possible in most cases due to the fact that the elements of the problem such as the coefficients and the source functions are not always continuous [9, 11]. For this reason, we aim to obtain a weak solution in the distribution space. Hilbert valued functions in the distribution space physically correspond to the quantum field [81]. There is a close relationship between DNA and quantum field. According to recent studies, DNA plays a distinctive role among quantum states known as spin [82, 83]. Therefore, the unique solvability of problem (1.26) in distribution space is important in the field of applied mathematics as well as in quantum physics and genetics. Thus, the method we propose for the solution of (1.26) has the potential to be used in more than one research area and to lead interdisciplinary studies.

The novelty of the current study can be emphasized by the following issues: In this study, the weak (generalized) solutions of the first order of accuracy unconditionally stable finite difference schemes for the coupled sine-Gordon system under damping effect will be obtained with the help of the energy method. With unconditionally stable difference schemes, stability is achieved in the grid space and time components without requiring conditions known as CFL conditions for step sizes. In this regard, unconditionally stable finite difference schemes are known as an effective method of obtaining approximate solution. Also, with the help of unconditionally stable finite difference scheme and the fixed point theory for nonlinear term of the problem, a unified approach will be developed for the solution of the problem we are dealing with. In addition, the existence and uniqueness of the solution of nonlinear coupled systems, which are the subject of active research in mathematics all around the world, the help of unconditionally stable finite difference schemes will bring a new approach to the literature in this field. We hope that, with the interpretation of our results in the field of biomathematics and biophysics, important findings can be obtained to examine DNA deformations and mutations, as well as to assist in the diagnosis and treatment of genetic-based diseases (albino, DOWN syndrome, hemophilia, etc.). (For the relationship between DNA mutations and the nonhomogenicity of the equation, see [9]).

2
THEORY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

In the present section we give some notations, definitions and preliminary results on the literature, which are necessary in the sequel. The concepts in this chapter are written on the basis of [6, 10, 11, 70, 81, 84–91].

2.1 Preliminaries

It is well known that the solution methods for linear equations can not be applied to nonlinear equations in most cases. Because there is not a conventional technique for determining the analytical solutions of nonlinear PDEs, it is often crucial to use numerical methods for their solutions. Similar with linear equations, the existence, uniqueness and stability issues for the solution of nonlinear PDEs are of essential importance. Therefore, nonlinear equations are among the most diverse and rewarding fields in modern mathematics.

Definition 2.1. [70, 92] A *nonlinear PDE* is stated in the following form

$$G(t,x,u,D^{\alpha}u)=0, \tag{2.1}$$

for $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$, $u=(u_1,u_2,...,u_m)\in\mathbb{R}^m$, $G=(G_1,G_2,...,G_s)\in\mathbb{R}^s$, $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ is multi-index consisting of non-negative integers $\alpha_1,\alpha_2,...,\alpha_n$, and the differential operator $D^{\alpha}=D_1^{\alpha_1}D_2^{\alpha_2}...D_n^{\alpha_n}$ with $D_i=\frac{\partial}{\partial x_i}$, $i=1,2,...,n$. If $s>1$, then (2.1) is said to be a system of non-linear PDEs.

Alternatively, equation (2.1) can be written in the operator form as

$$Lu(t,x)=g(t,x). \tag{2.2}$$

Here, L denotes a partial differential operator and $g(t,x)$ is the given function for $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$. Equation (2.2) is called nonlinear PDE if L is not a linear operator. For $g(t,x)=0$, (2.2) is said to be a *homogeneous equation* and for $g(t,x)\neq 0$, (2.2) is called an *nonhomogeneous equation*.

Definition 2.2. [4, 70] A *classical solution* of (2.1) is a function $u=u(t,x)$ on some domain D which satisfies the following properties

i) $u=u(t,x)$ is continuously differentiable over D in the sense that all of partial derivatives in equation (2.1) exist,

ii) the function $u=u(t,x)$ satisfies equation (2.1).

An extension of the classical solution can be obtained by relaxing the condition i.), i.e. the solution do not have to be continuously differentiable. In this case the solution is called a *weak (or generalized) solution* of (2.1).

In most cases, equation (2.1) is governed by initial and/or boundary conditions, which arise from the physical interpretation of the problem.

The conditions which determine the physical state of $u(t,x)$ at a specific time $t=t_0$ or $t=0$ are called the *initial (Cauchy) conditions*, and the problem of finding the solution of (2.1) with prescribed Cauchy data is said to be the *initial-value problem (IVP)* or *Cauchy problem*.

The conditions which determine the physical state of $u(t,x)$ on boundary ∂D of the given domain D are called the *boundary conditions*, and problem (2.1) governed by these kinds of data is said to be the *boundary-value problem*.

Definition 2.3. [70, 84] A boundary value problem is *well posed* if it satisfies the conditions:

1. existence of solutions,
2. uniqueness of the solution,
3. stability, in other words, continuous dependence of the solution on the initial data.

These conditions are also called *Hadamard principle*.

Definition 2.4. [70] *The linear superposition principle* is implemented to linear PDEs under certain convergence conditions. The principle states that any linear combination of solutions for a linear equation results in a new solution to the related problem. However, the principle can not be applied to nonlinear PDEs to generate a new solution.

2.2 Semigroup Theory

[11, 86, 93]

The linear system consisting of ordinary differential equations

$$\begin{cases} \frac{dv_1}{dt} = c_{11}v_1 + ... + c_{1n}v_n + g_1(t) \\ \frac{dv_2}{dt} = c_{21}v_1 + ... + c_{2n}v_n + g_2(t) \\ \qquad\vdots \\ \frac{dv_n}{dt} = c_{n1}v_1 + ... + c_{nn}v_n + g_n(t) \end{cases} \tag{2.3}$$

can be written in the matrix form

$$dv/dt = Av + g \tag{2.4}$$

and can be solved using the formula

$$v(t) = e^{At}v(0) + \int_0^t e^{A(t-s)}g(s)ds \tag{2.5}$$

by Giuseppe Peano in 1887. Here,

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k. \tag{2.6}$$

This method relies on transforming a complicated one dimensional problem to a relatively simple higher dimensional problem, and then solving it by using the methods on calculus of one-variable. This idea serves as a basis of the spectral theory for self-adjoint normal operators on a Hilbert space, and mainly of the semigroup theory. To work in much greater generality and for the application on nonlinear operators, the semigroup theory only requires considering the exponential of an operator A. The one-parameter semigroup theory, especially, have direct applications in the theory of PDEs.

Let us consider the evolution of a physical system in time, by an IVP for (an ordinary or a partial) differential equation in the following form

$$\begin{cases} \frac{du(t)}{dt} = A(u(t)) \quad (t > 0), \\ u(0) = f. \end{cases} \tag{2.7}$$

The system is called an *Abstract Cauchy Problem* with a linear operator A on a Banach space V. Here, the function $u(t)$ describes the state of some physical system at time t, and $u(0) = f$ is given initial data. The time rate of change of $u(t)$ is given by some operator A of the state of the system $u(t)$.

To investigate well-posedness of problem (2.7), a dynamical system which is a family of operators $T(t)_{t\geq 0}$ on V satisfying the properties

$$\begin{cases} T(t+\tau) = T(t)T(\tau), \\ T(0) = I \end{cases} \tag{2.8}$$

is considered.

The first equation in (2.8) corresponds to the uniqueness of solution, and the second equation refers to the initial condition.

Roughly speaking, $T(t)_{t\geq 0}$ is the solution of Abstract Cauchy Problem defined by system (2.7). Conversely, the question of under which conditions the given semigroup corresponds to a differential equation is also reasonable.

2.2.1 C_0 Semigroup

This subsection presents the notion of a one-parameter strongly continuous semigroup of bounded linear operators on a Banach space V, which is called a (C_0) semigroup.

Definition 2.5. A family $T = \{T(t) : 0 < t < \infty\}$ of linear operators from V to V is called a (C_0) semigroup if it fulfils the following properties:

(i) $\|T(t)\| < \infty$

(ii) $T(t+s)g = T(t)T(s)g$, $\forall g \in V$, $\forall t, s \geq 0$,

(iii) $T(0)g = g$, $\forall g \in V$,

(iv) $t \to T(t)g$ *is continuous for* $t \geq 0$ *for each* $g \in V$.

Let T be a (C_0) semigroup. Let us define its generator (or infinitesimal generator) A by the equation

$$Ag = lim_{t \to 0} \frac{T(t)g - g}{t} \tag{2.9}$$

where g is in the domain of A if and only if this limit exists. Formally, the semigroup property suggests that $T(t) = e^A$ where $A = (d/dt)T(t)\mid_{t=0}$. This also suggests that the solution of (2.7) is given by

$$u(t) = T(t)g, \tag{2.10}$$

where T is the semigroup generated by A. The following result is well known.

Theorem 2.1. *(Well-posedness theorem). The IVP (2.7) (with the linear operator A) is "well-posed" if and only if A is the generator of a (Co) semigroup T. Then the unique solution of system (2.7) is given by $u(t) = T(t)g$ for g in the domain of A.*

Theorem 2.2. *(Hille-Yosida generation theorem). A linear operator A generates a (Co) contraction semigroup if and only if the domain of A is dense in Y and for each $\lambda > 0$,*

$$\lambda I - A \text{ maps the domain of } A \text{ onto } Y \tag{2.11}$$

and

$$\left\| (\lambda I - A)^{-1} g \right\| \leq \frac{1}{\lambda} \|g\|, \ \forall g \in Y \tag{2.12}$$

holds.

A result of the Hille-Yosida generation theorem is Stone's theorem:

Theorem 2.3. *Let A be a densely defined operator on a complex Hilbert space. Then A and −A both generate (Co) contraction semigroups if and only if A generates a (Co) group of unitary operators if and only if iA is self-adjoint.*

The important implications in Theorems 2.1, 2.2 and 2.3 are:

(i) a densely defined operator A satisfying (2.11), (2.12) generates a (Co) semigroup.

(ii) if A generates a (Co) semigroup T then the IVP (2.7) is well-posed and is governed by T.

2.3 Abstract Linear Spaces

Definition 2.6. Let V be a vector space on the scalar field $\mathbb{R}$. A function $L : V \to \mathbb{R}$ is called *linear functional (or linear form)* on V if $\forall \lambda, \mu \in \mathbb{R}$ and $x, y \in V$ implies

$$L(\lambda x + \mu y) = \lambda L(x) + \mu L(y). \tag{2.13}$$

Definition 2.7. Let V be a vector space defined over the scalar field $\mathbb{R}$. A function $a : V \times V \to \mathbb{R}$ is said to be a bilinear form on V, which is a linear map on each argument. In other words, $\forall \lambda, \mu \in \mathbb{R}$ and $x, y, z \in V$,

$$a(\lambda x + \mu y, z) = \lambda a(x, z) + \mu a(y, z), \tag{2.14}$$

$$a(x, \lambda y + \mu z) = \lambda a(x, y) + \mu a(x, z). \tag{2.15}$$

Definition 2.8. The bilinear form $a(\cdot,\cdot)$ is said to be *symmetric* if

$$a(y, x) = a(x, y), \text{ for all } x, y \in V, \tag{2.16}$$

and *positive-definite* if

$$a(x, x) > 0, \text{ for all } x \in V. \tag{2.17}$$

Definition 2.9. A symmetric and positive-definite bilinear form constitutes an *inner product* on a vector space V, and V with an inner product $(\cdot,\cdot)$ is said to be an *inner product space*. For all $x \in V$, the corresponding *norm* for the inner product is defined as

$$\|x\| = (x, x)^{1/2}. \tag{2.18}$$

Definition 2.10. An infinite sequence $\{x_m\}_{m=1}^{\infty}$ in V converge to $x \in V$ if

$$\|x_m - x\| \to 0 \text{ as } m \to \infty. \tag{2.19}$$

Definition 2.11. A sequence $\{x_m\}_{m=1}^{\infty}$ in V is said to be a *Cauchy sequence* if

$$\|x_m - x_n\| \to 0 \text{ as } m, n \to \infty. \tag{2.20}$$

Definition 2.12. An inner product space V is said to be complete if every Cauchy sequence converges in V.

Definition 2.13. A *Hilbert space* is an inner product space which is complete.

Definition 2.14. A *norm* is a function $\|.\| : V \to \mathbb{R}$ which satisfies the following properties

1. $\|x\| > 0, \forall x \in V, x \neq 0,$
2. $\|\lambda x\| = |\lambda| \|x\|, \forall \lambda \in \mathbb{R}, x \in V,$
3. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in V.$

Remark 2.1. A function is said to be a *seminorm* if the above conditions are satisfied with the exception that property *1.* is replaced by $|x| \geq 0$, $\forall x \in V$.

Definition 2.15. The vector space V with a norm $\|.\|$ is said to be a *normed vector space*. It is easily seen that an inner product space is a normed vector space, but the converse is not always true. A complete normed space is said to be a *Banach space*.

Definition 2.16. Let V and W be two Hilbert spaces. A linear operator $L : V \to W$ is *bounded* provided there exists a constant K satisfying

$$\|Lx\|_W \leq K \|x\|_V, \forall x \in V. \tag{2.21}$$

The norm is defined as

$$\|L\| = \sup_{x \in V \setminus \{0\}} \frac{\|Lx\|_W}{\|x\|_V} \tag{2.22}$$

for a linear bounded operator L. Hence,

$$\|Lx\|_W \leq \|L\| \|x\|_V, \tag{2.23}$$

where $\|L\|$ is the smallest constant K such that (2.21) holds.

The set of all bounded linear operators from V into W is denoted by $B(V, W)$.

Definition 2.17. Let $L : V \to W$ be a linear bounded operator in a Hilbert space H. Then *the Hilbert adjoint operator* of L is denoted by L^*, which satisfies

$$(Lx, y) = (x, L^* y). \tag{2.24}$$

Definition 2.18. L is *self adjoint* or *Hamiltonian* if $L^* = L$ or equivalently

$$(Lx, y) = (x, Ly). \tag{2.25}$$

Remark 2.2. A bounded linear operator $L : V \to W$ is continuous. That is, $x_j \to x$ in V implies $Lx_j \to Lx$ in V as $j \to \infty$ since

$$\left\|Lx_j - Lx\right\|_W = \left\|L(x_j - x)\right\|_W \leq \|L\| \left\|x_j - x\right\| \to 0, \text{ as } j \to \infty. \tag{2.26}$$

The converse is also true.

In the case $W = \mathbb{R}$, the operator becomes a linear functional.

Definition 2.19. The set which contains the bounded linear functionals defined on V is referred to as the *dual space* of V. The dual space is denoted as V'. The norm on V' is defined by

$$\|L\|_{V'} = \sup_{x \in V \setminus \{0\}} \frac{|L(x)|}{\|x\|_V}. \tag{2.27}$$

Note that V' is already a vector space.

Remark 2.3. A bilinear form $a(.,.)$ defined on V is bounded if there is a constant C_0 satisfying

$$|a(x,y)| \leq C_0 \|x\| \|y\|. \tag{2.28}$$

Theorem 2.4. *(Riesz representation theorem). Let L be a bounded linear functional defined on a Hilbert space V. Then there exists unique x in V satisfying*

$$L(y) = (y,x), \forall x \in V. \tag{2.29}$$

Moreover,

$$\|L\|_{V'} = \|x\|_V. \tag{2.30}$$

Using this result one can determine the linear functionals L in V' associated with $x \in V$. Here, by means of Hilbert space, V' is equivalent to V.

Definition 2.20. A bilinear form $a(.,.)$ is called *coercive* if there exists a positive constant α_0 satisfying the inequality

$$a(x,x) \geq \alpha_0 \|x\|_V^2. \tag{2.31}$$

Remark 2.4. One may associate the bounded linear functional L defined in (2.29) to a coercive, symmetric bilinear form $a(.,.)$ and solve the equation for x in a Hilbert space V such that

$$a(x,y) = L(y), \forall y \in V. \tag{2.32}$$

Then the Riesz representation theorem implies existence of a unique solution $x \in V$ for each $L \in V'$. Furthermore, setting $y = x$ in (2.29) yields

$$\alpha_0 \|x\|_V^2 \leq a(x,x) = L(x) \leq \|L\|_{V'} \|x\|_V. \tag{2.33}$$

Cancelling one factor $\|x\|_V$ gives

$$\|x\|_V \leq K \|L\|_{V'} \tag{2.34}$$

where $K = 1/\alpha_0$ is a constant. This procedure serves as an example of *energy estimate*. For a symmetric bilinear form $a(.,.)$ which satisfies (2.28) and (2.31), the energy norm $\|y\|_e$ is defined as

$$\|y\|_e = a(y,y)^{1/2}, \text{ for } y \in V. \tag{2.35}$$

It is convenient to express the solution of (2.32) by constructing a *minimization problem*.

Theorem 2.5. *Let $a(\cdot,\cdot)$ be positive definite, symmetric bilinearform, and L is a bounded linear form defined on a Hilbert space V. Then $x \in V$ implies (2.31) if and only if*

$$E(x) \leq E(y),\ \forall y \in V, \text{ for } E(y) = \frac{1}{2}a(y,y) - L(y). \tag{2.36}$$

Here, $x \in V$ satisfies (2.31) iff x minimizes energy functional E.

Definition 2.21. The method for considering the minimization problem, which is achieved by modifying the argument of functional E about the given vector x is said to be a *variational method*. Equation (2.32) is said to be the *variational equation of E*.

The subsequent theorem extends the results of Riesz representation theorem for bilinear nonsymmetric forms.

Theorem 2.6. *(Lax-Milgram Lemma). Let $a(\cdot,\cdot)$ be a bounded, coercive bilinear form, and L be a bounded linear form in a Hilbert space V.Then there is a unique vector $x \in V$ which satisfies (2.32). Moreover, the energy estimate (2.34) holds.*

Note that there is no characterization by energy minimization for the solution in the unsymmetric case.

2.4 Function Spaces

2.4.1 The Space of Continuous Functions

Let $\Omega \subset \mathbf{R}^n$. For $k \in \mathbf{Z}^+$, $C^k(\Omega)$ the vector space which consists of all functions g having all partial derivatives $D^\alpha g$ for $|\alpha| \leq k$, which of all are continuous on Ω. We denote

$$C^0(\Omega) = C(\Omega) \tag{2.37}$$

$$C^\infty(\Omega) = \cap_{m=0}^{\infty} C^m(\Omega). \tag{2.38}$$

The norm for $C(\Omega)$ is denoted by

$$\|g\|_{C(\Omega)} = \sup_{x\in\Omega} |g(x)|\,. \tag{2.39}$$

The norm for $C^k(\Omega)$ is denoted by

$$\|g\|_{C^k(\Omega)} = \sum_{n=0}^{k} \sup_{x\in\Omega} \left|g^{(n)}(x)\right|. \tag{2.40}$$

Definition 2.22. *(compact support)* Let $\Omega \subset R^n$ be a nonempty domain. A function u is said to have a *compact support* if it vanishes in the exterior of its domain Ω. That is

$$supp\ u = \overline{\{x : u(x) \neq 0\}}. \tag{2.41}$$

In some cases, a function vanishes near the boundary of its domain.

Then, with the aid of the above definition, we can define the function space $C_0^k(\Omega)$. It represents the space of functions which has compact support in Ω.

Remark 2.5. The space of continuous functions does not constitute a Hilbert space since the supremum-norm can not be associated with an inner product as in (2.18).

2.4.2 Integrability, the Lebesgue Spaces

In the previous subsection we have stated that $C(\Omega)$ is not a Hilbert space, then we should study integrals of functions $g = g(x)$ in Ω and these are more convenient than functions in $C(\Omega)$. Let us introduce the *Lebesgue integral*

$$I_\Omega(g) = \int_\Omega g(x)dx, \tag{2.42}$$

for a nonnegative measurable function g. This integral could be finite or infinite, and it coincides with the *Riemann integral* when $g \in C(\Omega)$.

Let us define the Lebesgue spaces

$$L^p(\Omega) = \left\{ g : \left(\int_\Omega |g(x)|^p\, dx \right)^{1/p} < \infty \text{ and } g \text{ is measurable} \right\} \tag{2.43}$$

for $1 \le p < \infty$ and

$$L^\infty(\Omega) = \left\{ g : \sup_{x\in\Omega} |g(x)| < \infty \text{ and } g \text{ is measurable} \right\}. \tag{2.44}$$

The norm defined on $L^p(\Omega)$ is

$$\|g\|_{L^p(\Omega)} = \left(\int_\Omega |g|^p \, dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty. \tag{2.45}$$

Theorem 2.7. *(Riesz-Fischer). The space $L^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.*

The norm on $L^\infty(\Omega)$ is defined by

$$\|g\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |g(x)|. \tag{2.46}$$

Here, the function sup is the supremum, which disregards the values in null sets.

$L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(g,h) = \int_\Omega g(x)h(x)dx, \tag{2.47}$$

and the corresponding norm

$$\|g\|_{L^2(\Omega)} = \left(\int_\Omega |g|^2 \, dx \right)^{1/2}, \forall g,h \in L^2(\Omega). \tag{2.48}$$

Definition 2.23. Let $M \subset X$. The set M is dense in X if for any $x \in M$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in M satisfying

$$\|x_n - x\| \to 0 \text{ as } n \to \infty. \tag{2.49}$$

Remark 2.6. $C(\Omega)$ is dense in $L^p(\Omega)$ for $1 < p < \infty$, which means any function in $L^p(\Omega)$ can be approximated closely by using the elements of $C(\Omega)$.

2.4.3 The Weak Derivative, the Sobolev Spaces

Let us introduce certain Hilbert spaces which are commonly used in the theory of PDEs. These spaces are composed of some functions which belong to $L^2(\Omega)$, and also their partial derivatives up to a certain order. To construct the spaces, let us introduce the general notion of partial derivatives.

Let $\Omega \subset \mathbf{R}^n$ and $g \in C^1(\Omega)$. Let us consider an integral using integration by parts

$$\int_\Omega \varphi \frac{\partial g}{\partial x_i} dx = -\int_\Omega g \frac{\partial \varphi}{\partial x_i} dx, \ \forall \varphi \in C_0^1(\Omega). \tag{2.50}$$

Here, if we let $g \in L_2(\Omega)$, the partial derivative $\frac{\partial g}{\partial x_i}$ does not necessarily mean to be classical derivative. Let us consider the term $\frac{\partial g}{\partial x_i}$ as a linear functional

$$L(\varphi) = \frac{\partial g}{\partial x_i}(\varphi) = -\int_\Omega g \frac{\partial \varphi}{\partial x_i} dx, \; \forall \varphi \in C_0^1(\Omega). \tag{2.51}$$

Here, the functional $L(\varphi)$ is called *weak (generalized) derivative* of g. If this functional is bounded, then by Riesz representation theorem the weak derivative is unique and it belongs to $L_2(\Omega)$. The weak derivative coincides with the classical derivative if $g \in C^1(\Omega)$.

In a similar manner with the classical partial derivative, let us introduce the weak partial derivative of order a, $D^a g$ as a linear functional of the form

$$D^a g(\varphi) = (-1)^{|a|} \int_\Omega g D^a \varphi dx \;, \; \forall \varphi \in C_0^{|a|}(\Omega). \tag{2.52}$$

If the functional is bounded in $L_2(\Omega)$, then by Riesz representation theorem there is a unique $D^a f \in L_2(\Omega)$ satisfying

$$(D^a g, \varphi) = (-1)^{|a|}(g, D^a \varphi), \; \forall \varphi \in C_0^{|a|}(\Omega). \tag{2.53}$$

Now we can define the space $H^k(\Omega)$, $k \geq 0$. The space is constructed by the functions whose partial derivatives, in the weak sense, of order $\leq k$ belong to $L^2(\Omega)$. That is,

$$H^k(\Omega) = \left\{ u \in L^2(\Omega) : D^a u \in L^2(\Omega), |a| \leq k \right\}. \tag{2.54}$$

The space is equipped with the inner product and the norm

$$(u, v) = \sum_{|a| \leq k} \int_\Omega D^a u(x) D^a v(x) dx, \tag{2.55}$$

$$\|u\|_{H^k(\Omega)} = (u, u)^{1/2}, \; \forall u, v \in H^k(\Omega). \tag{2.56}$$

Derived from $H^k(\Omega)$, of prime importance is attributed to the spaces

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : D^1 u \in L^2(\Omega), 1 \leq i \leq n \right\}, \tag{2.57}$$

and

$$H_0^1(\Omega) = \text{ the closure of } C_0^\infty(\Omega) \text{ on } H^1(\Omega) \tag{2.58}$$

with its dual space $H^{-1}(\Omega)$. These are Hilbert spaces equipped with the inner product

$$((u,v)) = \sum_{i=1}^{n} \int_{\Omega} D_i u(x) D_i v(x) dx, \tag{2.59}$$

and the corresponding norm

$$\|u\| = ((u,u))^{1/2}, \text{ for all } u,v \in H_0^1(\Omega). \tag{2.60}$$

In a similar manner, let us define the Banach space $W_p^k(\Omega)$ induced by the norm

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |D^{\alpha} u(x)|^p \, dx \right)^{1/p}, 1 \leq p < \infty. \tag{2.61}$$

Note that $W_2^k(\Omega) = H^k(\Omega)$ for $p = 2$, and $\|u\|_{W_{\infty}^k(\Omega)} = \|u\|_{C^k(\Omega)}$ for any $u \in C^k(\Omega)$.

Remark 2.7. The space $C^m(\Omega)$ is dense in $H^k(\Omega)$ for any $m \geq k$ and for sufficiently smooth boundary $\partial\Omega$. This consequence enables us to carry out the proofs for $H^k(\Omega)$ in $C^m(\Omega)$ easily, and then extending the result to the functions in $H^k(\Omega)$ by density property.

2.5 Variational Method

We have briefly mentioned the term "energy" in the first section. However, we will consider this method in detail in this section, because of its importance.

The energy method or the variational approach, is an effective tool in theory of PDEs. This method is useful in solving nonlinear problems with complicated structure and for linear or semilinear problems which don't have a corresponding mild formulation due to low regularity of coefficients. It is a modern fundamental approach to boundary value problems for PDEs. The method is originated from theory of linear PDEs, but different approaches are also valid.

The term “variational” represents small changes in functions or functionals, and this refer to the characterization of extremum (minimum, maximum, saddle point) for physical systems. The functional stands as a physical quantity in the system, and it appears as an integral in the given space. This kind of characterization is called a “variational principle” which leads to the Euler-Lagrange equation. This equation optimizes a functional, that is, a boundary value problem is considered as an optimization problem. It is evident since the main issue in real world problems is to optimize some physical quantities such as time, distance, energy, etc. The mathematical branch which deals with these optimization problems is called the *calculus of variations*.

The variational method is applied by means of integral inequalities. It is originally based on the variations of the classical energy and the kinetic energy of the fluid motion. The new functionals are similar with the Lyapunov method in PDEs and it is appeared in the literature as generalized energy methods. It is related to a-priori estimates, and this term has a physical meaning as “bound of an energy”. This method is based on compactness of bounded sets in Banach space in weak topologies, the weak formulation and some results in Sobolev spaces.

2.5.1 Fundamental Tools of Energy Method

A-priori estimates

A priori estimate or a priori bound is an estimate which determines the size of solution for a given problem. “A priori” is a Latin word that means “from before”, which refers to the fact that the estimate for the solution is obtained before the knowledge of existence of solution. That is, the estimate can be obtained by using the form of the equation, without knowledge of the exact solution. Finding a priori estimates is an essential issue encountered with nonlinear problems. A priori estimates are of great importance in the proof for the existence of solution. In this thesis’ concept, a priori estimate means an estimate which guarantees that all positive solutions of the related problem are bounded, by some positive constant.

Gronwall’s inequality in differential form

Let $u(t)$ be an absolutely continuous and nonnegative function defined on the interval $[0,T]$, satisfying the differential inequality almost everywhere (a.e.)

$$u'(t) \leq a(t)u(t) + b(t), \tag{2.62}$$

for summable nonnegative functions $a(t), b(t)$ on$[0,T]$. Then the following inequality holds: [85]

$$u(t) \leq e^{\int_0^t a(s)ds}\left[u(0) + \int_0^t b(s)ds\right], \forall t \in [0,T]. \tag{2.63}$$

Gronwall’s inequality in integral form.

Let $u(t)$ be a summable, nonnegative function defined on the interval $[0,T]$, satisfying the integral inequality

$$u(t) \leq K_1 \int_0^t u(s)ds + K_2, \tag{2.64}$$

where K_1, K_2 are constants. Then the following inequality holds: [85]

$$u(t) \leq K_2\left(1 + K_1 t e^{K_1 t}\right), \text{ for a.e } t \in [0,T]. \quad (2.65)$$

Discrete Gronwall Lemma

Let us assume that the mesh functions $\{u_n\}$ and $\{f_n\}$ satisfy the following statement

$$\frac{u_n - u_{n-1}}{\tau} \leq A u_n u_n + B u_{n-1} + f_n, \text{ for n=1,2,...,N,} \quad (2.66)$$

then we have

$$\max_{1\leq n\leq N} u_n \leq \left(u_0 + \tau \sum_{k=1}^{n} f_n\right) e^{2(A+B)T}, \quad (2.67)$$

for $0 \leq t \leq T$, $n = 1,2,...,N$ and nonnegative constants A, B satisfying $(A+B)\tau \leq \frac{N-1}{2N}$ [28], [94], [95].

Remark 2.8. Modifying the Discrete Gronwall's Inequality given above, we can obtain a similar statement for the central difference. By this inequality, we have

$$\frac{u_n - u_{n-1}}{\tau} \leq A_0 u_n u_n + B_0 u_{n-1} + f_n \implies \max_{1\leq n\leq N} u_n \leq \left(u_0 + \tau \sum_{k=1}^{n} f_n\right) e^{2(A+B)T}, \quad (2.68)$$

$$\frac{u_{n+1} - u_n}{\tau} \leq A_1 u_{n+1} + B_1 u_n + f_{n+1} \implies \max_{1\leq n\leq N} u_{n+1} \leq \left(u_1 + \tau \sum_{k=1}^{n+1} f_{n+1}\right) e^{2(A+B)T}. \quad (2.69)$$

Adding this two statement side by side, and dividing both sides by 2, we obtain

$$\frac{u_{n+1} - u_{n-1}}{2\tau} \leq \frac{A_1}{2} u_{n+1} + \frac{(A_0 + B_1)}{2} u_n + \frac{B_0}{2} u_{n-1} + \frac{1}{2}(f_n + f_{n+1})$$

$$\implies \max_{1\leq n\leq N}(u_n + u_{n+1}) \leq \left(u_0 + \tau \sum_{k=1}^{n} f_n\right) e^{(A_0+B_0)T} + \left(u_1 + \tau \sum_{k=1}^{n+1} f_{n+1}\right) e^{(A_1+B_1)T}$$

$$\leq e^{C_1} u_0 + e^{C_2} u_1 + 2\tau e^{C_3} \sum_{k=1}^{n} f_n + \tau e^{C_4} f_{n+1}$$

$$\leq K_1 u_0 + K_2 u_1 + 2\tau K_3 \sum_{k=1}^{n} f_n + \tau K_4 f_{n+1}. \quad (2.70)$$

Thus,

$$\frac{u_{n+1} - u_{n-1}}{2\tau} \leq k_1 u_{n+1} + k_2 u_n + k_3 u_{n-1} + \frac{1}{2}(f_n + f_{n+1})$$

$$\Longrightarrow (u_n + u_{n+1}) \leq K_1 u_0 + K_2 u_1 + \tau K_3 \sum_{k=1}^{n} f_n + \tau K_4 f_{n+1}, \tag{2.71}$$

where

$$k_1 = \frac{A_1}{2}, k_2 = \frac{(A_0 + B_1)}{2}, k_3 = \frac{B_0}{2}. \tag{2.72}$$

Sobolev-Poincaré's Inequality

Let $\Omega \subset \mathbb{R}^n$ be an open set, $2 \leq q < \infty$ $(n = 1, 2)$ and $2 \leq q \leq \frac{2n}{n-2}$ $(n \geq 3)$. Then for $u \in H_0^1(\Omega)$ we have [88]

$$\|u\|_q \leq c(\Omega, p) \|\nabla u\|_2 . \tag{2.73}$$

Poincaré's Inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, then there exists constant $K = K(\Omega)$ satisfying the inequality [84]

$$\|u\| \leq K(\Omega) \|\nabla u\| , \text{ for all } u \in H_0^1(\Omega). \tag{2.74}$$

Discrete Poincaré-Friedrichs Inequality

For any mesh function $\{u_k : k = 1, ..., m\}$ there exists a constant C satisfying the inequality [96]

$$a(u_k, u_k) \geq \|\nabla u_k\|^2 \geq C \|u_k\|^2 . \tag{2.75}$$

Discrete Green's Identity

For any two mesh functions $\{u_k : k = 1, ..., m\}$ and $\{v_k : k = 1, ..., m\}$, the following identity is satisfied: [94]

$$\sum_{i=1}^{m} u_k(-\Delta v_k)\tau = \sum_{i=1}^{m} \nabla u_k \nabla v_k \tau + u_0 \nabla v_0 - v_0 \nabla u_0. \tag{2.76}$$

The ε-Young inequality

For any two mesh functions $\{u_k : k = 1, ..., m\}$ and $\{v_k : k = 1, ..., m\}$, the following inequality is satisfied: [85]

$$(u_{k+1}, v_{k+1}) \leq \frac{\|u_{k+1}\|^2}{2\varepsilon} + \frac{\varepsilon}{2} \|v_{k+1}\|^2 , \forall \varepsilon > 0. \tag{2.77}$$

2.5.2 Density Theorems

Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^m for $m \geq 1, 1 \leq p < \infty$, then

$$C^m(\bar{\Omega}) \text{ is dense in } W^{m,p}(\Omega). \tag{2.78}$$

The result is valid even under weaker regularity of Ω, particularly the validation occurs whenever there is a linear continuous prolongation operator

$$\Pi \in \mathscr{L}\left(W^{m,p}(\Omega), W^{m,p}(\mathbb{R}^n)\right), (\Pi u x) = u(x) \tag{2.79}$$

for almost all x in Ω. From that it follows

$$W^{m,p} \text{ is dense in } W^{m-1,p}(\Omega), \tag{2.80}$$

$$H^m \text{ is dense in } H^{m-1}(\Omega), \tag{2.81}$$

for sufficiently regular Ω [11].

2.5.3 Sobolev Embedding Theorem

Definition 2.24. [85] Let U, V be two normed spaces satisfying

i) $U \subset V$,

ii) $\forall x \in U$ there is a constant c independent of u such that

$$\|x\|_V \leq c \|x\|_U. \tag{2.82}$$

Then U is said to be embedded in V, denoted by $U \hookrightarrow V$.

Theorem 2.8. *[88] Let $\Omega \subset \mathbb{R}^n$ bean open set of class C^{m+1}, $p \geq 1$ be a real number, and $m \geq 1$ be an integer. Then*

i) if $mp > n$, then

$$W^{j+m,p}(\Omega) \hookrightarrow C_b^j(\Omega) \tag{2.83}$$

ii) if $mp = n$, then

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \text{ for } p \leq q < \infty, \tag{2.84}$$

and if $j = 0$, then

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \text{ for } p \leq q < \infty, \tag{2.85}$$

iii) if $mp < n$, then

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \text{ for } p \le q < p_0, \tag{2.86}$$

and if $j = 0$, then

$$W^{m,p}(\Omega) \hookrightarrow L^{q}(\Omega), \text{ for } p \le q < p_0, \tag{2.87}$$

where

$$p_0 = \begin{cases} \frac{np}{n-mp}, & n > mp \\ \infty, & n \le mp. \end{cases} \tag{2.88}$$

Note that the above embeddings also holds for $W_0^{m,p}(\Omega)$ without any restriction on Ω.

2.5.4 Compactness Theorems

Definition 2.25. [85] Let the embedding $U \hookrightarrow V$ be given. U is said to be *compactly embedded* in V, denoted by

$$U \subset\subset V \tag{2.89}$$

if for every bounded sequence in U there exists a convergent subsequence in V.

Theorem 2.9. *[11] Let Ω be a bounded set of class C^1. Then the embedding*

$$W^{1,p}(\Omega) \subset L^{q_1}(\Omega) \tag{2.90}$$

is compact,

i)if $p \ge n$, for any $q_1 \in (1, \infty)$,

ii) if $1 \le p < n$, for any $q_1 \in (1, q)$ satisfying $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$.

If $p > n$, the embedding

$$W^{1,p}(\Omega) \subset C^{0,\alpha_1}(\Omega) \tag{2.91}$$

is compact for all $\alpha_1 < \alpha = 1 - \frac{n}{p}$.

The embeddings

$$W_0^{1,p}(\Omega) \subset L^{q_1}(\Omega), \text{ for } p \le n, \tag{2.92}$$

$$W_0^{1,p}(\Omega) \subset C^{0,\alpha_1}(\Omega), \text{ for } p \le n, \tag{2.93}$$

are also compact for any open bounded Ω, with the same values of q_1, α_1.

Theorem 2.10. *[6, 85, 88] (Rellich-Kondrachov Compactness Theorem) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with boundary C^1, and let $1 \le p < n$. Then $W^{1,p}(\Omega)$ is compactly* *embedded into $L^q(\Omega)$, and is denoted by*

$$W^{1,p}(\Omega) \subset\subset W^{1,p}(\Omega) \tag{2.94}$$

for each $1 \leq q < p_*$, where $p_* = \frac{np}{n-p}$.

The following theorem enables us to obtain strong convergence results for the abstract evolution problems.

Theorem 2.11. *[97, 98] (Aubin-Lions-Simon) Let $U_0 \subset U_1 \subset U_2$ be Banach spaces. Let us assume the embedding of U_1 into U_2 is continuous and the embedding of U_0 into U_1 is compact. For $1 \leq p, q \leq \infty$ and $T > 0$ we define the space*

$$E_{p,q} = \{u \in L^p(0,T;U_0), \frac{du}{dt} \in L^q(0,T;U_2)\} \tag{2.95}$$

which have the compactness results:

i) the embedding of $E_{p,q}$ into $L^p(0,T;U_1)$ is compact for $p < \infty$,

ii) the embedding of $E_{p,q}$ into $C^0(0,T;U_1)$ is compact for $p = \infty$, $q > 1$.

3
MAIN RESULTS

3.1 Problem Settings

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ be two Hilbert spaces, which are equipped with the following inner products and norms

$$(\xi,\eta) = \int\limits_{\Omega} \xi(x)\eta(x)dx,\ \ \|\xi\| = (\xi,\xi)^{1/2}, \forall \eta,\xi \in L^2(\Omega), \tag{3.1}$$

$$((\xi,\eta)) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial \xi(x)}{\partial x_i}\frac{\partial \eta(x)}{\partial x_i} dx, \|\xi\|_V = ((\xi,\xi))^{1/2},\ \forall \eta,\xi \in H_0^1(\Omega). \tag{3.2}$$

Let us define the dual spaces of V and H as V' and H', respectively. Here, the pair (V,H) of the Hilbert spaces forms a Gelfand triple, which is denoted by $V \hookrightarrow H \equiv H' \hookrightarrow V'$ where $V' = H^{-1}(\Omega)$. The embeddings $V \subset H$, $H \subset V'$ are continuous, dense, compact. The unique solvability results are presented in the setting of the triple space. The bilinear form

$$a(\eta,\varphi) = \int_{\Omega} \nabla\eta.\nabla\varphi dx = (\nabla\eta,\nabla\varphi) = ((\eta,\varphi)), \forall \eta,\varphi \in V = H_0^1(\Omega), \tag{3.3}$$

will be used in variational formulation. This form is bounded, symmetric on $V \times V = H_0^1(\Omega)^2$ and coercive, that is,

$$a(\eta,\eta) \geq \|\eta\|_V^2, \forall \eta \in V. \tag{3.4}$$

The form is associated with the operator $A = -\Delta$ defined by

$$(A\eta,\varphi) = a(\eta,\varphi) \tag{3.5}$$

where $A : V \to V'$ is an isomorphism. It is an unbounded self-adjoint operator with dense domain $D(A) = \{\eta \in V |\, A\eta \in H\}$ in V and in H. In the present study we investigate unique solvability of the following problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha_{11}\frac{\partial u}{\partial t} + \alpha_{12}\frac{\partial v}{\partial t} + \beta_1 Au + \gamma_1 sin\left(\delta_{11}u + \delta_{12}v\right) \\ +\rho_{11}u + \rho_{12}v = f, 0 < t < 1, \\ \frac{\partial^2 v}{\partial t^2} + \alpha_{21}\frac{\partial u}{\partial t} + \alpha_{22}\frac{\partial v}{\partial t} + \beta_2 Av + \gamma_2 \sin\left(\delta_{21}u + \delta_{22}v\right) \\ +\rho_{21}u + \rho_{22}v = g, 0 < t < 1, \\ u(0) = u_0 \in V, \frac{du}{dt}(0) = u_0' \in H, \\ v(0) = v_0 \in V, \frac{dv}{dt}(0) = v_0' \in H. \end{cases} \tag{3.6}$$

We consider system (3.6) in the following vector form

$$\begin{cases} \mathbf{w}'' + \alpha\mathbf{w}' + \beta\mathbf{B}\mathbf{w} + \gamma\sin\delta\mathbf{w} + \rho\mathbf{w} = \mathbf{f}, 0 < t < T, \\ \mathbf{w}(\mathbf{0}) = \mathbf{w}_0, \mathbf{w}'(\mathbf{0}) = \mathbf{w}_0', \end{cases} \tag{3.7}$$

with

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \ \mathbf{w}' = \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}, \ \mathbf{w}'' = \begin{bmatrix} \frac{d^2u}{dt^2} \\ \frac{d^2v}{dt^2} \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} f \\ g \end{bmatrix}, \ \sin\mathbf{w} = \begin{bmatrix} \sin u \\ \sin v \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}, \ \beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}, \ \delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix},$$

$$\gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \ \rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix},$$

$$\mathbf{w}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \ \mathbf{w}_0' = \begin{bmatrix} u_0' \\ v_0' \end{bmatrix}.$$

The norm $|\delta|$ of a 2×2 matrix is defined by

$$\sum_{i,j=1,2} \left|\delta_{ij}\right|. \tag{3.8}$$

Let us introduce the product spaces of functions by $\mathscr{V}=V\times V$ and $\mathscr{H}=H\times H$, equipped with the inner products

$$((\eta,\xi))=((\eta_1,\xi_1))+((\eta_2,\xi_2)),\eta=[\eta_1,\eta_2]^T,\xi=[\xi_1,\xi_2]^T\in\mathscr{V}, \tag{3.9}$$

$$(\eta,\xi)=(\eta_1,\xi_1)+(\eta_2,\xi_2),\eta=[\eta_1,\eta_2]^T,\xi=[\xi_1,\xi_2]^T\in\mathscr{H}, \tag{3.10}$$

respectively. Here $[.,.]^T$ is the transpose of $[.,.]$. The dual pairing between $\mathscr{V}'$ and $\mathscr{V}$ are

$$\langle\eta,\xi\rangle=\langle\eta_1,\xi_1\rangle+\langle\eta_2,\xi_2\rangle,\eta=[\eta_1,\eta_2]^T\in\mathscr{V}',\xi=[\xi_1,\xi_2]^T\in\mathscr{V}, \tag{3.11}$$

where $\mathscr{V}'=V'\times V'$ is the dual space of $\mathscr{V}$.

Let us assume that the operator A has a square root D satisfying $D=\sqrt{-\Delta}$. D is a self adjoint, positive definite operator and it generates a C_0 semigroup in problem (3.7). Therefore, the operator matrix $\mathbf{B}$ in (3.7) which contains operator entries A is also self-adjoint, positive definite with a dense domain $\mathscr{D}(\mathbf{B})=D(A)\times D(A)$ in $\mathscr{V}$ and in $\mathscr{H}$. Thus, $\mathbf{B}$ generates a C_0 semigroup [11, 86, 87].

By the embeddings $V\hookrightarrow H\hookrightarrow V'$, the pair $(\mathscr{V},\mathscr{H})$ is a Gelfand triple denoted by $\mathscr{V}\hookrightarrow\mathscr{H}\hookrightarrow\mathscr{V}'$. Norms on $\mathscr{V}$ and $\mathscr{H}$ are denoted as $\|\xi\|$ and $|\xi|$, respectively. The weak solvability of (3.7) is stated in the following form.

Definition 3.1. [9] A function $\mathbf{w}$ is a weak (variational, or generalized) solution for problem (3.7) if

$$\mathbf{w}\in\mathbf{W}(0,T)=W(0,T)\times W(0,T) \tag{3.12}$$

and $\mathbf{w}$ satisfies the variational (weak) formulation

$$\begin{cases}\langle\mathbf{w}''(.),\omega\rangle+(\alpha\mathbf{w}'(.),\omega)+((\beta\mathbf{w}(.),\omega))\\ +(\gamma\sin\delta\mathbf{w}(.),\omega)+(\rho\mathbf{w}(.),\omega)=(\mathbf{f}(.),\omega),\\ \mathbf{w}(\mathbf{0})=\mathbf{w}_0,\mathbf{w}'(\mathbf{0})=\mathbf{w}_0'\end{cases} \tag{3.13}$$

for all $\omega\in\mathscr{V}$. Here, the solution space is

$$W(0,T)=\left\{\varphi|\varphi\in L^2(0,T;V),\varphi'\in L^2(0,T;H),\varphi''\in L^2\left(0,T;V'\right)\right\}. \tag{3.14}$$

Here, ω is a test function, which belongs to a dense topological vector subspace of $\mathscr{H}$, so that the dual space of test functions enhances $\mathscr{V}$ to a larger topological vector space whose elements can be considered as generalized eigenvectors for the continuous spectrum of unbounded operators.

The following theorem states the continuous dependence of weak solutions for (1.26), which will be used in proof of uniqueness.

Theorem 3.1. [9] *Suppose that assumption (1.24) holds. Let* $\mathbf{w}_A = [u_A, v_A]^t$ *(resp.,* $\mathbf{w}_B = [u_B, v_B]^t$ *) be the weak solution of (3.7) with initial values* $(\mathbf{w}_{A0}, \mathbf{w}_{A1}) \in \mathcal{V} \times \mathcal{H}$ *(resp.,* $(\mathbf{w}_{B0}, \mathbf{w}_{B1}) \in \mathcal{V} \times \mathcal{H}$*) and* $\mathbf{f}_A \in L^2(0,T;\mathcal{H})$ $\left(\text{resp., } \mathbf{f}_B \in L^2(0,T;\mathcal{H})\right)$*. Then there is a constant* $K > 0$ *which depends only on* T *and the constants* $\alpha, \beta, \gamma, \delta$ *such that, for all* t *in* $[0,T]$ *the inequality*

$$\|\mathbf{w}_A(t) - \mathbf{w}_B(t)\|^2 + \left|\mathbf{w}_A'(t) - \mathbf{w}_B'(t)\right|^2$$

$$\leq K\left(\|\mathbf{w}_{A0} - \mathbf{w}_{B0}\|^2 + |\mathbf{w}_{A1} - \mathbf{w}_{B1}|^2 + \int_0^t |\mathbf{f}_A(\sigma) - \mathbf{f}_B(\sigma)|^2 \, d\sigma \right) \tag{3.15}$$

holds.

Let us consider system (3.7) in difference form. Using the family of grid points Ω_h defined in (1.29), the system can be written as

$$\begin{cases} \tau^{-2}(\mathbf{w}_{k+1} - 2\mathbf{w}_k + \mathbf{w}_{k-1}) + \alpha(2\tau)^{-1}(\mathbf{w}_{k+1} - \mathbf{w}_{k-1}) \\ +\beta\mathbf{B}\mathbf{w}_k + \gamma \sin\delta\mathbf{w}_k + \rho\mathbf{w}_k = \mathbf{f}_k, \ 0 < t < T, \\ \mathbf{w}_0 = \varphi, \tau^{-1}(\mathbf{w}_1 - \mathbf{w}_0) = \xi \end{cases} \tag{3.16}$$

where

$$\mathbf{w}_k = \begin{bmatrix} u_k \\ v_k \end{bmatrix}, \ \sin\mathbf{w_k} = \begin{bmatrix} \sin u_k \\ \sin v_k \end{bmatrix}, \ \mathbf{f_k} = \begin{bmatrix} f_k \\ g_k \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ \alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix},$$

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}, \ \gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix},$$

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}, \rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix},$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}, \ \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},$$

and

$$f_k \to f \ \text{and} \ g_k \to g \ \text{in} \ L^2(0,T;\mathscr{V}). \tag{3.17}$$

Here, A is an unbounded self-adjoint positive definite operator with domain

$$D(A) = \left\{ u_k : -\Delta u_k \in L^2(0,T;H), \ u_0 = u_M = 0 \right\}. \tag{3.18}$$

We will obtain unique solvability results for system of difference equation (3.16) in the weak sense by constructing variational formulation of the equation. We use some strong convergence properties of the sequences which are obtained by compactness theorems. The unique weak solvability of (3.16) by means of difference scheme (1.28) is presented in the next sections.

3.2 Unique Solvability of First Order of Accuracy Difference Scheme

In this section, theoretical statements on weak approximate solution to (3.16) is established by the unconditionally stable difference scheme (1.28). Applying variational formulation, it will be shown that difference problem (3.16) converges to a unique weak solution.

Let us consider the variational formulation of (1.28) in the following form

$$\begin{cases} (u_{k+1},\bar{u})+(u_{k-1},\bar{u})-2(u_k,\bar{u}) \\ +\frac{\alpha_{11}}{2}\tau\left[(u_{k+1},\bar{u})-(u_{k-1},\bar{u})\right]+\frac{\alpha_{12}}{2}\tau\left[(u_{k+1},\bar{u})-(u_{k-1},\bar{u})\right] \\ +\tau^2\beta_1(Au_{k+1},\bar{u})+\tau^2\gamma_1(\sin(\delta_{11}u_k+\delta_{12}v_k),\bar{u}) \\ +\tau^2\rho_{11}(u_k,\bar{u})+\tau^2\rho_{12}(v_k,\bar{u})=\tau^2(f_k,\bar{u}), \\ (v_{k+1},\bar{v})+(v_{k-1},\bar{v})-2(v_k,\bar{v}) \\ +\frac{\alpha_{21}}{2}\tau\left[(u_{k+1},\bar{v})-(u_{k-1},\bar{v})\right]+\frac{\alpha_{22}}{2}\tau\left[(v_{k+1},\bar{v})-(v_{k-1},\bar{v})\right] \\ +\beta_2\tau^2(Av_{k+1},\bar{v})+\gamma_2\tau^2(\sin(\delta_{21}u_k+\delta_{22}v_k),\bar{v}) \\ +\rho_{21}\tau^2(u_k,\bar{v})+\rho_{22}\tau^2(v_k,\bar{v})=\tau^2(g_k,\bar{v}) \end{cases} \tag{3.19}$$

where $\bar{u}$ and $\bar{v}$ are test functions in V.

Definition 3.2. The set of mesh functions $\{u_k\}$ and $\{v_k\}$ are said to be the approximate weak solutions of (1.28) if $u_k, v_k \in V$ satisfy (3.19).

Theorem 3.2. *Suppose that assumptions (1.24), (3.17) and*

$$\frac{\tau^2}{4}|\gamma||\delta|+\frac{\tau}{8}|\alpha|\le\frac{N-1}{2N} \tag{3.20}$$

$$-\frac{\tau}{4}|\alpha|+\frac{9}{4}\tau^2|\gamma||\delta|+\tau^2(-|\beta|-|\rho|+2)\le\frac{N-1}{2N} \tag{3.21}$$

for $k=1,2,...,N$ are satisfied, then there exists a positive constant K such that

$$\max_{1\le k\le N}\left(\|\mathbf{w}_{k-1}\|^2+\|\mathbf{w}_k\|^2+\|\mathbf{w}_{k+1}\|^2\right)\le K \tag{3.22}$$

where K does not depend on the grid parameters τ and h, for all $k\in\mathbb{N}$.

Throughout this paper, K represents a generic constant, having possibly different values at different occurrences.

Proof. Setting $\bar{u} = u_{k+1}$ and $\bar{v} = v_{k+1}$ in (3.19), we have

$$\begin{aligned}
&(u_{k+1}, u_{k+1}) + (u_{k-1}, u_{k+1}) - 2(u_k, u_{k+1}) \\
&+\tfrac{\alpha_{11}}{2}\tau\left[(u_{k+1}, u_{k+1}) - (u_{k-1}, u_{k+1})\right] \\
&+\tfrac{\alpha_{12}}{2}\tau\left[(u_{k+1}, u_{k+1}) - (u_{k-1}, u_{k+1})\right] \\
&+\tau^2\beta_1(Au_{k+1}, u_{k+1}) + \tau^2\gamma_1(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}) \\
&+\tau^2\rho_{11}(u_k, u_{k+1}) + \tau^2\rho_{12}(v_k, u_{k+1}) = \tau^2(f_k, u_{k+1}),
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&(v_{k+1}, v_{k+1}) + (v_{k-1}, v_{k+1}) - 2(v_k, v_{k+1}) \\
&+\tfrac{\alpha_{21}}{2}\tau\left[(u_{k+1}, v_{k+1}) - (u_{k-1}, v_{k+1})\right] \\
&+\tfrac{\alpha_{22}}{2}\tau\left[(v_{k+1}, v_{k+1}) - (v_{k-1}, v_{k+1})\right] \\
&+\beta_2\tau^2(Av_{k+1}, v_{k+1}) + \gamma_2\tau^2(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1}) \\
&+\rho_{21}\tau^2(u_k, v_{k+1}) + \rho_{22}\tau^2(v_k, v_{k+1}) = \tau^2(g_k, v_{k+1}).
\end{aligned} \tag{3.24}$$

Substituting the operator $A = -\Delta$ into (3.23) - (3.24), applying Discrete Green's Identity, and using the boundary conditions in (3.18) yields

$$\begin{aligned}
&(u_{k+1}, u_{k+1}) + (u_{k-1}, u_{k+1}) - 2(u_k, u_{k+1}) \\
&+\tfrac{\alpha_{11}}{2}\tau\left[(u_{k+1}, u_{k+1}) - (u_{k-1}, u_{k+1})\right] \\
&+\tfrac{\alpha_{12}}{2}\tau\left[(u_{k+1}, u_{k+1}) - (u_{k-1}, u_{k+1})\right] \\
&+\tau^2\beta_1(\nabla u_{k+1}, \nabla u_{k+1}) + \tau^2\gamma_1(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}) \\
&+\tau^2\rho_{11}(u_k, u_{k+1}) + \tau^2\rho_{12}(v_k, u_{k+1}) = \tau^2(f_k, u_{k+1}),
\end{aligned} \tag{3.25}$$

$$(v_{k+1},v_{k+1})+(v_{k-1},v_{k+1})-2(v_k,v_{k+1})$$

$$+\tfrac{\alpha_{21}}{2}\tau\left[(u_{k+1},v_{k+1})-(u_{k-1},v_{k+1})\right]$$

$$+\tfrac{\alpha_{22}}{2}\tau\left[(v_{k+1},v_{k+1})-(v_{k-1},v_{k+1})\right] \tag{3.26}$$

$$+\beta_2\tau^2(\nabla v_{k+1},\nabla v_{k+1})+\gamma_2\tau^2(\sin(\delta_{21}u_k+\delta_{22}v_k),v_{k+1})$$

$$+\rho_{21}\tau^2(u_k,v_{k+1})+\rho_{22}\tau^2(v_k,v_{k+1})=\tau^2(g_k,v_{k+1}).$$

For the terms $\beta_1\tau^2(\nabla u_{k+1},\nabla u_{k+1})$ and $\beta_2\tau^2(\nabla v_{k+1},\nabla v_{k+1})$ we assign the bilinear form $a(u_k,v_k)$. By coercivity property of $a(u_k,v_k)$ and the embedding $V\hookrightarrow H$, the system is written in the following form

$$(u_{k+1},u_{k+1})+(u_{k-1},u_{k+1})-2(u_k,u_{k+1})$$

$$+\tfrac{\alpha_{11}}{2}\tau\left[(u_{k+1},u_{k+1})-(u_{k-1},u_{k+1})\right]$$

$$+\tfrac{\alpha_{12}}{2}\tau\left[(u_{k+1},u_{k+1})-(u_{k-1},u_{k+1})\right] \tag{3.27}$$

$$+\tau^2\beta_1(u_{k+1},u_{k+1})+\tau^2\gamma_1(\sin(\delta_{11}u_k+\delta_{12}v_k),u_{k+1})$$

$$+\tau^2\rho_{11}(u_k,u_{k+1})+\tau^2\rho_{12}(v_k,u_{k+1})\le\tau^2(f_k,u_{k+1}),$$

$$(v_{k+1},v_{k+1})+(v_{k-1},v_{k+1})-2(v_k,v_{k+1})$$

$$+\tfrac{\alpha_{21}}{2}\tau\left[(u_{k+1},v_{k+1})-(u_{k-1},v_{k+1})\right]$$

$$+\tfrac{\alpha_{22}}{2}\tau\left[(v_{k+1},v_{k+1})-(v_{k-1},v_{k+1})\right] \tag{3.28}$$

$$+\beta_2\tau^2(v_{k+1},v_{k+1})+\gamma_2\tau^2(\sin(\delta_{21}u_k+\delta_{22}v_k),v_{k+1})$$

$$+\rho_{21}\tau^2(u_k,v_{k+1})+\rho_{22}\tau^2(v_k,v_{k+1})\le\tau^2(g_k,v_{k+1}).$$

The system of equation (3.27)-(3.28) can be rearranged as

$$\left(1+\tau^2\beta_1+\tau^2\rho_{11}\right)(u_{k+1},u_{k+1})-2(u_k,u_{k+1})+(u_{k-1},u_{k+1})$$

$$+\frac{\alpha_{11}}{2}\tau(u_{k+1},u_{k+1})-\frac{\alpha_{11}}{2}\tau(u_{k-1},u_{k+1})$$

$$+\frac{\alpha_{12}}{2}\tau\left(v_{k+1},u_{k+1}\right)-\frac{\alpha_{12}}{2}\tau\left(v_{k-1},u_{k+1}\right)$$
$$+\tau^2\rho_{12}\left(v_{k+1},u_{k+1}\right)+\tau^2\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),u_{k+1}\right)$$

$$\leq \tau^2\left(f_k,u_{k+1}\right), \tag{3.29}$$

$$\left(1+\tau^2\beta_2+\tau^2\rho_{22}\right)\left(v_{k+1},v_{k+1}\right)-2\left(v_k,v_{k+1}\right)+\left(v_{k-1},v_{k+1}\right)$$
$$+\frac{\alpha_{21}}{2}\tau\left(u_{k+1},v_{k+1}\right)-\frac{\alpha_{11}}{2}\tau\left(u_{k-1},v_{k+1}\right)$$
$$+\frac{\alpha_{22}}{2}\tau\left(v_{k+1},v_{k+1}\right)-\frac{\alpha_{22}}{2}\tau\left(v_{k-1},v_{k+1}\right)$$
$$+\tau^2\rho_{21}\left(v_{k+1},u_{k+1}\right)+\tau^2\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),v_{k+1}\right)$$

$$\leq \tau^2\left(g_k,v_{k+1}\right). \tag{3.30}$$

Moreover, system (3.29)-(3.30) can be written as

$$c_1\left(u_{k+1},u_{k+1}\right)-2\left(u_k,u_{k+1}\right)+\left(u_{k-1},u_{k+1}\right)+c_2\left(v_{k+1},u_{k+1}\right)$$

$$+c_3\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),u_{k+1}\right)+\frac{\alpha_{11}}{2}\tau\left(u_{k+1},u_{k+1}\right)-\frac{\alpha_{11}}{2}\tau\left(u_{k-1},u_{k+1}\right)$$

$$+\frac{\alpha_{12}}{2}\tau\left(v_{k+1},u_{k+1}\right)-\frac{\alpha_{12}}{2}\tau\left(v_{k-1},u_{k+1}\right)\leq \tau^2\left(f_k,u_{k+1}\right), \tag{3.31}$$

$$d_1\left(v_{k+1},v_{k+1}\right)-2\left(v_k,v_{k+1}\right)+\left(v_{k-1},v_{k+1}\right)+d_2\left(u_{k+1},v_{k+1}\right)$$
$$+d_3\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),v_{k+1}\right)+\frac{\alpha_{21}}{2}\tau\left(u_{k+1},v_{k+1}\right)-\frac{\alpha_{21}}{2}\tau\left(u_{k-1},v_{k+1}\right)$$

$$+\frac{\alpha_{22}}{2}\tau\left(v_{k+1},v_{k+1}\right)-\frac{\alpha_{22}}{2}\tau\left(v_{k-1},v_{k+1}\right)\leq \tau^2\left(g_k,v_{k+1}\right), \tag{3.32}$$

with

$$c_1=1+\tau^2\beta_1+\tau^2\rho_{11},\ c_2=\tau^2\rho_{12},\ c_3=\tau^2\gamma_1,$$
$$d_1=1+\tau^2\beta_2+\tau^2\rho_{22},\ d_2=\tau^2\rho_{21},\ d_3=\tau^2\gamma_2.$$

Taking sum of (3.31) and (3.32), we get

$$c_1\left(u_{k+1},u_{k+1}\right)+d_1\left(v_{k+1},v_{k+1}\right)-2\left(u_k,u_{k+1}\right)-2\left(v_k,v_{k+1}\right)$$

$$+\left(u_{k-1},u_{k+1}\right)+\left(v_{k-1},v_{k+1}\right)+c_2\left(v_{k+1},u_{k+1}\right)+d_2\left(u_{k+1},v_{k+1}\right)$$
$$+c_3\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),u_{k+1}\right)+d_3\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),v_{k+1}\right)$$
$$+\frac{\alpha_{11}}{2}\tau\left(u_{k+1},u_{k+1}\right)-\frac{\alpha_{11}}{2}\tau\left(u_{k-1},u_{k+1}\right)$$
$$+\frac{\alpha_{12}}{2}\tau\left(v_{k+1},u_{k+1}\right)-\frac{\alpha_{12}}{2}\tau\left(v_{k-1},u_{k+1}\right)$$
$$+\frac{\alpha_{21}}{2}\tau\left(u_{k+1},v_{k+1}\right)-\frac{\alpha_{21}}{2}\tau\left(u_{k-1},v_{k+1}\right)$$
$$+\frac{\alpha_{22}}{2}\tau\left(v_{k+1},v_{k+1}\right)-\frac{\alpha_{22}}{2}\tau\left(v_{k-1},v_{k+1}\right)$$

$$\leq\tau^2\left(f_k,u_{k+1}\right)+\tau^2\left(g_k,v_{k+1}\right). \tag{3.33}$$

Using the product space $\mathscr{H}$ and the corresponding inner product (3.10), system (3.33) can be written in the vector form

$$\left(A_1\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)-2\left(\mathbf{w}_k,\mathbf{w}_{k+1}\right)+\left(\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)$$
$$+\left(A_2\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)+\left(A_3\sin\delta\mathbf{w}_k,\mathbf{w}_{k+1}\right)$$
$$+\tau\left(A_4\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)-\tau\left(A_4\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)$$
$$+\tau\left(A_5\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)-\tau\left(A_5\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)$$

$$\leq\tau^2\left(F_k,\mathbf{w}_{k+1}\right) \tag{3.34}$$

where

$$\mathbf{w}_k=\begin{bmatrix}u_k\\ v_k\end{bmatrix},A_1=\begin{bmatrix}c_1&0\\0&d_1\end{bmatrix},A_2=\begin{bmatrix}c_2&0\\0&d_2\end{bmatrix},$$
$$A_3=\begin{bmatrix}c_3&0\\0&d_3\end{bmatrix},A_4=\begin{bmatrix}\frac{\alpha_{11}}{2}&0\\0&\frac{\alpha_{22}}{2}\end{bmatrix},$$
$$A_5=\begin{bmatrix}0&\frac{\alpha_{12}}{2}\\\frac{\alpha_{21}}{2}&0\end{bmatrix},F_k=\begin{bmatrix}f_k&0\\0&g_k\end{bmatrix}.$$

Let us rewrite system (3.35) in the following form

$$\left(A_1\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)+\left(A_2\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)+\tau\left(A_4\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)+\tau\left(A_5\mathbf{w}_{k+1},\mathbf{w}_{k+1}\right)$$
$$\leq 2\left(\mathbf{w}_k,\mathbf{w}_{k+1}\right)-\left(\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)-\left(A_3\sin\delta\mathbf{w}_k,\mathbf{w}_{k+1}\right)$$

$$+\tau\left(A_4\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)+\tau\left(A_5\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)+\tau^2\left(F_k,\mathbf{w}_{k+1}\right), \tag{3.35}$$

or equivalently,

$$(|A_1|+|A_2|)\|\mathbf{w}_{k+1}\|^2+\tau\left(\left|A_4\right|+|A_5|\right)\|\mathbf{w}_{k+1}\|^2$$

$$\leq 2(\mathbf{w}_k,\mathbf{w}_{k+1})-(\mathbf{w}_{k-1},\mathbf{w}_{k+1})-(A_3\sin\delta\mathbf{w}_k,\mathbf{w}_{k+1})$$

$$+\tau\left(A_4\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)+\tau(A_5\mathbf{w}_{k-1},\mathbf{w}_{k+1})+\tau^2(F_k,\mathbf{w}_{k+1}). \tag{3.36}$$

Taking the absolute value of right-hand side of (3.37) and then using triangle inequalities, the matrix norm defined in (3.8), we have

$$(|A_1|+|A_2|)\|\mathbf{w}_{k+1}\|^2+\tau\left(\left|A_4\right|+|A_5|\right)\|\mathbf{w}_{k+1}\|^2$$

$$\leq 2|(\mathbf{w}_k,\mathbf{w}_{k+1})|+|(\mathbf{w}_{k-1},\mathbf{w}_{k+1})|+|(A_3\sin\delta\mathbf{w}_k,\mathbf{w}_{k+1})|$$

$$+\tau\left|\left(A_4\mathbf{w}_{k-1},\mathbf{w}_{k+1}\right)\right|+\tau|(A_5\mathbf{w}_{k-1},\mathbf{w}_{k+1})|+\tau^2|(F_k,\mathbf{w}_{k+1})|. \tag{3.37}$$

Using Cauchy-Schwarz and triangle inequalities, the ε-Young inequality (2.77) with $\varepsilon=1$, and the inequality

$$(\sin(\delta_{11}u_k+\delta_{12}v_k),u_{k+1})\leq\|\delta_{11}u_k+\delta_{12}v_k\|\|u_{k+1}\|, \tag{3.38}$$

the matrix norm defined in (3.8), the terms (3.37) are estimated as in the following

$$\left[\frac{1}{2}+\tau^2\left(|\beta|+|\rho|-\frac{|\delta||\gamma|}{2}-\frac{1}{2}\right)\right]\|\mathbf{w}_{k+1}\|^2$$

$$-\left[1+\frac{\tau^2}{2}|\delta||\gamma|\right]\|\mathbf{w}_k\|^2-\frac{1}{2}\|\mathbf{w}_{k-1}\|^2$$

$$\leq\frac{\tau}{4}|\alpha|\left(\|\mathbf{w}_{k+1}\|^2+\|\mathbf{w}_{k-1}\|^2\right)+\frac{\tau^2}{2}\|\mathbf{F}_k\|^2, \tag{3.39}$$

Then we have

$$\frac{1}{2}\left(\|\mathbf{w}_{k+1}\|^2-\|\mathbf{w}_{k-1}\|^2\right)-\frac{\tau}{8}|\alpha|\|\mathbf{w}_{k-1}\|^2$$

$$\leq\left[-\frac{\tau}{4}|\alpha|+\tau^2(-|\beta|-|\rho|+|\delta||\gamma|)\right]\|\mathbf{w}_{k+1}\|^2$$

$$\left(1+\frac{\tau^2}{2}|\delta||\gamma|\right)\|\mathbf{w}_k\|^2+\frac{\tau}{8}|\alpha|\|\mathbf{w}_{k-1}\|^2+\frac{\tau^2}{2}\|\mathbf{F}_k\|^2. \tag{3.40}$$

Dividing both sides of (3.40) by τ, we obtain the inequality in difference form

$$\frac{\|\mathbf{w}_{k+1}\|^2-\|\mathbf{w}_{k-1}\|^2}{2\tau}-\frac{|\alpha|}{8}\|\mathbf{w}_{k-1}\|^2$$

$$\leq k_1 \left\| \mathbf{w}_{k+1} \right\|^2 + k_2 \left\| \mathbf{w}_k \right\|^2 + k_3 \left\| \mathbf{w}_{k-1} \right\|^2 + \frac{\tau}{2} \left\| \mathbf{F}_k \right\|^2, \tag{3.41}$$

where

$$k_1 = -\frac{|\alpha|}{4} + \tau\left(-|\beta| - |\rho| + |\delta|\,|\gamma|\right),$$

$$k_2 = \frac{1}{\tau} + \frac{\tau}{2}|\delta|\,|\gamma|,\ k_3 = \frac{|\alpha|}{8}.$$

Then we can apply the discrete Gronwall Lemma obtained as the statement (2.71), in Remark 2.8. In (2.71) we set

$$u_{n+1} = \left\| \mathbf{w}_{k+1} \right\|^2,\ u_{n-1} = \left\| \mathbf{w}_{k-1} \right\|^2,\ f_n = \frac{\tau}{2} \left\| \mathbf{F}_k \right\|^2,$$

$$A_0 = B_1 = \frac{\tau}{4}|\delta|\,|\gamma|,\ A_1 = k_1,\ B_0 = k_3.$$

Then we have

$$\max_{1\leq k\leq N}\left(\left\| \mathbf{w}_{k+1} \right\|^2 + \left\| \mathbf{w}_k \right\|^2\right) \leq e^{2C_1}\left\| \mathbf{w}_1 \right\|^2 + e^{2C_2}\left\| \mathbf{w}_0 \right\|^2 + e^{2C_3}\tau^2\sum_{k=1}^{n}\left\| \mathbf{F}_k \right\|^2 + e^{2C_1}\frac{\tau^2}{2}\left\| \mathbf{F}_{k+1} \right\|^2 \tag{3.42}$$

where

$$C_1 = A_1 + B_1,\ C_2 = A_2 + B_2,\ C_3 = max\{C_1, C_2\}.$$

Equation (3.42) can be written as

$$\max_{1\leq k\leq N}\left(\left\| \mathbf{w}_{k+1} \right\|^2 + \left\| \mathbf{w}_k \right\|^2\right) \leq K_1\left\| \mathbf{w}_1 \right\|^2 + K_2\left\| \mathbf{w}_0 \right\|^2 + K_3\tau^2\sum_{k=1}^{n}\left\| \mathbf{F}_k \right\|^2 + K_4\frac{\tau^2}{2}\left\| \mathbf{F}_{k+1} \right\|^2. \tag{3.43}$$

Let

$$K_0 = \max\{K_1, K_2, K_3, K_4\},$$

then

$$\left\| \mathbf{w}_{k+1} \right\|^2 + \left\| \mathbf{w}_k \right\|^2 \leq K_0\left(\left\| \mathbf{w}_1 \right\|^2 + \left\| \mathbf{w}_0 \right\|^2 + \tau^2\sum_{k=1}^{n}\left\| \mathbf{F}_k \right\|^2 + \frac{\tau^2}{2}\left\| \mathbf{F}_{k+1} \right\|^2\right). \tag{3.44}$$

Without loss of generality, let us assume that there exists a constant $\tilde{K}$ in terms of the given values $\mathbf{w}_1$, $\mathbf{w}_0$, $\mathbf{F}_k$ and $\mathbf{F}_{k+1}$, that is

$$\tilde{K} = K_0\left(\left\| \mathbf{w}_0 \right\|, \left\| \mathbf{w}_1 \right\|, \left\| \mathbf{F}_k \right\|, \left\| \mathbf{F}_{k+1} \right\|\right).$$

Then we have

$$\max_{1\leq k\leq N}\left(\left\| \mathbf{w}_{k+1} \right\|^2 + \left\| \mathbf{w}_k \right\|^2\right) \leq \tilde{K} \tag{3.45}$$

Since the right-hand side of (3.41) is bounded by the result of the discrete Gronwall inequality, we obtain

$$-\frac{|\alpha|}{8}\left\|\mathbf{w}_{k-1}\right\|^2 \leq M,$$

and so

$$\max_{1\leq k\leq N}\left\|\mathbf{w}_{k-1}\right\|^2 \leq M_0, \tag{3.46}$$

for any nonnegative constants M_0, M. Thus,

$$\max_{1\leq k\leq N}\left(\left\|\mathbf{w}_{k+1}\right\|^2+\left\|\mathbf{w}_{k}\right\|^2+\left\|\mathbf{w}_{k+1}\right\|^2\right) \leq \tilde{K}+M_0=K \tag{3.47}$$

Hence, Theorem 3.2 is proved. ■

Theorem 3.3. *Suppose that assumptions (1.24) and (3.17) hold. Then there is a positive constant K, which does not depend on the grid parameters τ and h, such that for all $k\in\mathbb{N}$*

$$\begin{aligned}&\left(\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2+\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2\right.\\&+\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2+\left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2\\&\left.+\left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2+\left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2\right)\leq K.\end{aligned} \tag{3.48}$$

Proof. Applying the method of proving Theorem 3.2, we construct the weak formulation for the derivative terms by modifying (1.28). Multiplying by τ and taking the inner product for first equation of (1.28) by $\frac{u_{k+1}-u_k}{\tau}$ and $\frac{u_k-u_{k-1}}{\tau}$, we obtain

$$\begin{aligned}&\left(\frac{u_{k+1}-u_k}{\tau}-\frac{u_k-u_{k-1}}{\tau},\frac{u_{k+1}-u_k}{\tau}\right)\\&+\tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_{k+1}-u_k}{\tau}\right)\\&+\tau\beta_1\left(Au_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),\frac{u_{k+1}-u_k}{\tau}\right)\\&+\tau\rho_{11}\left(u_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)\\&=\tau\left(f_k,\frac{u_{k+1}-u_k}{\tau}\right)\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}&\left(\frac{u_{k+1}-u_k}{\tau}-\frac{u_k-u_{k-1}}{\tau},\frac{u_k-u_{k-1}}{\tau}\right)\\&+\tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)\end{aligned}$$

$$+\tau\beta_1\left(Au_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),\frac{u_k-u_{k-1}}{\tau}\right)$$
$$+\tau\rho_{11}\left(u_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)$$
$$=\tau\left(f_k,\frac{u_k-u_{k-1}}{\tau}\right), \tag{3.50}$$

respectively. Similar to the proof of Theorem 3.2, for the terms containing the operator A we use the discrete Green formula and we assign the bilinear form $a(u_k, v_k)$. By coercivity property of $a(u_k, v_k)$ and the embedding $V \hookrightarrow H$, the system can be written in the following form

$$\left(\frac{u_{k+1}-u_k}{\tau}-\frac{u_k-u_{k-1}}{\tau},\frac{u_{k+1}-u_k}{\tau}\right)$$
$$+\tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_{k+1}-u_k}{\tau}\right)$$
$$+\tau\beta_1\left(u_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),\frac{u_{k+1}-u_k}{\tau}\right)$$
$$+\tau\rho_{11}\left(u_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_{k+1}-u_k}{\tau}\right)$$
$$\leq\tau\left(f_k,\frac{u_{k+1}-u_k}{\tau}\right) \tag{3.51}$$

and

$$\left(\frac{u_{k+1}-u_k}{\tau}-\frac{u_k-u_{k-1}}{\tau},\frac{u_k-u_{k-1}}{\tau}\right)$$
$$+\tau\alpha_{11}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_k-u_{k-1}}{\tau}\right)$$
$$+\tau\beta_1\left(u_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),\frac{u_k-u_{k-1}}{\tau}\right)$$
$$+\tau\rho_{11}\left(u_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)$$
$$\leq\tau\left(f_k,\frac{u_k-u_{k-1}}{\tau}\right), \tag{3.52}$$

Taking the sum of (3.51) and (3.52) yields

$$\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2-\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2\right\}$$
$$+2\tau\alpha_{11}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2+\tau\alpha_{12}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{u_{k+1}-u_{k-1}}{\tau}\right)$$
$$+\tau\beta_1\left(Au_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\gamma_1\left(\sin\left(\delta_{11}u_k+\delta_{12}v_k\right),\frac{u_k-u_{k-1}}{\tau}\right)$$

$$+\tau\rho_{11}\left(u_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)+\tau\rho_{12}\left(v_{k+1},\frac{u_k-u_{k-1}}{\tau}\right)$$

$$\leq\tau\left(f_k,\frac{u_{k+1}-u_k}{\tau}\right). \tag{3.53}$$

Similarly, modifying (1.28), multiplying by τ, and taking the inner product for the second equation of (1.28) by $\frac{v_{k+1}-v_k}{\tau}$ and $\frac{v_k-v_{k-1}}{\tau}$, we get

$$\left(\frac{v_{k+1}-v_k}{\tau}-\frac{v_k-v_{k-1}}{\tau},\frac{v_{k+1}-v_k}{\tau}\right)$$

$$+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\alpha_{22}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{v_{k+1}-v_k}{\tau}\right)$$
$$+\tau\beta_2\left(Av_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),\frac{v_{k+1}-v_k}{\tau}\right)$$
$$+\tau\rho_{21}\left(u_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\rho_{22}\left(v_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)$$

$$=\tau\left(g_k,\frac{v_{k+1}-v_k}{\tau}\right) \tag{3.54}$$

and

$$\left(\frac{v_{k+1}-v_k}{\tau}-\frac{v_k-v_{k-1}}{\tau},\frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{v_k-v_{k-1}}{\tau}\right)+\tau\alpha_{22}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\beta_2\left(Av_{k+1},\frac{v_k-v_{k-1}}{\tau}\right)+\tau\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),\frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\rho_{21}\left(u_{k+1},\frac{v_k-v_{k-1}}{\tau}\right)+\tau\rho_{22}\left(v_{k+1},\frac{v_k-v_{k-1}}{\tau}\right)$$

$$=\tau\left(g_k,\frac{v_k-v_{k-1}}{\tau}\right), \tag{3.55}$$

respectively. For the terms containing the operator A we use the discrete Green formula and we assign the bilinear form $a(u_k,v_k)$. By coercivity property of $a(u_k,v_k)$ and the embedding $V\hookrightarrow H$, the system is written in the following form

$$\left(\frac{v_{k+1}-v_k}{\tau}-\frac{v_k-v_{k-1}}{\tau},\frac{v_{k+1}-v_k}{\tau}\right)$$

$$+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\alpha_{22}\left(\frac{v_{k+1}-v_{k-1}}{2\tau},\frac{v_{k+1}-v_k}{\tau}\right)$$
$$+\tau\beta_2\left(v_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right),\frac{v_{k+1}-v_k}{\tau}\right)$$
$$+\tau\rho_{21}\left(u_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)+\tau\rho_{22}\left(v_{k+1},\frac{v_{k+1}-v_k}{\tau}\right)$$

$$\leq \tau\left(g_k, \frac{v_{k+1}-v_k}{\tau}\right) \tag{3.56}$$

and

$$\left(\frac{v_{k+1}-v_k}{\tau}-\frac{v_k-v_{k-1}}{\tau}, \frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau}, \frac{v_k-v_{k-1}}{\tau}\right)+\tau\alpha_{22}\left(\frac{v_{k+1}-v_{k-1}}{2\tau}, \frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\beta_2\left(v_{k+1}, \frac{v_k-v_{k-1}}{\tau}\right)+\tau\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right), \frac{v_k-v_{k-1}}{\tau}\right)$$
$$+\tau\rho_{21}\left(u_{k+1}, \frac{v_k-v_{k-1}}{\tau}\right)+\tau\rho_{22}\left(v_{k+1}, \frac{v_k-v_{k-1}}{\tau}\right)$$

$$\leq \tau\left(g_k, \frac{v_k-v_{k-1}}{\tau}\right). \tag{3.57}$$

Taking the sum of (3.56) and (3.57)

$$\left\{\left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2-\left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2\right\}$$

$$+2\tau\alpha_{22}\left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2+\tau\alpha_{21}\left(\frac{u_{k+1}-u_{k-1}}{2\tau}, \frac{v_{k+1}-v_{k-1}}{\tau}\right)$$
$$+\tau\beta_2\left(Av_{k+1}, \frac{v_{k+1}-v_{k-1}}{\tau}\right)+\tau\gamma_2\left(\sin\left(\delta_{21}u_k+\delta_{22}v_k\right), \frac{v_{k+1}-v_{k-1}}{\tau}\right)$$
$$+\tau\rho_{21}\left(u_{k+1}, \frac{v_{k+1}-v_{k-1}}{\tau}\right)+\tau\rho_{22}\left(v_{k+1}, \frac{v_{k+1}-v_{k-1}}{\tau}\right)$$

$$\leq \tau\left(g_k, \frac{v_{k+1}-v_{k-1}}{\tau}\right). \tag{3.58}$$

is obtained. Multiplying (3.53) by α_{21} and (3.58) by α_{12}, and subtracting these terms, we get

$$\Delta_1=\alpha_{21}\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2-\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2\right\}$$
$$+2\tau\alpha_{11}\alpha_{21}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2$$
$$-\alpha_{12}\left\{\left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2-\left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2\right\}$$
$$+2\tau\alpha_{12}\alpha_{22}\left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2+s_2\left(u_{k+1}, v_{k-1}\right)$$
$$+2\tau\left(\|u_{k+1}\|^2+\|v_{k+1}\|^2\right)+s_1\left(u_{k+1}, v_{k+1}\right)$$

$$+s_3\left(u_{k+1}, u_{k-1}\right)+s_4\left(v_{k+1}, v_{k-1}\right)+s_5\left(v_{k+1}, u_{k-1}\right) \tag{3.59}$$

where

$$s_1 = \alpha_{21}\rho_{12} - \alpha_{12}\rho_{21},\ s_2 = \alpha_{21}\beta_1 - \alpha_{21}\rho_{11},$$

$$s_3 = \alpha_{12}\beta_2 - \alpha_{12}\rho_{22},\ s_4 = \alpha_{12}\rho_{21}\tau,\ s_5 = -\alpha_{21}\rho_{12}.$$

Then we obtain

$$\Delta_1 \leq \alpha_{21}(f_k, u_{k+1}) - \alpha_{21}(f_k, u_{k-1}) - \alpha_{12}(g_k, v_{k+1})$$

$$+\alpha_{21}(g_k, v_{k-1}) + s_6(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1})$$

$$-s_6(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k-1}) - s_7(\sin(\delta_{21}u_k + \delta_{22}v_k), v_{k+1})$$

$$+s_7(\sin(\delta_{11}u_k + \delta_{12}v_k), v_{k-1}), \tag{3.60}$$

where

$$s_6 = \alpha_{21}\gamma_1,\ s_7 = \alpha_{12}\gamma_2.$$

Here, the coefficients s_i, $i = 1, 2, ..., 7$ in (3.59) and (3.60) are bounded constants by (1.24). Using the inequalities

$$(\sin(\delta_{11}u_k + \delta_{12}v_k), u_{k+1}) \leq \|\delta_{11}u_k + \delta_{12}v_k\| \|u_{k+1}\|, \tag{3.61}$$

$$(u_{k+1}, v_{k+1}) \leq \frac{1}{2}\left(\|u_{k+1}\|^2 + \|v_{k+1}\|^2\right), \tag{3.62}$$

(3.59) and (3.60) can be rewritten as

$$\alpha_{21}\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2 - \left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2\right\}$$

$$+2\tau\alpha_{11}\alpha_{21}\left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2$$

$$+\alpha_{12}\left\{\left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2 - \left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2\right\}$$

$$+2\tau\alpha_{12}\alpha_{22}\left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2$$

$$+a_1\|u_{k+1}\|^2 + a_2\|u_k\|^2 + a_3\|u_{k-1}\|^2 + a_4\|f_k\|^2$$

$$+b_1\|v_{k+1}\|^2 + b_2\|v_k\|^2 + b_3\|v_{k-1}\|^2 + b_4\|g_k\|^2 \tag{3.63}$$

where

$$a_1 = a_3 = \frac{\tau s_6}{2}\left(|\delta_{11}| + |\delta_{12}|\right),\ a_2 = \tau(s_6\,|\delta_{11}| + s_7\,|\delta_{12}|),\ a_4 = \alpha_{21},$$

$$b_1 = b_3 = \frac{\tau s_7}{2}\left(|\delta_{21}| + |\delta_{22}|\right),\ b_2 = \tau(s_6\,|\delta_{12}| + s_7\,|\delta_{22}|),\ b_4 = \alpha_{12}.$$

We denote

$$N = max\left\{\alpha_{21}, 2\tau\alpha_{11}\alpha_{21}, 2\tau\alpha_{12}\alpha_{22}, \alpha_{12},\right\}, \tag{3.64}$$

Then using Theorem 3.2, assumptions (1.24) and (3.17) the estimation for (3.63) can be written as

$$N\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2 - \left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2 + \left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2 + \left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2 + \left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2 - \left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2\right\} \le K_0. \tag{3.65}$$

Let $K = \frac{K_0}{N}$, then we have

$$\left\{\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2 - \left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2 + \left\|\frac{u_{k+1}-u_{k-1}}{2\tau}\right\|^2 + \left\|\frac{v_{k+1}-v_{k-1}}{2\tau}\right\|^2 + \left\|\frac{v_{k+1}-v_k}{\tau}\right\|^2 - \left\|\frac{v_k-v_{k-1}}{\tau}\right\|^2\right\} \le K. \tag{3.66}$$

Thus, Theorem 3.3 is proved. ■

Corollary 3.1. *Under the hypotheses of Theorem 3.2 and Theorem 3.3 there exist subsequences*

$$\left\{u_{k_m}\right\} \subset \{u_k\} \text{ and } \left\{v_{k_m}\right\} \subset \{v_k\} \tag{3.67}$$

which converge in V to bounded measurable functions u and v, respectively. Moreover, the limit functions u and v are unique weak solutions satisfying (3.22) and (3.48).

Proof. Estimates (3.22), (3.48), and Discrete Gronwall's Lemma [28] imply that

$$\{u_k\} \text{ and } \{v_k\} \text{ are bounded in } L^\infty(0,T;V). \tag{3.68}$$

Then, by Rellich Theorem [6, 85, 88] there exists a subsequence $\mathbf{w}_{k_m} = \left[u_{k_m}, v_{k_m}\right]^T$ of

$\mathbf{w}_k = [u_k, v_k]^T$ and $\tilde{\mathbf{w}}_k \in L^\infty(0, T; \mathscr{V})$ such that

$$\tilde{\mathbf{w}}_k \in L^\infty(0, T; \mathscr{V}) \subset L^2(0, T; \mathscr{V}) \tag{3.69}$$

and

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \text{ weak* in } L^\infty(0, T; \mathscr{V}), \text{ weakly in } L^2(0, T; \mathscr{V}). \tag{3.70}$$

By the Aubin Compactness Theorem [97, 98], the above convergence results imply

$$\mathbf{w}_{k_m} \to \tilde{\mathbf{w}}_k \text{ strongly in } L^2(0, T; \mathscr{H}) \tag{3.71}$$

and by (3.71),

$$\sin\delta \mathbf{w}_{k_m} \to \sin\delta \tilde{\mathbf{w}}_k \text{ strongly in } L^2(0, T; \mathscr{H}), \tag{3.72}$$

which shows the existence of $\tilde{\mathbf{w}}_k$ a.e. in $\mathscr{H}$ and $\tilde{\mathbf{w}}_0 = \mathbf{w}_0$. Uniqueness follows from convergence of difference scheme (1.28), and by Theorem 3.1. Hence, the proof is completed. ∎

Note that similar convergence and compactness results for weak solvability of systems can be found in [6], [10], [11], and [81].

4
NUMERICAL ANALYSIS

In the present section, we verify the theoretical results of our study by numerical experiments. A unified numerical method that is based on finite difference method equipped with fixed point iteration is employed. The fixed point iteration is applied for nonlinear part of the problem. We propose a unified numerical method to obtain more accurate results for the solution of an initial-boundary value problem (IBVP) for the one dimensional coupled sine Gordon equations. We choose an exact solution

$$w(t,x) = \{u(t,x), v(t,x)\} \tag{4.1}$$

with

$$u(t,x) = e^{-2t}\sin \pi x, v(t,x) = e^{-t}\sin \pi x \tag{4.2}$$

and we formulate a boundary value problem that leads to this solution. Let us consider the IBVP

$$\begin{cases} u_{tt} - u_{xx} + u_t + u = -\sin(u-v) + (\pi^2+3)e^{-2t}\sin \pi x \\ +\sin(e^{-2t}\sin \pi x - e^{-t}\sin \pi x), \\ 0 < t < 1, 0 < x < 1, \\ v_{tt} - v_{xx} + v_t + v = \sin(u-v) + (\pi^2+1)e^{-t}\sin \pi x \\ -\sin(e^{-2t}\sin \pi x - e^{-t}\sin \pi x), \\ 0 < t < 1,\ 0 < x < 1, \\ u(0,x) = \sin \pi x, u_t(0,x) = -2\sin \pi x,\ 0 \le x \le 1, \\ v(0,x) = \sin \pi x, v_t(0,x) = -\sin \pi x, 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, v(t,0) = v(t,1) = 0,\ 0 \le t \le 1. \end{cases} \tag{4.3}$$

System (4.3) is used for modelling the wave propagation on an infinite chain of elastically bound atoms lying on a fixed lower chain of the similar atoms. The second order derivative terms describe the elastic interaction of energy between the neighboring atoms, and their kinetic energy, respectively. The nonlinear terms containing sine stand for the potential energy by virtue of the fixed lower chain. The remaining terms are damping terms and source functions.

For the approximate solution of problem (4.3), difference scheme (1.28) is considered. The modified Gauss elimination method is utilised for the solution of system (4.3). The set of a family of grid points

$$\Omega_h = [0,1]_\tau \times [0,1]_h = \{(t_k, x_n) : t_k = k\tau,\ \ 0 \le k \le N,$$

$$N\tau = 1, x_n = nh, 0 \le n \le M, Mh = 1\} \tag{4.4}$$

is considered.

4.1 First Order of Accuracy Difference Scheme

Using difference scheme (1.28), the system

$$\begin{cases} \frac{{}_m u_n^{k+1} - 2{}_m u_n^k + {}_m u_n^{k-1}}{\tau^2} - \frac{{}_m u_{n+1}^{k+1} - 2{}_m u_n^{k+1} + {}_m u_{n-1}^{k+1}}{h^2} + \frac{{}_m u_n^{k+1} - {}_m u_n^{k-1}}{2\tau} +_m u_n^k \\ = -\sin({}_{m-1}u_n^k -_{m-1} v_n^k) + \sin(e^{-2t_k}\sin\pi x_n - e^{-t_k}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_k}\sin\pi x_n, \ \ x_n = nh, 1 \le n \le M-1, \\ \frac{{}_m v_n^{k+1} - 2{}_m v_n^k + {}_m v_n^{k-1}}{\tau^2} - \frac{{}_m v_{n+1}^{k+1} - 2{}_m v_n^{k+1} + {}_m v_{n-1}^{k+1}}{h^2} + \frac{{}_m v_n^{k+1} - {}_m v_n^{k-1}}{2\tau} +_m v_n^k \\ = \sin({}_{m-1}u_n^k -_{m-1} v_n^k) - \sin(e^{-2t_k}\sin\pi x_n - e^{-t_k}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_k}\sin\pi x_n, \ \ x_n = nh, 1 \le n \le M-1, \\ t_{k+1} = (k+1)\tau,\ 0 \le k \le N, N\tau = 1, \\ x_n = nh,\ 1 \le n \le M-1, Mh = 1, \\ {}_m u_n^0 = \sin x_n, \tau^{-1}({}_m u_n^1 -_m u_n^0) \ = -2\sin x_n, \\ {}_m v_n^0 = \sin x_n, \tau^{-1}({}_m v_n^1 -_m v_n^0) \ = -\sin x_n, \\ {}_m u_0^k =_m u_M^k = 0,_m v_0^k =_m v_M^k = 0, 0 \le k \le N \end{cases} \tag{4.5}$$

is obtained, where m is the index representing the number of fixed point iterations. Rewriting problem (4.5) with matrix coefficients, an $(N+1)\times(N+1)$ sized system of linear equations

$$\begin{cases} A\, {}_m u_{n+1} + B\, {}_m u_n + C\, {}_m u_{n-1} = D_{m-1}\varphi_n, \quad 1 \le n \le M-1, \\ A\, {}_m v_{n+1} + B\, {}_m v_n + C\, {}_m v_{n-1} = E_{m-1}\tilde{\varphi}_n, \quad 1 \le n \le M-1, \\ {}_m u_0 = 0, \ {}_m u_M = 0, \\ {}_m v_0 = 0, \ {}_m v_M = 0 \end{cases} \tag{4.6}$$

is obtained. Here,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b & c & d & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & b & c & d & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & b & c & \ddots & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \ddots & \ddots & d & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & b & c & d \\ -1 & 1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(M+1)},$$

$$C = A, D = E = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

where

$$a = -\frac{1}{h^2}, \ b = \frac{1}{\tau^2} - \frac{1}{2\tau}, \ c = 1 - \frac{2}{\tau^2}, \ d = \frac{1}{\tau^2} + \frac{1}{2\tau} + \frac{2}{h^2},$$

and

$$
{}_m\varphi_n^k = \begin{bmatrix} \sin(\pi x_n) \\ -\sin\left({}_{m-1}u_n^1 -_{m-1} v_n^1\right) + \sin(e^{-2t_1}\sin\pi x_n - e^{-t_1}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_1}\sin\pi x_n \\ -\sin\left({}_{m-1}u_n^2 -_{m-1} v_n^2\right) + \sin(e^{-2t_2}\sin\pi x_n - e^{-t_2}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_2}\sin\pi x_n \\ \vdots \\ -\sin\left({}_{m-1}u_n^{N-1} -_{m-1} v_n^{N-1}\right) + \sin(e^{-2t_{N-1}}\sin\pi x_n - e^{-t_{N-1}}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_{N-1}}\sin\pi x_n \\ -2\sin(\pi x_n) \end{bmatrix}_{(N+1)\times 1},
$$

$$
{}_0\varphi_n^k = \begin{bmatrix} \sin(\pi x_n) \\ -\sin\left({}_0u_n^1 -_0 v_n^1\right) + \sin(e^{-2t_1}\sin\pi x_n - e^{-t_1}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_1}\sin\pi x_n \\ -\sin\left({}_0u_n^2 -_0 v_n^2\right) + \sin(e^{-2t_2}\sin\pi x_n - e^{-t_2}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_2}\sin\pi x_n \\ \vdots \\ -\sin\left({}_0u_n^{N-1} -_0 v_n^{N-1}\right) + \sin(e^{-2t_{N-1}}\sin\pi x_n - e^{-t_{N-1}}\sin\pi x_n) \\ +(\pi^2+3)e^{-2t_{N-1}}\sin\pi x_n \\ -2\sin(\pi x_n) \end{bmatrix}_{(N+1)\times 1},
$$

$$
{}_m\tilde{\varphi}_n^k = \begin{bmatrix} \sin(\pi x_n) \\ \sin\left({}_{m-1}u_n^1 -_{m-1} v_n^1\right) - \sin(e^{-2t_1}\sin\pi x_n - e^{-t_1}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_1}\sin\pi x_n \\ \sin\left({}_{m-1}u_n^2 -_{m-1} v_n^2\right) - \sin(e^{-2t_2}\sin\pi x_n - e^{-t_2}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_2}\sin\pi x_n \\ \vdots \\ \sin\left({}_{m-1}u_n^{N-1} -_{m-1} v_n^{N-1}\right) - \sin(e^{-2t_{N-1}}\sin\pi x_n - e^{-t_{N-1}}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_{N-1}}\sin\pi x_n \\ -\sin(x_n) \end{bmatrix}_{(N+1)\times 1},
$$

$$
{}_0\tilde{\varphi}_n^k = \begin{bmatrix} \sin(x_n) \\ -\sin\left({}_0u_n^1 - {}_0 v_n^1\right) + \sin(e^{-2t_1}\sin\pi x_n - e^{-t_1}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_1}\sin\pi x_n \\ -\sin\left({}_0u_n^2 - {}_0 v_n^2\right) + \sin(e^{-2t_2}\sin\pi x_n - e^{-t_2}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_2}\sin\pi x_n \\ \vdots \\ -\sin\left({}_0u_n^{N-1} - {}_0 v_n^{N-1}\right) + \sin(e^{-2t_{N-1}}\sin\pi x_n - e^{-t_{N-1}}\sin\pi x_n) \\ +(\pi^2+1)e^{-t_{N-1}}\sin\pi x_n \\ -\sin(\pi x_n) \end{bmatrix}_{(N+1)\times 1}
$$

for $0 \le k \le N$ with

$$
u_s = \begin{bmatrix} {}_mu_s^0 \\ {}_mu_s^1 \\ \vdots \\ {}_mu_s^{N-1} \\ {}_mu_s^N \end{bmatrix}_{(N+1)\times(1)}, v_s = \begin{bmatrix} {}_mv_s^0 \\ {}_mv_s^1 \\ \vdots \\ {}_mv_s^{N-1} \\ {}_mv_s^N \end{bmatrix}_{(N+1)\times(1)} \quad \text{where} \quad s = n-1,\ n,\ n+1.
$$

The numerical algorithm is performed for $m = 1, 2, ..., p$, where p depends on a certain given error tolerance ε such that

$$
\left|{}_pu_n - {}_{p-1} u_n\right| < \varepsilon \text{ and } \left|{}_pv_n - {}_{p-1} v_n\right| < \varepsilon. \tag{4.7}
$$

The modified Gauss elimination method is employed for the solution of system (4.6). The solution is obtained by the following formulas

$$
{}_mu_n = {}_m \alpha_{n+1m}u_{n+1} + {}_m \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0, \tag{4.8}
$$

$$
{}_mv_n = {}_m \alpha_{n+1m}v_{n+1} + {}_m \beta_{n+1}, \quad n = M-1, ..., 2, 1, 0, \tag{4.9}
$$

where ${}_m\alpha_j$, ${}_m\beta_j$, $j = 1, ..., M-1$ are square matrices with size $(N+1)\times(N+1)$ and α_1, β_1 are

$$
{}_m\alpha_1 = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix}_{(N+1)\times(N+1)}, {}_m\beta_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(N+1)\times 1}.
$$

4.2 Error Analysis

We consider problem (4.3). For the approximate solutions of the problem we use the first order of accuracy difference scheme. The figures of exact and numerical solutions are presented for $N = M = 80$ values.

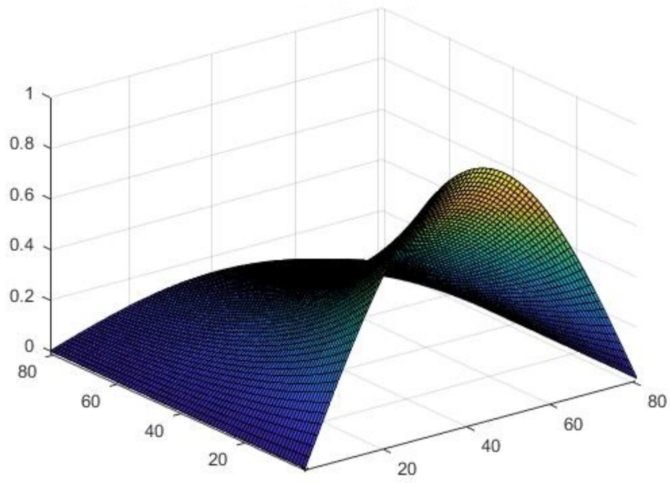

Figure 4.1 exact solution $u(t,x) = e^{-2t} \sin \pi x$ of problem (4.3)

.

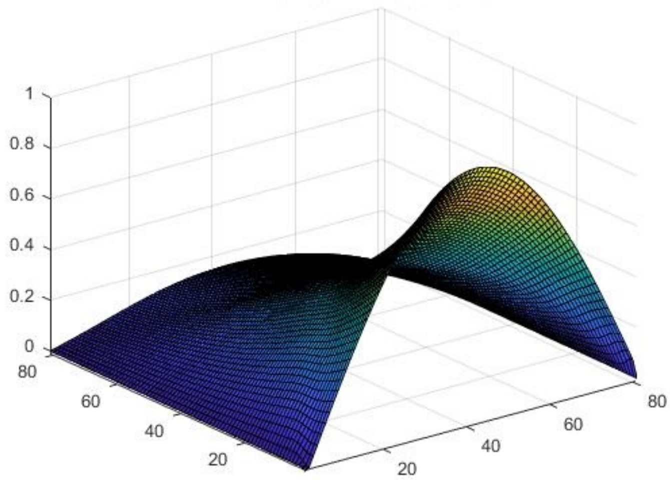

Figure 4.2 difference scheme solution $u(t,x) = e^{-2t} \sin \pi x$ of problem (4.3)

.

Errors, rate of convergence, number of iterations, and the related CPU times are shown in tables for different $N = M$ values.

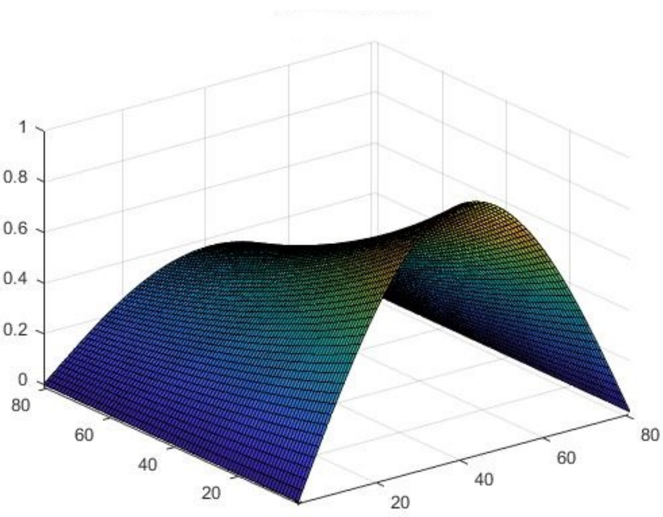

Figure 4.3 exact solution $v(t,x) = e^{-t}\sin\pi x$ of problem (4.3)

.

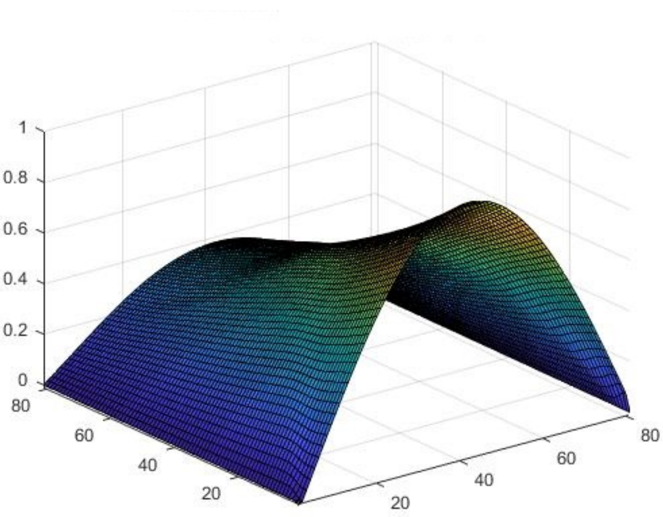

Figure 4.4 difference scheme solution $v(t,x) = e^{-t}\sin\pi x$ of problem (4.3)

.

The errors in the approximation are computed by the following formulas

$$E_u = \max_{\substack{1\le k\le N-1 \\ 1\le n\le M-1}} \left| u\left(t_k, x_n\right) - u_n^k \right| \text{ and } E_v = \max_{\substack{1\le k\le N-1 \\ 1\le n\le M-1}} \left| u\left(t_k, x_n\right) - u_n^k \right|, \tag{4.10}$$

where $u\left(t_k, x_n\right), v\left(t_k, x_n\right)$ are the exact solutions, and u_n^k, v_n^k are the numerical solutions for the approximate solution of problem (4.3) at (t_k, x_n). The rate of convergence for the solutions u and v are computed by the following formulas

$$r_u = \log_2\left(\frac{E_u(N)}{E_u(2N)}\right) \text{ and } r_v = \log_2\left(\frac{E_v(N)}{E_v(2N)}\right). \tag{4.11}$$

Here m is the index representing the number of fixed point iteration. The numerical results are presented in the following table.

Table 1. Error analysis for the approximate solution of problem (4.3) by (4.5)

N=M	E_u	E_v	r_u	r_v	m	CPU times
10	0.3090	0.3090	0.98236	0.98236	7	2.715307
20	0.1564	0.1564	0.99447	0.99447	9	2.802261
40	0.0785	0.0785	0.99816	0.99816	10	3.263671
80	0.0393	0.0393	1.00367	1.00367	11	4.767175
160	0.0196	0.0196	–	–	12	14.558062

Table 1 shows the error analysis for the approximate solution of (4.3), with $\varepsilon = 10^{-20}$. In the iteration, the initials are chosen as matrices of the form

$$_0u_n^k = I(N+1, 1), \tag{4.12}$$

$$_0v_n^k = I(N+1, 1). \tag{4.13}$$

Numerical solutions of problem (4.3) are obtained by first order of accuracy difference scheme (4.5) jointly with fixed point iteration. The difference scheme converges for different iteration numbers m, $N = M$ values, initial vectors $_0u_n^k$, $_0v_n^k$, and termination criteria ε. When the maximum difference at grid points of two successive results gets less than ε, the iterative process is stopped.

As it is obvious from the table that if values of N and M are doubled, then the errors decrease approximately by a factor of $1/2$ for difference scheme (4.5). The errors and the rate of convergence presented in the table indicates the convergence of the difference scheme and the accuracy of the results. It is observed that the difference scheme has first order of convergence as it is expected.

5
RESULTS AND DISCUSSION

In this work, the unique solvability for the system of finite difference schemes for the coupled sine-Gordon equations based on the variational formulation is constructed. The theoretical statements on the existence and uniqueness of the system are proved by the energy method, known as the variational formulation. A novel combined iterative method based on unconditionally stable finite difference schemes with the fixed point iteration is constructed. In the iterations, a first order of accuracy stable difference scheme is employed. Numerical implementations are carried out to confirm the theoretical results and to demonstrate the efficiency of the method.

Our future work on coupled sine-Gordon equations and other nonlinear wave systems will likely concentrate on employing high order of accuracy unconditionally stable difference schemes for the weak formulation, as these schemes improves the theoretical and numerical results both in accuracy and convergence. Other feasible directions for the future study would depend upon modelling nonlinear wave systems having nonlocal boundary conditions and establishing the weak solvability of these systems.

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PUBLICATIONS FROM THE THESIS

Papers

1. O. Yildirim and M. Uzun, "Weak solvability of the unconditionally stable difference scheme for the coupled sine-Gordon system", *Nonlinear Analysis: Modelling and Control*, vol. 25, no. 6, p. 997–1014, 2020.

Conference Papers

1. O. Yildirim and M. Uzun, "A composite numerical method on the solution of coupled sine-Gordon equations based on the fixed point theory", "Banach Spaces and Their Applications", 26-29 June 2019, Lviv, Ukraine.

2. O. Yildirim and M. Uzun, "On the numerical solution of nonlinear system of coupled sine-Gordon equations", "The Fourth International Conference on Analysis and Applied Mathematics (ICAAM 2018)", 6-9 September 2018, Near East University, Lefkosa, Mersin, Turkey.

Projects

1. Özgür Yıldırım, DNA Dinamiğindeki Problemlerde Kararlılık Analizi, Bilimsel Araştırma Projeleri Koordinatörlüğü (BAP), 2016-01-03-DOP02, Araştırmacı.