REPUBLIC OF TURKEY
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

ON THE SOLUTION OF FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH WAVELET BASIS FUNCTIONS

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February, 2022

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Jumana H.S. ALKHALISSI

Signature

Dedicated this thesis and all its knowledge to the first professor, the prophet Muhammad (peace be upon him), and the person who supported me always through the dark night and worst times, and my mother owing to feeling I lived far from her.

ACKNOWLEDGEMENTS

I would like to express sincere gratitude to my supervisor, Prof. Ibrahim EMIROGLU. for his support along through this research way, for his patience, motivation and enthusiasm. I could not have imagined having a better advisor and mentor for my Ph.D study.

I would say thank you for my second supervisor Prof. Mustafa BAYRAM, for his guiding me to choose the right way for this thesis.

I would like to special thank you to Prof. Aydin SECER for guiding me to his important publications and for technical support on my study.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Aydin SECER and Prof. Coşkun YAKAR, for their encouragement, insightful comments, and hard questions which incented me to widen my research from various perspectives.

I am also thankful to the graduate school of science and engineering at Yildiz Technical University and all its member's staff for giving me the chance to complete my Ph.D in it and for all the considerate guidance.

Finally, I would like to thank my mother and my big brother for supporting me during the compilation of this dissertation.

Jumana H.S. ALKHALISSI

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LIST OF SYMBOLS

$F^{\alpha}_{m\times m}$	Block- pulse matrix of integration
${}^{C}\mathrm{D}^{\alpha}_{a}$	Caputo fractional derivative
$P^{\lambda,y,c}_{m}$	Generalized Gegenbauer -Humberts polynomial
$\psi^{y,c}_{n,m}$	Generalized Gegenbauer -Humberts wavelet
$P^{y,c,\alpha}_{m\times m}$	Generalized Gegenbauer -Humberts wavelet operational matrix of fractional integration
I^{α}_{a}	Riemann- Liouville fractional integral
D^{α}_{a}	Riemann- Liouville fractional derivative

LIST OF ABBREVIATIONS

ADM	Adomian's decomposition method
ALW	Approximate long wave equation
FPDEs	Fractional partial differential equations
GHW	Generalized Gegenbauer-Humberts wavelet
KdV	Korteweg-de Vries equation
MB	Modified Boussinseq equation
OHAM	Optimal homotopy asymptotic method
WBK	Whitham- Broer- Kaup equation

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ABSTRACT

On The Solution of Fractional Order Partial Differential Equations with Wavelet Basis Functions

Jumana H.S. ALKHALISSI

Department of Mathematical Engineering
Doctor of Philosophy Thesis

Supervisor: Prof. Dr. Ibrahim EMİROĞLU
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A vast application of partial differential equations in different physical and engineering sciences, and the main role that be playing by fractional differential equations to the best representation of various phenomena and real world problems therefore derived and developed a new numerical techniques is necessity. The aim of this thesis, is to introduce new wavelet technique based on the generalized Gegenbauer- Humbert polynomials; we call this method generalized Gegenbauer- Humbert wavelet. Utilized the proposed method to solve fractional differential equations (linear and non-linear) with initial and boundary- initial conditions. According to this new technique allows us to examine and select the best method to solve the problems under discussion; this method unifies some known wavelet methods in one formula.

The proposed method established the efficiency and accuracy when used to solve fractional differential equations (linear and non-linear) with ordinary, partial and coupled systems of fractional partial differential equations. The performance of our method is analyzed by comparing it with other different numerical methods; the convergence analysis is inspected in addition.

Keywords: The generalized Gegenbauer- Humbert polynomial, operational matrix, fractional partial differential equations, the systems of fractional partial differential equations

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ÖZET

Kesirli Mertebeden Kısmi Diferansiyel Denklemlerin Dalgacık Bazlı Fonksiyonlarla Çözümü

Jumana H.S. ALKHALISSI

Matematik Mühendisliği Anabilim Dalı
Doktora Tezi

Danışman: Prof. Dr. Ibrahim EMİROĞLU
Eş-Danışman: Prof. Dr. Mustafa BAYRAM

Kısmi diferansiyel denklemlerin fizik ve mühendislik bilimlerinde geniş bir uygulaması vardır ve bu tür denklemlerin sayısal çözümleri fizik ve mühendislik problemlerinin çözümünde önemli rol almaktadır.

Bu tezin amacı, genelleştirilmiş Gegenbauer- Humbert polinomlarına dayanan yeni bir yöntem geliştirmektir ve bu yönteme genelleştirilmiş Gegenbauer- Humbert dalgacığı denir. Doğrusal ve doğrusal olmayan başlangıç ve sınır değer problemlerinin çözümü için yeni bir metot önerisi yapılmıştır. Bu yöntem doğrusal ve doğrusal olmayan başlangıç ve sınır değer problemlerinin çözülmesi için geliştirilmiş olup doğruluğu diğer farklı sayısal yöntemlerle karşılaştırılmıştır. Ayrıca önerilen metodun yakınsama analizi de tartışılmıştır.

Anahtar Kelimeler: Genelleştirilmiş Gegenbauer-Humbert polinomu, işlem matrisi, kesirli kısmi diferansiyel denklemler, kesirli kısmi diferansiyel denklem sistemleri

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Review

In spite of, history of the fractional calculus backs to 1695 when L'Hospital asked Lebiniz what does it mean the derivative of order one half, it still attract frequently of curiosity and interest among researchers [1]. The first researcher who gave the fractional derivative particular definition was Laplace in 1812 [2]. Upwards 10 years after 1923 when Abel solved a physical problem (tautochrone problem) by using the fractional operations at the first time, Liouville derived a definition of fractional integration that is called today Liouville formula of fractional integration. The formula of which is under the name Riemann- Liouville fractional integral obtained by N. Ya. Sonin in 1869 [3].

The fractional calculus has played a special role to simplify considerable phenomena in different science and engineering, such as colored noise, control theory, visco-elasticity, electrical networks, fluid mechanics, anomalous diffusion, electromagnetism, etc. In additions, to simulate the behavior some of real- world problems by differential equations with fractional order are more preferable than the integer order for example, influenza A, fractional model of HIV, Dengue and Covid-19 recently. Caputo derivative founded to overtake the troubles that appeared when using Riemann- Liiouville derivative to solve real- world problems.

Owing to the analytical solutions of the fractional differential equations (FDEs) are not available often, the numerical methods to find an approximate solutions are needed. The challenging of investigate and develop techniques to find the solutions of FDEs attract a lot of scientists and researchers recently. some of the recent methods and techniques are transform methods (Laplace [4] and Fourier [5]), the Adomian decomposition method [6–8], homotopy analysis method [9], collocation method [10], homotopy perturbation method [11, 12], Sumudu transform method [13] and variational iteration method [14].

One of the most coming techniques that is used in different sciences and engineering is the orthogonal functions [15, 16] and [17, 18]. Many sets of functions are frequently used such as the Sine–cosine functions, block-pulse functions, Legendre, Laguerre and Chebyshev orthonormal. In the field of sciences and engineering, the orthogonal functions have shown many successes to solve the FDEs such as wavelets method.

Over the last years, methods based on wavelets have been acquiring vast interest for solving differential equations in different sciences and engineering numerically because of their features like orthogonality and capability of representing a various functions with variate levels of resolution. Therefore, developed wavelet to solve difficult problems with accurately numerical algorithms receiving attention of the researchers in the last decades. Wavelet basis is transformed the underlying problem to a system algebraic equations by evaluating the integrals using operational matrices [19] and [20]. Haar wavelet was constructed by Haar in 1909 is the modest of the orthogonal wavelets, Chen et al. [21] was the first who derived the operational matrix of Haar wavelet of fractional integration and used to solve the differential equations. The Legendre and Chebyshev wavelets gained more attractive from a lot of researchers too [20, 22, 23] and CAS wavelet [24]. The generalize of Legendre, Chebyshev and other polynomials is Gegenbauer (ultraspherical) polynomials [25] which are orthogonal on the interval [-1,1]. To obtain the operational matrix for the Gegenbauer wavelet method, Rehman and Saeed [26] did the main role to investigate it. Also, Srivastava et al. [27] applied the Gegenbauer wavelet to find the solution of the fractional Bagley-Torvik equation.

In this thesis, we developed a new algorithm of wavelets based on generalized Gegenbauer- Humbert polynomials to solve fractional partial differential equations. We organized this thesis as follows, we consider some basic mathematical definitions and preliminaries about fractional calculus, orthogonal polynomials, generalized Gegenbauer- Humbert polynomials and wavelets in Chapter 2. In Chapter 3, we have constructed a generalized Gegenbauer- Humbert wavelet abbreviated (GHW) and their operational matrix of fractional integration then are utilized to solve (linear and non-linear) fractional differential equations. We derived the operational matrix of fractional derivative of GHW and enforcement the proposed method for (linear and non- linear) fractional differential equations are described in Chapter 4. The aim of Chapter 5 is to evolve the GHW method for solving the partial fractional differential equations with boundary and initial- boundary conditions. While in Chapter 6, we extend the GHW method to solve systems of partial fractional differential equations. The numerical results of some problems to test the accuracy and efficiency of the proposed method are considered in each above chapters. The conclusion are covered in Chapter 7.

1.2 Objective of the Thesis

The goal of this thesis is to construct a new technique of wavelets for solving partial fractional differential equations based on the orthogonal functions of generalized ultraspherical polynomials. The proposed method unify some of wavelet methods as one formula, therefore allowed to examine which one best to use for solving the problem under studying. This method effort advancing the study on various wavelets in order to solve differential equations of arbitrary order of an effective way and more accurate.

1.3 Hypothesis

This thesis discussed for the first time the following:

- A new modification in the Gegenbauer wavelet method by combinations with other orthogonal polynomial.
- Investigate the operational matrices wither related to integration and derivation of fractional order and utilized to solve fractional differential equations.
- The convergence and error-bound analysis provided in our study to show the credibility of the suggested algorithm and support the mathematical formulation of the algorithm.
- The proposed method compared with other wavelet method and observed that, the proposed algorithm is an efficient tool to tackle the fractional order problems of complex nature.

2
MATHEMATICAL PRELIMINARY

2.1 Fractional Calculus

Fractional calculus history started from attempting to generalize the principle of conventional calculus to arbitrary order. A significant number of authors have shed-light on the fractional calculus, are more suitable to represent different real phenomena including their properties. Numerous definitions of differentiations and integrations of fractional order such as the Riemann–Liouville, the Liouville-Grünwald, the Grünwald-Letnikov, the Hadamarod, the Weyl, the Marchaud, the Hadamard, the Love-Young, the Erdélyi-Kober, the Riesz-Feller and the Caputo fractional derivatives and integrals some of these definitions are equivalent but in general not. Some of these definitions are ineffective owing to the insufficiency performance of representing the initial and boundary conditions containing derivatives of fractional order. The formula of Riemann -Liouville and Caputo definitions are famous and commonly used.

2.1.1 The Euler Gamma Function

Euler generalized the factorial function to non-integer numbers in 1729 which is called the gamma function $\Gamma(\cdot)$ (see [28–30]).

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dt, \; Re(x) \geq -1. \tag{2.1}$$

By integrating Eq.2.1 by parts, given the following formula:

$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} e^{-t}\, t^{x}\, dt, \\ &= \left[e^{-t}\, t^{x}\right]_0^{\infty} + x \int_0^{\infty} e^{-t}\, t^{x-1}\, dt, \\ &= x\,\Gamma(x). \end{aligned} \tag{2.2}$$

Its obvious $\Gamma(1)=1$, therefore repeating Eq.2.2 yield

$$\Gamma(n+1)=n!,\ n\in\mathbb{N}. \tag{2.3}$$

Also, we referred some of properties of the gamma function as the following (see [31], [28] and [29])

- The reflection formula of Euler's gamma function is

$$\Gamma(x)\,\Gamma(x-1)=\frac{\pi}{sin(\pi x)},\ x\in\mathbb{C},\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}. \tag{2.4}$$

- The Legendre duplication formula is

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right),\ x\in\mathbb{C}. \tag{2.5}$$

- The Stirling's formula is

$$\Gamma(x)=\sqrt{2\pi}\,e^{-x}\,x^{x-1/2}\,(1+O(1/x)),\ (|arg(x)|<\pi-\epsilon,|x|\to\infty). \tag{2.6}$$

- For $n\in\mathbb{N}$,

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!}{2^n}\sqrt{\pi},\ (2n-1)!!=1.3\ldots(2n-1). \tag{2.7}$$

- For $x\in\mathbb{C}$

$$\Gamma(x+1)=\frac{e^{x\,(\gamma+\frac{1}{n})}}{x\,\prod_{n=1}^{\infty}(1+\frac{x}{n})}, \tag{2.8}$$

  where γ is the Euler constant.

2.1.2 Beta Function

Euler investigated the definition of beta function in 1772 as (see [29])

$$B(a,b)=\int_0^1 t^{a-1}\,(1-t)^{b-1}dt,\ (\Re(a),\Re(b)>0). \tag{2.9}$$

There is a relation between beta function and gamma function can be expressed as

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \alpha,\beta\notin\mathbb{Z}_0^-. \tag{2.10}$$

The binomial coefficient $\binom{\alpha}{\beta}$ definition is

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}, \tag{2.11}$$

where $\alpha, \beta \in \mathbb{C}, \alpha \notin \mathbb{Z}^{-}$.

2.1.3 Riemann-Liouville Fractional Integral

The generalized form of Cauchy's integral is called Riemann-Liouville fractional integral. The Cauchy's iterated integral formula for m-fold integral where $m \in \mathbb{N}$ is

$$\begin{aligned} \mathrm{I}_a^n g(x) &= \int_a^x \mathrm{I}_a^{n-1} g(\tau)\, d\tau, \quad n = 1, 2, \ldots. \\ &= \frac{1}{(n-1)!} \int_a^x (x-\tau)^{n-1} g(\tau)\, d\tau, \end{aligned} \tag{2.12}$$

and by mathematical induction can be prove it.

Generalized the formula Eq.2.12 by replaced n with an arbitrary number α and use the Gamma function to replace $(n-1)!$ with $\Gamma(\alpha)$, we obtain the following definition of Riemann-Liouville fractional integral

Definition 2.1. The Riemann- Liouville fractional integration operator of order $\alpha \geq 0$ of a function $g(x)$ is defined as [30]:

$$\mathrm{I}_a^{\alpha} g(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} g(\tau) d\tau, & \alpha > 0, \\ g(x), & \alpha = 0. \end{cases} \tag{2.13}$$

for $x \in [a, b]$.

We consider some properties of the Riemann-Liouville fractional integral as :

- For $\gamma > -1$ the Riemann-Liouville fractional integral of the power function $(x-a)^{\gamma}$ is

$$(\mathrm{I}_a^{\alpha}(y-a)^{\gamma})(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}. \tag{2.14}$$

- For $\gamma > 0$ then

$$\mathrm{I}_a^{\alpha}\, \mathrm{I}_a^{\gamma} g(x) = \mathrm{I}_a^{\gamma}\, \mathrm{I}_a^{\alpha} g(x) = \mathrm{I}_a^{\alpha+\gamma} g(x). \tag{2.15}$$

2.1.4 Riemann-Liouville Fractional Derivative

Definition 2.2. The definition of Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}$ is [32], [33]

$$\mathrm{D}_a^\alpha\, g(x) = \mathrm{D}_a^m \mathrm{I}_a^{m-\alpha} g(x) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_a^x (x-\tau)^{m-\alpha-1} g(\tau) d\tau, \tag{2.16}$$

where $m-1 < \alpha < m, m \in \mathbb{N}$ and $m = \lceil \alpha \rceil$.

If $\alpha = 0$ then the Riemann- Liouville fractional derivative represent the identity operator. While the Riemann-Liouville fractional derivative of the function $(x-a)^\gamma$ for $\gamma > -1$ is as

$$(\mathrm{D}_a^\alpha (y-a)^\gamma)(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}. \tag{2.17}$$

2.1.5 Caputo Fractional Derivative

In spite of the main role Riemann- Liouville fractional definitions played in the growth, the theory of fractional calculus and their applications, it has a lack for modelling the real-life phenomena. The procedure of Riemann- Liouville leads to initial conditions having fractional order at lower limits, therefore necessitating an approach to fractional derivatives for modelling real-life problems by utilizing initial conditions with derivatives of integer order. M. Caputo investigated another formula for fractional derivative definition and used it for realizing seismological phenomena in 1967; then in viscoelasticity theory with F. Mainardi in 1969.

Definition 2.3. The Caputo fractional derivative operator of order $\alpha \geq 0$ of a function $g(x)$ is defined as [30]:

$$^C\mathrm{D}_a^\alpha g(x) = \begin{cases} \frac{d^m g(x)}{dt^m}, & \alpha = m \in \mathbb{N} \\ \mathrm{I}_a^{m-\alpha} \left(\frac{d}{dx}\right)^m g(x), & m-1 < \alpha < m, \end{cases} \tag{2.18}$$

where $m = \lceil \alpha \rceil$ and $x > a$.

The following are some properties of Caputo fractional derivatives [34]:

- Let ζ is a constant, then

$$^C\mathrm{D}_a^\alpha \zeta = 0. \tag{2.19}$$

- For $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α.

$$
{}^{C}\mathrm{D}_{a}^{\alpha}(x^{\beta}) = \begin{cases} 0, & \beta \in N \bigcup \{0\} \ and \ \beta < \lceil \alpha \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \beta \in N \bigcup \{0\} \ and \ \beta \geq \lceil \alpha \rceil or \ \beta \notin N and \ \beta < \lfloor \alpha \rfloor . \end{cases} \tag{2.20}
$$

- For $m = \lceil \alpha \rceil, \alpha > 0$

$$
\mathrm{I}_{a}^{\alpha}\left[{}^{C}\mathrm{D}_{a}^{\alpha} g(x) \right] = g(x) - \sum_{n=0}^{m-1} g^{(n)}(a) \frac{(x-a)^{n}}{n!}. \tag{2.21}
$$

- If g is a continuous function, then ${}^{C}\mathrm{D}_{a}^{\alpha}\, \mathrm{I}_{a}^{\alpha} g(x) = g(x)$.

The operator ${}^{C}\mathrm{D}_{a}^{\alpha}$ is represent a linear operator, since,

$$
{}^{C}\mathrm{D}_{a}^{\alpha}(\lambda\, g(x) + \mu\, u(x)) = \lambda\, {}^{C}\mathrm{D}_{a}^{\alpha} g(x) + \mu\, {}^{C}\mathrm{D}_{a}^{\alpha} u(x), \tag{2.22}
$$

where λ and μ are constants.

2.2 The Orthogonal Polynomials

Legendre, who discovered the Legendre polynomials in 1784; since then the orthogonal polynomials have appeared widely in the mathematical and scientific research. The reason of gained a big attention of scientists, features of this technique by reducing a various problems to a system of algebraic equations can be solve easily.

Definition 2.4. (The orthogonality) [35] Let $f(x)$ and $g(x)$ functions, then the inner or scalar product of these functions can be defined by the following integral

$$
\int_{a}^{b} w(x) f(x) g(x)\, dx, \tag{2.23}
$$

where $w(x) \geq 0, a \leq x \leq b$. If the the above integral equal to zero, then we called $f(x)$, $g(x)$ are orthogonal. For n th order polynomials $Q_n(x)$ and satisfies the orthogonality relation

$$
\int_{a}^{b} w(x) Q_n(x) Q_m(x)\, dx, \ \ n \neq m, \tag{2.24}
$$

where $w(x)$ is a weight function and non-negative in the interval (a, b) and the integral is well-defined for all finite order polynomials $Q_n(x)$, these polynomials form a set of orthogonal polynomials. It is obvious that

$$\int_a^b w(x)\,[Q_n(x)]^2\,dx = h_n \geq 0. \tag{2.25}$$

2.2.1 The Generalized Gegenbauer- Humbert Polynomials

The generalized Gegenbauer -Humbert polynomials $P_m^{\lambda,y,c}(x)$,$m \geq 0$, which are defined by the generation function as([36], [37] and [38]):

$$\Phi(t) = (c - 2x\,t + y\,t^2)^{-\lambda} = \sum_{m\geq 0} P_m^{\lambda,y,c}(x)\,t^m, \tag{2.26}$$

where $\lambda > 0$, y and $c \neq 0$ are real number. As a special cases of Eq.(2.26) we consider $P_m^{\lambda,y,c}(x)$ as follows:

- $P_m^{1,1,1}(x) = U_m(x)$, Chebyshev polynomial of the second kind.
- $P_m^{1/2,1,1}(x) = \psi_m(x)$, Legendre polynomial.
- $P_m^{1,1,1}(\frac{x}{2}+1) = B_m(x)$, Morgan- Voyc polynomial.
- $P_m^{1,2,1}(\frac{x}{2}) = \phi_{m+1}(x)$, Fermat polynomial of the first kind.
- $P_m^{1,2a,2}(x) = D_m(x,a)$, Dickson polynomial and $a > 0$ where a is a real parameter.
- If $y = c = 1$, the corresponding polynomials are called Gegenbauer polynomials.

The class of the generalized Gegenbauer -Humbert polynomial sequences satisfy the following recurrence relation [36]:

$$P_m^{\lambda,y,c}(x) = 2\,x\,\frac{\lambda+m-1}{c_m}\,P_{m-1}^{\lambda,y,c}(x) - y\,\frac{2\lambda+m-2}{c_m}\,P_{m-2}^{\lambda,y,c}(x), \forall m \geq 2, \tag{2.27}$$

with initial conditions: $P_0^{\lambda,y,c}(x) = \Phi(0) = c^{-\lambda}$, $P_1^{\lambda,y,c}(x) = \Phi'(0) = 2\,\lambda\,x\,c^{-\lambda-1}$. The generalized Gegenbauer -Humbert polynomial sequence in Eq.(2.27) is an orthogonal polynomial iff $y\,c > 0$.

The explicit formula of generalized Gegenbauer– Humbert polynomial is [39]:

$$P_m^{\lambda,y,c}(x) = \sum_{k=0}^{\lceil n/2 \rceil} \frac{(-y)^k\,c^{-\lambda-n+k}\,(\lambda)_{n-k}\,(2x)^{n-2k}}{k!\,(n-2k)!}, \tag{2.28}$$

where the falling fractional rotation $x^{\underline{n}}$ (some times also denoted $(x)_r$) is defined by:

$$x^{\underline{r}} = x(x-1)^{\underline{r-1}}, (r \geq 1), x^{\underline{0}} = 1.$$

Moreover, the generalized Gegenbauer– Humbert polynomials satisfy the following equalities :

$$\mathrm{D}\, P_n^{\lambda,y,c}(x) = 2\,\lambda\, P_{n-1}^{\lambda+1,y,c}(x), \tag{2.29}$$

$$\mathrm{D}^k\, P_n^{\lambda,y,c}(x) = 2^k\,(\lambda)_k\, P_{n-k}^{\lambda+k,y,c}(x), \tag{2.30}$$

where D represents the standard differentiation operator and $D^k \equiv \frac{d^k}{dt^k}$.

2.3 Wavelets

Alfred Haar was the first mathematician who introduced the Haar functions in his thesis in 1909; then it is called today Haar wavelets. On the other hand, Haar wavelets it still un-useful in some applications due to not giving a smooth curve of representation. In 1982, the geologist Jean Morlet first derived a method dealing with seismic signals that change when it pass different layers of earth by constructing windows for each component of frequency using the dilation, compression or shifting of an individual window. These windows functions called wavelets of constant shape by Morlet. Because of unable Fourier representing varying frequency components throughout the time without being sensitive to any small error, Fourier was replaced with wavelet transforms in the physical and engineering problems. In 1984, Morlet and Grossman published their paper that used wavelet in the first time. Based on the principle that the information gained by different types of wavelets is independent of each other (i.e., orthogonality), Meyer found a new form of wavelet which made the deals with wavelets more easier. Stephane Mallat was a student of higher education under Meyer's supervision, wavelets are implied a multiresolution process at 1986. Ingrid Daubechies played a great role in wavelet theory when he introduced a new class of wavelets functions employing the multiresolution principle in 1988. The suggested method by Daubechies overcomes the jumping that happens when using Haar wavelet (see [40] for more history of wavelets).

Wavelets method have a wide applications in a lot of sciences, and engineering because of their affectively features to model various of problems, for instance, data compression, computer graphics, image processing, wave propagation, differential equations, biomedical technology, etc.

Wavelets constitute a family of functions constructed from dilation and translation of a function called the mother wavelet $\psi(t)$. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi(\frac{t-b}{a}), \quad a,b \in \Re, a \neq 0. \tag{2.31}$$

If we restrict the parameters a and b to discrete values as $a = r_0^{-k}, b = ns_0 r_0^{-k}, r_0 > 1, s_0 > 0$,where n,k are positive integers, the family of discrete wavelets are defined as:

$$\psi_{k,n}(t) = r_0^{k/2}\psi(r_0^k t - ns_0). \tag{2.32}$$

2.3.1 Multiresolution Analysis (MRA)

The basic idea of MRA is to represent a function in $L^2(\Re)$ as successive approximations at different levels of resolution.

Definition 2.5. [32, 41] A set of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ is called a MRA of the Hilbert space $L^2(\Re)$ if satisfied the following properties:

1. $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$.
2. $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\Re)$.
3. $\bigcap_{j\in\mathbb{Z}} V_j = 0$.
4. The orthogonal complement subspace of W_j of V_j in V_{j+1} i.e. $V_{j+1} = V_j \oplus W_j$.
5. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}, \ \forall j \in \mathbb{Z}$.
6. There exists a scaling function $\varphi(t) \in V_0$ such that $\{\varphi(t-n) \mid n \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Properties (2)-(5) explain that $\{V_j\}_{j\in\mathbb{Z}}$ is a nested sequence of subspaces V_j covers $L^2(\Re)$.

2.3.2 The generalized Gegenbauer– Humbert wavelets

Here, we introduce generalized Gegenbauer– Humbert wavelets (GHW),

$$\psi_{n,m}^{y,c}(t) = \psi^{y,c}(k,n,m,t), \tag{2.33}$$

are defined on the interval [0,1], where $k = 1, 2, ...$ is the level of resolution, $n = 1, 2, ..., 2^{k-1}$ is the translation parameter, $m = 0, 1, ..., M-1$ represent the order of the generalized Gegenbauer– Humbert polynomial. It can be defined as follow as:

$$\psi_{n,m}^{y,c}(t) = \begin{cases} \frac{1}{\sqrt{h_m}} 2^{k/2} P_m^{\lambda,y,c}(2^k t - 2n + 1), & \frac{2n-2}{2^k} \leq t \leq \frac{2n}{2^k} \\ 0, & o.w. \end{cases} \tag{2.34}$$

where h_m is the normalization factor defined as in Eq.2.37 and $M > 0$, $y\,c > 0$. Corresponding to each λ, y and c we have a different family of wavelets.

$$h_m = \int_s (P_m^{\lambda,y,c}(t))^2 \, d\mu(t), \forall m \geq 1, \tag{2.35}$$

$$= \left(\frac{y}{c}\right)^m \frac{(\lambda + m - 1)^{\underline{m}} (2\lambda + m - 1)^{\underline{m}}}{m! \, (\lambda + m)^{\underline{m}}} h_0, \tag{2.36}$$

where h_m is the normalization factor defined as follows:

$$h_m = \left(\frac{y}{c}\right)^m c^{-\lambda} \sqrt{c\,y} \, \frac{\sqrt{\pi}\, 2^{(2-2\lambda)} \Gamma(2\lambda + m) \Gamma(\lambda + 1)}{m! \, (\lambda + m) (\Gamma(\lambda))^2 \Gamma(\lambda + \frac{1}{2})}, \tag{2.37}$$

where the falling fractional rotation $x^{\underline{n}}$ (some times also denoted $(x)_r$).

The weight function of the generalized Gegenbauer- Humbert wavelets can be defined as

$$\vartheta_n^{\lambda}(t) = (cy - (2^k t - 2n + 1)^2)^{\lambda - 1/2}. \tag{2.38}$$

2.4 Function Approximations and the Generalized Gegenbauer – Humbert Wavelets Matrix

Theorem 2.1. *A function $f(t) \in L^2(R)$ can be expanded into truncated generalized Gegenbauer -Humbert Wavelets series as:*

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \, \psi_{n,m}^{y,c}(t) = C^T \, \Psi^{y,c}(t), \tag{2.39}$$

where $c_{nm} = \int_{-1}^{1} f(t) \psi_{n,m}^{y,c}(t) \vartheta_n^{\lambda}(t) \, dt$. In addition , C and $\Psi^{y,c}(t)$ are $2^{k-1}M \times 1$ matrices given by:

$$C=[c_{10},c_{11},...,c_{1M-1},c_{20},c_{21},...,c_{2M-1},...,c_{2^{k-1}0},c_{2^{k-1}1},...,c_{2^{k-1}M-1}]^T,$$

$$\Psi^{y,c}(t)=\left[\psi^{y,c}_{10}(t),...,\psi^{y,c}_{1M-1}(t),\psi^{y,c}_{20}(t),...,\psi^{y,c}_{2M-1}(t),...,\psi^{y,c}_{2^{k-1}0}(t),...,\psi^{y,c}_{2^{k-1}M-1}(t)\right]^T. \tag{2.40}$$

To find a numerical approximate solution, we need to build a system of $2^{k-1}M$ algebraic equations by using a collocation points of the generalized Gegenbauer -Humbert wavelets are taken as $t_i=\frac{2i-1}{2^kM}$, where $i=1,2,...,2^{k-1}M$. The (GHW) matrix is given by:

$$\Psi^{y,c}_{2^{k-1}M\times 2^{k-1}M}=\left[\Psi^{y,c}\left(\frac{1}{2^kM}\right),\Psi^{y,c}\left(\frac{3}{2^kM}\right),\cdots,\Psi^{y,c}\left(\frac{2^kM-1}{2^kM}\right)\right], \tag{2.41}$$

or

$$\Psi^{y,c}_{2^{k-1}M\times 2^{k-1}M}=\begin{bmatrix}
\Psi^{y,c}_{1,0}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{1,0}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{1,0}\left(\frac{2^kM-1}{2^kM}\right)\\
\Psi^{y,c}_{1,1}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{1,1}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{1,1}\left(\frac{2^kM-1}{2^kM}\right)\\
\vdots & \vdots & \cdots & \vdots\\
\Psi^{y,c}_{1,M-1}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{1,M-1}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{1,M-1}\left(\frac{2^kM-1}{2^kM}\right)\\
\Psi^{y,c}_{2,0}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{2,0}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{2,0}\left(\frac{2^kM-1}{2^kM}\right)\\
\vdots & \vdots & \cdots & \vdots\\
\Psi^{y,c}_{2,M-1}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{2,M-1}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{2,M-1}\left(\frac{2^kM-1}{2^kM}\right)\\
\vdots & \vdots & \cdots & \vdots\\
\Psi^{y,c}_{2^{k-1},0}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{2^{k-1},0}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{2^{k-1},0}\left(\frac{2^kM-1}{2^kM}\right)\\
\vdots & \vdots & \cdots & \vdots\\
\Psi^{y,c}_{2^{k-1},M-1}\left(\frac{1}{2^kM}\right) & \Psi^{y,c}_{2^{k-1},M-1}\left(\frac{3}{2^kM}\right) & \cdots & \Psi^{y,c}_{2^{k-1},M-1}\left(\frac{2^kM-1}{2^kM}\right)
\end{bmatrix}. \tag{2.42}$$

In particular, we fix $k=2, M=3$, we have $n=1,2$ and $m=0,1,2$, for fix value of $y=3$, $c=1$, $\lambda=12$ the GHW matrix is given as:

$$\Psi^{3,1}_{6\times 6}=\begin{bmatrix}
1.074567 & 1.074567 & 1.074567 & 0. & 0. & 0.\\
-2.108965 & 0. & 2.108965 & 0. & 0. & 0.\\
2.293272 & -.804134 & 2.293272 & 0. & 0. & 0.\\
0. & 0. & 0. & 1.074567 & 1.074567 & 1.074567\\
0. & 0. & 0. & -2.108965 & 0. & 2.108965\\
0. & 0. & 0. & 2.293272 & -.804134 & 2.293272
\end{bmatrix}. \tag{2.43}$$

Similarly, we get different generalized Gegenbauer- Humbert wavelet matrices for different value of y, c and λ.

In the same way, an arbitrary function $u(x,t) \in [0,1) \times [0,1)$ of two variables may be expanded into GHW basis as:

$$u(x,t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i^{y,c}(x) \psi_j^{y,c}(t) = \Psi^{y,c\,T}(x) U \Psi^{y,c}(t), \tag{2.44}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$, $\hat{m} = 2^{k-1}M$.

2.5 GHW Operational matrix of fractional order integration

We write $f(t) \approx C^T \Psi^{y,c}(t)$, an arbitrary function $f \in L_2[0,1)$ can be expanded into block -pulse functions as:

$$f(t) \approx \sum_{i=0}^{m-1} f_i b_i(t) = f^T B(t), \quad m = 2^{k-1}M, \tag{2.45}$$

where f_i is the coefficients of the block -pulse function.The generalized Gegenbauer-Humbert wavelets can be expanded into m-set of block-pulse functions as :

$$\psi^{y,c}(t) = \Psi^{y,c}_{m\times m} B(t). \tag{2.46}$$

The fractional integral of block -pulse function vector can be written as:

$$(\mathrm{I}^{\alpha} B)(t) = F^{\alpha}_{m\times m} B(t), \tag{2.47}$$

where $F^{\alpha}_{m\times m}$ is the block- pulse matrix of integration given in [19] as follows:

$$F^{\alpha}_{m\times m} = \frac{1}{m^{\alpha}\,\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \tag{2.48}$$

$\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$, with

$$P^{y,c,\alpha}_{m\times m} = \Psi^{y,c}_{m\times m}\, F^{\alpha}\, (\Psi^{y,c}_{m\times m})^{-1}, \tag{2.49}$$

where $P^{y,c,\alpha}_{m\times m}$ is the (GHW) operational matrix of integration of fractional order α. In particular,for $k = 2, M = 3$, for fix value of $y = 3, c = 1, \lambda = 5, \alpha = 0.5$ the GHW matrix is given as:

$$P_{6\times 6}^{3,1,0.5} = \begin{bmatrix} 0.53680 & 0.15761 & -0.31336 & 0.43691 & -0.7547 & 0.26957 \\ -0.21066 & 0.22434 & 0.16149 & 0.85907 & -0.44957 & 0.24122 \\ 0.40907 & -0.37608 & 0.16046 & 0.75705 & -0.20247 & 0.10034 \\ 0. & 0. & 0. & 0.53680 & 0.15761 & -0.31336 \\ 0. & 0. & 0. & -0.21066 & 0.22434 & 0.16149 \\ 0. & 0. & 0. & 0.40907 & -0.37608 & 0.16046 \end{bmatrix}. \tag{2.50}$$

3
THE GENERALIZED GEGENBAUER- HUMBERTS WAVELET FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

In this chapter, employ a new method of wavelets (GHW that we presented in Chapter 2), based on our paper [42] to solve linear and non-linear fractional differential equation. The main purpose of this chapter is to introduce new technique of wavelets and applied to convert the FDEs problem to a system of algebraic equations.

3.1 Description of the GHW Technique

Using the (GHW) operational matrix to solve non -linear Riccati fractional equation of the form:

$$\mathrm{D}^{\alpha}u(t) = N(t)u^2 + Q(t)u + R(t), \quad t > 0, \ \ 0 < \alpha \leq 1, \tag{3.1}$$

with the initial condition $u(0) = h$. We suppose that the functions $\mathrm{D}^{\alpha}u(t)$, $N(t)$, $Q(t)$ and $R(t)$ are approximated using (GHW) as follows :

$$\mathrm{D}^{\alpha}u(t) = U^T \Psi^{y,c}(t), \tag{3.2}$$

$$u(t) \approx U^T P^{y,c,\alpha} \Psi^{y,c}(t) + U_0^T \Psi^{y,c}(t) = C^T \Psi^{y,c}(t), \tag{3.3}$$

$$N(t) = V^T \Psi^{y,c}(t), \quad Q(t) = W^T \Psi^{y,c}(t), \quad R(t) = X^T \Psi^{y,c}(t). \tag{3.4}$$

Now, substituting Eqs.(3.2–3.4) in Eq.(3.1), we have

$$U^T \Psi^{y,c}(t) = V^T \Psi^{y,c}(t)[C^T \Psi^{y,c}(t)]^2 + W^T \Psi^{y,c}(t) C^T \Psi^{y,c}(t) + X^T \Psi^{y,c}(t). \tag{3.5}$$

Substituting Eq.2.48 into Eq.3.5, we have

$$C^T \Psi^{y,c}_{m\times m}(t) = V^T [C^T \Psi^{y,c}_{m\times m}(t)]^2 + W^T C^T \Psi^{y,c}_{m\times m}(t) + X^T, \tag{3.6}$$

where V, W, X and $\Psi^{y,c}_{m\times m}(t)$ are known, Eq.3.6 represents a system of a non -linear equations with unknown vector C. This system of non -linear equations can be solved approximately using some numerical methods like Newton iteration methods.

Algorithm:

input: $M \in \mathbb{N}$, $k \in \mathbb{N}\bigcup\{0\}$, $\mu \in \mathbb{N}/\{1\}$, $0 < \alpha \leq 1$ and the functions $N(t)$, $Q(t)$, $R(t)$ and h.

1. Define the basis function $\psi^{y,c}_{nm}$ by Eq.2.34 and the vector $\Psi^{y,c}$ defined in Eq.2.40.
2. Compute the (GHW) matrix $\psi^{y,c}_{m\times m}$ and by Eq.2.46.
3. Compute the (GHW) operational matrix $P^{y,c,\alpha}$ and $P^{y,c,2\alpha}$ using Eq.2.49.
4. Define the unknown matrix $U = [u_{ij}]_{m\times m}$ where $m = \mu^k M$.
5. Compute the vectors V, W, X in Eq.3.3 and Eq.3.4.
6. Solve the non-linear system of algebraic equations in Eq.3.6 for the unknown vector C.

Output: The approximate solution : $u(t) \approx C^T\,\Psi^{y,c}(t)$.

3.1.1 Convergence of the GHW

Theorem. The series $f(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\,\psi^{y,c}_{n,m}(x)$ is converges to $f(x)$, when $2^{k-1}, M \to \infty$.

Proof. To prove this theorem, we will use the fact that is every Cauchy sequence is convergent. Since the wavelet basis represent a family of orthonormal functions in the space $L^2(R)$, take the inner product of $f(x)$ and $\psi^{y,c}_{n,m}(x)$, where

$$c_{nm} = \left\langle f(x), \psi^{y,c}_{n,m}(x)\right\rangle.$$

We assume that $\hat{l} = 2^{k-1}$, $l = 2^{a-1}$, $\hat{d} = M$ and $d = N$, where k, a the resolutions level, and M, N the order of the generalized Gegenbauer -Humbert polynomials.

Let $B_{\hat{l},\hat{d}}$ represent a sequence of partial sums of $c_{ij}\,\psi^{y,c}_{i,j}(x)$, we need to prove that $B_{\hat{l},\hat{d}}$ is a Cauchy sequence converges to $f(x)$ when $\hat{l}, \hat{d} \to \infty$. Firstly, we prove that $B_{\hat{l},\hat{d}}$ is a Cauchy sequence, suppose that $B_{l,d}$ be an arbitrary sums of $c_{ij}\,\psi^{y,c}_{i,j}(x)$ with $\hat{l} > l$, $\hat{d} > d$.

$$\left\|B_{\hat{l},\hat{d}}-B_{l,d}\right\|^2=\left\|\sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1}c_{ij}\ \psi_{i,j}^{y,c}(x)\right\|^2=\left\langle\sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1}c_{ij}\ \psi_{i,j}^{y,c}(x),\ \sum_{s=l+1}^{\hat{l}}\sum_{r=d}^{\hat{d}-1}c_{sr}\ \psi_{s,r}^{y,c}(x)\right\rangle$$

$$=\sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1}\sum_{s=l+1}^{\hat{l}}\sum_{r=d}^{\hat{d}-1}c_{ij}\bar{c_{sr}}\left\langle\psi_{i,j}^{y,c}(x),\psi_{s,r}^{y,c}(x)\right\rangle \tag{3.7}$$

$$=\sum_{i=l+1}^{\hat{l}}\sum_{j=d}^{\hat{d}-1}\left|c_{ij}\right|^2.$$

As $\hat{l},\hat{d}\to\infty$, by the definition of the Bessel's inequality, we have $\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left|c_{ij}\right|^2$ is convergent. This implies $B_{\hat{l},\hat{d}}$ is a Cauchy sequence converges to, say $y(x)\in L^2[0,1)$. Now, to show that $y(x)=f(x)$,

$$\left\langle y(x)-f(x),\psi_{i,j}^{y,c}(x)\right\rangle=\left\langle y(x),\psi_{i,j}^{y,c}(x)\right\rangle-\left\langle f(x),\psi_{i,j}^{y,c}(x)\right\rangle \tag{3.8}$$

$$=\lim_{\hat{l},\hat{d}\to\infty}\left\langle B_{\hat{l},\hat{d}},\psi_{i,j}^{y,c}(x)\right\rangle-c_{ij}=c_{ij}-c_{ij}=0. \tag{3.9}$$

This implies $\sum_{i=l}^{\hat{l}}\sum_{j=0}^{\hat{d}-1}c_{ij}\ \psi_{i,j}^{y,c}(x)$ converges to $f(x)$ as $\hat{l},\ \hat{d}\to\infty$.

3.2 Applications of the GHW

In this section, we implement the GHW method to solve several examples of linear and non -linear fractional differential equations.

Example 3.1. Consider the equation [43]

$$\mathrm{D}^{\alpha}y(t)+y(t)=\frac{\Gamma(3)}{\Gamma(3-\alpha)}\,t^{2-\alpha}+t^2,0<\alpha<1, \tag{3.10}$$

subject to initial condition $y(0)=0$. The exact solution of the above problem is given by $y(t)=t^2$. Now, take the fractional integration of order α of Eq.3.10 as:

$$y(t)=t^2+t^{2+\alpha}\,\frac{\Gamma(3)}{\Gamma(\alpha+3)}-I^{\alpha}y(t). \tag{3.11}$$

Let

$$y(t)=C^T\Psi^{y,c}(t), \tag{3.12}$$

then

$$\mathrm{I}^{\alpha}y(t)=C^T\ \mathrm{I}^{\alpha}\Psi^{y,c}(t)=C^T\ P_{m\times m}^{y,c,\alpha}\,\Psi^{y,c}(t). \tag{3.13}$$

Substituting Eqs.(3.12) and (3.13) into Eq.(3.11), we get the following system of algebraic equations:

$$C^T\Psi^{y,c}(t)=t^2+t^{2+\alpha}\,\frac{\Gamma(3)}{\Gamma(\alpha+3)}-C^T P^{y,c,\alpha}_{m\times m}\,\Psi^{y,c}(t). \tag{3.14}$$

When $\alpha = 0.8$, $\lambda = 9$, $y = 3$, $c = 1$ with $k = 2$, $M = 3$ the operational matrix of integration is:

$$P^{3,1,0.8}_{6\times 6}=\begin{bmatrix} 0.3478675 & 0.1133429 & -0.007008237 & 0.5038849 & -0.02498075 & 0.005500101 \\ -0.2925878 & 0.05265004 & 0.09257616 & 0.04520844 & -0.01490000 & 0.005573679 \\ 0.2267984 & 0.03117652 & 0.02638249 & 0.3789326 & -0.02172524 & 0.005715148 \\ 0. & 0. & 0. & 0.3478675 & 0.1133429 & -0.007008237 \\ 0. & 0. & 0. & -0.2925878 & 0.05265004 & 0.09257616 \\ 0. & 0. & 0. & 0.2267984 & 0.03117652 & 0.02638249 \end{bmatrix} \tag{3.15}$$

To find the unknown vector C by solving the above system of linear equations, where the coefficients vector C if $k = 2$, $M = 3$ is as:

$$C^T=\begin{bmatrix}0.06573362 & 0.04470530 & 0.01152018 & 0.5298675 & 0.1349996 & 0.01147601\end{bmatrix} \tag{3.16}$$

and

$$\Psi^{y,c}(t)=\begin{bmatrix} \frac{\sqrt{2}\,3^{3/4}}{3} \\ \frac{(72\,t-18)\sqrt{30}\,3^{3/4}}{81} \\ \frac{(2880\,t^2-1440\,t+153)\sqrt{418}\,3^{3/4}}{1539} \\ \frac{\sqrt{2}\,3^{3/4}}{3} \\ \frac{(72\,t-54)\sqrt{30}\,3^{3/4}}{81} \\ \frac{(2880\,t^2-4320\,t+1593)\sqrt{418}\,3^{3/4}}{1539} \end{bmatrix} \tag{3.17}$$

Table 3.1 consider the approximate solutions obtained by applying the presented method for $\alpha = 0.8$, $\lambda = 9$, $y = 3$, $c = 1$ with $k = 2$, $M = 3$ and $k = 2$, $M = 5$. For $\alpha = 0.8$ Fig.3.1 shown the results.

Example 3.2. The second example covers the inhomogeneous linear equation

$$D^\alpha y(t)=\frac{2}{\Gamma(3-\alpha)}\,t^{2-\alpha}-\frac{1}{\Gamma(2-\alpha)}\,t^{1-\alpha}-y(t)+t^2-t, 0<\alpha\le 1,\ t<0, \tag{3.18}$$

with initial condition $y(0) = 0$. To solve Eq.3.18, converting the above problem by

Table 3.1 Exact and Approximate solution for different values of k, M in Example 3.1.

t	Exact Solution	GHW Method k=2, M=3	Absolute Error	GHW Method k=2, M=5	Absolute Error
0	0.	-0.20E-4	0.25335E-4	0.13745E-4	0.85887E-5
0.1	0.01	0.9403E-2	0.59943E-3	0.97656E-2	0.23730E-3
0.2	0.04	0.38921E-1	0.10793E-2	0.39597E-1	0.40507E-3
0.3	0.09	0.88533E-1	0.14648E-2	0.89465E-1	0.53493E-3
0.4	0.16	0.15824	0.17561E-2	0.15934	0.64695E-3
0.5	0.25	0.24799	0.20097E-2	0.24929	0.74516E-3
0.6	0.36	0.35778	0.22201E-2	0.35920	0.83500E-3
0.7	0.49	0.48758	0.24133E-2	0.48909	0.91970E-3
0.8	0.64	0.63740	0.25892E-2	0.63899	0.10010E-2
0.9	0.81	0.80724	0.27479E-2	0.80890	0.10801E-2

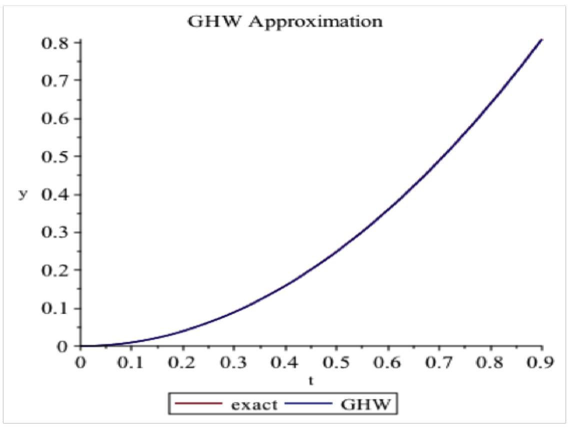


Figure 3.1 Exact and approximate solution of Example 3.1.when $\alpha = 0.8$, $\lambda = 9$, $K = 2$ and $M = 5$.

using the same procedure of GHW in Section 3.1 to the following system:

$$C^T\Psi^{y,c}(t)=t^2+t^{2+\alpha}\,\frac{\Gamma(3)}{\Gamma(\alpha+3)}-t^{(1+\alpha)}\frac{\Gamma(2)}{\Gamma(\alpha+2)}-C^T P^{y,c,\alpha}_{m\times m}\Psi^{y,c}(t). \tag{3.19}$$

Solving the last system for the unknown vector C, we approach to the exact solution that is $y(t)=t^2-t$ see Fig.3.2.

$$\begin{aligned}C^T=[&-0.1666447,-0.04131518,0.009681520,-0.5015621e-6,0.1863884E-7,\\&-0.1674811,0.04101341,0.009677169,-0.3040405E-6,0.1139629E-7],\end{aligned} \tag{3.20}$$

and

$$\Psi^{y,c}(t)=\begin{bmatrix}(1/3)\sqrt{2}\,3^{3/4}\\(2/33)(88\,t-22)\,3^{3/4}\\((128/69)\,t^2-(64/69)\,t+7/69)\sqrt{598}\,3^{3/4}\\((6656/621)\,t^3-(1664/207)\,t^2+(16/9)\,t-68/621)\sqrt{966}*3^{3/4}\\((372736/3105)\,t^4-(372736/3105)\,t^3+(8320/207)\,t^2-(15808/3105)\,t+574/3105)\sqrt{345}*3^{3/4}\\(1/3)\sqrt{2}\,3^{3/4}\\(2/33)\,(88\,t-66)\,3^{3/4}\\((128/69)\,t^2-(64/23)\,t+71/69)\sqrt{598}\,3^{3/4}\\((6656/621)\,t^3-(1664/69)\,t^2+(1232/69)\,t-100/23)\sqrt{966}\,3^{3/4}\\((372736/3105)\,t^4-(372736/1035)\,t^3+(138112/345)\,t^2-(22464/115)\,t+4058/115)\sqrt{345}*3^{3/4}\end{bmatrix} \tag{3.21}$$

For a different values of α when $k=2$, $M=5$, $y=3$, $c=1$ and $\lambda=11$ Table 3.2 shows the absolute errors of the approximate solutions obtained by the GHW method.

Example 3.3. Let consider the following FDE

$$D^{\alpha}y(t)+y(t)+y^2(t)=\frac{8}{3\sqrt{\pi}}\,t^{3/2}+t^2+t^4,\ 0<t<1 \tag{3.22}$$

with initial condition $y(0)=0$ and exact solution when $\alpha=1/2$ is $y(t)=t^2$. For solving the above problem by GHW procedure as: Suppose that

$$y(t)=C^T\Psi^{y,c}(t). \tag{3.23}$$

Table 3.2 The absolute error of the approximate solution in Example 3.2. for a different values of α.

t	**Exact Solution**	**Absolute Error** $\alpha = 0.3$	**Absolute Error** $\alpha = 0.7$	**Absolute Error** $\alpha = 1$
0	0.	0.55395E-2	0.32650E-2	0.12536E-2
0.1	0.01	0.10153E-2	0.12743E-2	0.97553E-3
0.2	0.04	0.21675E-4	0.53432E-3	0.72389E-3
0.3	0.09	0.12233E-3	0.19669E-3	0.49622E-3
0.4	0.16	0.36957E-3	0.79304E-4	0.29023E-3
0.5	0.25	0.40140E-3	0.24908E-3	0.10385E-3
0.6	0.36	0.47173E-3	0.39335E-3	0.64764E-4
0.7	0.49	0.52497E-3	0.51013E-3	0.21733E-3
0.8	0.64	0.56633E-3	0.60549E-3	0.35536E-3
0.9	0.81	0.59949E-3	0.68272E-3	0.48025E-3

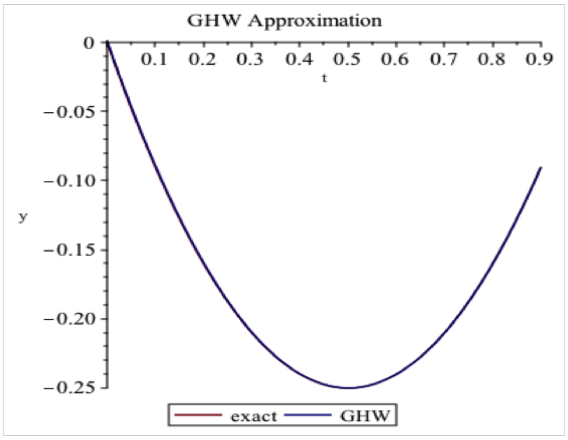


Figure 3.2 Exact and approximate solution of Example 3.2.when $\alpha = 1$, $\lambda = 11$, $K = 2$ and $M = 5$.

By integrating Eq.3.22 of fractional order α, yield that

$$y(t) = \mathrm{I}^{\alpha}\left[\frac{8}{3\sqrt{\pi}}t^{3/2} + t^2 + t^4 - y(t) - y^2(t)\right], \tag{3.24}$$

then

$$C^T\Psi^{y,c}(t) = \frac{8\,\Gamma(\frac{5}{2})}{3\sqrt{\pi}\,\Gamma(\alpha+\frac{5}{2})}\,t^{\frac{3}{2}+\alpha} + \frac{\Gamma(3)}{\Gamma(\alpha+3)}\,t^{2+\alpha} + \frac{\Gamma(5)}{\Gamma(\alpha+5)}\,t^{4+\alpha} - \mathrm{I}^{\alpha}y(t) - \mathrm{I}^{\alpha}y^2(t). \tag{3.25}$$

Note that, by integrating Eq.3.23 of order α, we have

$$\mathrm{I}^{\alpha}\,y(t) = C^T\mathrm{I}^{\alpha}\,(\Psi^{y,c}(t)) + y(0) = C^T\,P^{y,c,\alpha}\,\Psi^{y,c}(t). \tag{3.26}$$

See Fig.3.3 that explain the results obtained by GHW method approach to the exact solutions for $\alpha = .5, c = 1, y = 2, k = 2, M = 5$ and $\lambda = 15$. Table 3.3 shows the

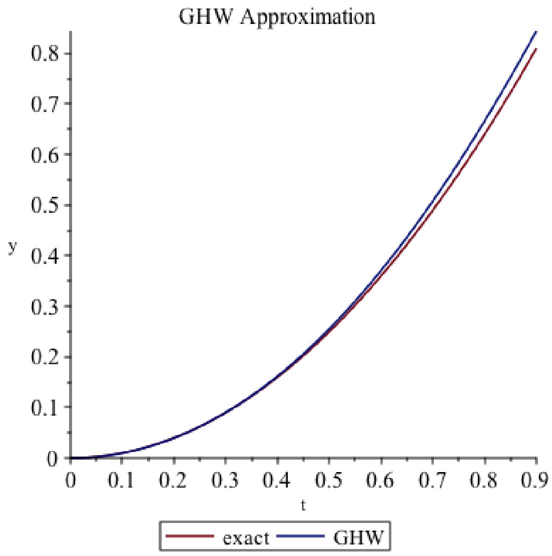


Figure 3.3 Exact and approximate solution of Example 3.3.when $\alpha = 0.5$, $\lambda = 15$, $K = 2$ and $M = 5$.

absolute errors for the GHW method for a different values of α with $y = 2, c = 1, k = 2, M = 5, \lambda = 15$, it is clear the less errors obtained when $\alpha = .5$. While in Table 3.4 we find the absolute error of the present method (GHW) for $\alpha = .5, y = 2, c = 1, k = 2, M = 5$ and for a different values of λ, and the result when $\lambda = 17$ gives the best absolute errors.

Example 3.4. Consider the following fractional order Riccati differential equation

$$\mathrm{D}^{\alpha}y(t) = 1 - y^2(t), 0 < \alpha \leq 1, \tag{3.27}$$

Table 3.3 The absolute error of the approximate solution in Example 3.3. for a different values of α.

t	$\alpha = .25$	$\alpha = .5$	$\alpha = .75$	$\alpha = 1$
0	0.1172514599E-2	**0.297050939E-4**	0.3633578440E-3	0.2469652125E-3
0.1	0.778174150E-2	**0.343969155E-3**	0.5310653779E-2	0.7883891932E-2
0.2	0.2121463137E-1	**0.36323837E-3**	0.1673747724E-1	0.2734067720E-1
0.3	0.360717884E-1	**0.25044733E-3**	0.3097430105E-1	0.5413349666E-1
0.4	0.504128450E-1	**0.19887655E-2**	0.454245600E-1	0.8461789039E-1
0.5	7.26624445	**0.54497237E-2**	0.578701233E-1	0.1155391731
0.6	3.064269124	**0.108002565E-1**	0.657509515E-1	0.1428928519
0.7	2.269968316	**0.179650984E-1**	0.675592881E-1	0.1635426545
0.8	3.470860037	**0.261734180E-1**	0.620599800E-1	0.1737822387
0.9	7.634607550	**0.341873632E-1**	0.493940572E-1	0.1707559887

Table 3.4 The absolute error of the approximate solution in Example 3.3. for a different values of λ.

t	$\lambda = .5$	$\lambda = 1.5$	$\lambda = 5$	$\lambda = 17$
0	0.2970540587E-4,	0.2970522087E-4	0.2970538837E-4	**0.297052474E-4**
0.1	0.343969206E-3	0.343969189E-3	0.343969144E-3	**0.343969175E-3**
0.2	0.36323836E-3	0.36323835E-3	0.36323833E-3	**0.36323838E-3**
0.3	0.25044733E-3	0.25044734E-3	0.25044733E-3	**0.25044730E-3**
0.4	0.19887655E-2	0.19887655E-2	0.19887656E-2	**0.19887654E-2**
0.5	7.238114432	7.238103534	7.23811359	**0.54497228E-2**
0.6	1.045595543	1.045592768	1.045595016	**0.108002562E-1**
0.7	0.6029125468	0.6029127621	0.602911758	**0.179650982E-1**
0.8	1.060967997	1.060967582	1.060968178	**0.261734178E-1**
0.9	7.209375707	7.209383721	7.209375969	**0.341873628E-1**

subject to initial condition

$$y(0) = 0. \tag{3.28}$$

The exact solution for the given problem when $\alpha = 1$ as

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{3.29}$$

The integral representation of Eq.3.27 and the initial condition are given by:

$$y(t) = y(0) + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} - \mathrm{I}^{\alpha} y^2(t). \tag{3.30}$$

Let

$$y(t) = C^T \Psi^{y,c}(t), \tag{3.31}$$

then

$$\mathrm{I}^{\alpha} y(t) = C^T \mathrm{I}^{\alpha} \Psi^{y,c}(t) = C^T P^{y,c,\alpha}_{m \times m} \Psi^{y,c}(t). \tag{3.32}$$

By substituting Eqs.3.31 and Eq.3.32 into Eq.3.30, we get the following system of algebraic equations:

$$C^T \Psi^{y,c}(t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)} - \begin{bmatrix} r_1^2 & r_2^2 & \cdots & r_{2^{k-1}M}^2 \end{bmatrix}, \tag{3.33}$$

where

$$\begin{bmatrix} r_1^2 & r_2^2 & \cdots & r_{2^{k-1}M}^2 \end{bmatrix} = C^T P^{y,c,\alpha}_{m \times m} \Psi^{y,c}_{2^{k-1}M \times 2^{k-1}M}(t). \tag{3.34}$$

Solving the non-linear system for an unknown vector C using the Newton iteration method. When $\alpha = 1$, $\lambda = 7$, $y = 3$, $c = 1$ with $k = 2$, $M = 3$ then the vector of coefficients C is as:

$$\begin{aligned} C^T = [&0.6096543999, 0.2601812904, -0.003785834691, 1.646279549, \\ &0.1665864971, -0.02303331226], \end{aligned} \tag{3.35}$$

and $\Psi^{y,c}(t)$, $P^{y,c,\alpha}_{m\times m}$ are as follow as

$$\Psi^{y,c}(t)=\begin{bmatrix} \frac{\sqrt{14}\,3^{3/4}}{21} \\ \frac{(16\,t-4)\sqrt{42}\,3^{3/4}}{63} \\ \frac{(256\,t^2-128\,t+13)\,\sqrt{210}\,3^{3/4}}{315} \\ \frac{\sqrt{14}\,3^{3/4}}{21} \\ \frac{(16\,t-12)\sqrt{42}\,3^{3/4}}{63} \\ \frac{(256\,t^2-384\,t+141)\,\sqrt{210}\,3^{3/4}}{315} \end{bmatrix}, \tag{3.36}$$

$$P^{3,1,1}_{6\times 6}=\begin{bmatrix} 0.03384441767 & 0.101549499 & 0.1692545804 & 0.203115244 & 0.203115244 & 0.2031152440 \\ -0.05210688975 & -0.1042387918 & -0.05213190206 & 0. & 0. & 0. \\ 0.03592531883 & 0.04565210929 & 0.05534907084 & 0.09129163453 & 0.09129163453 & 0.09129163453 \\ 0. & 0. & 0. & 0.03384441767 & 0.1015494990 & 0.1692545804 \\ 0. & 0. & 0. & -0.05210688975 & -0.1042387918 & -0.05213190206 \\ 0. & 0. & 0. & 0.03592531883 & 0.04565210929 & 0.05534907084 \end{bmatrix}. \tag{3.37}$$

By applying the presented method for $\alpha=1$, $\lambda=7$, $y=3$, $c=1$ with $k=2$, $M=3$ and $k=4$, $M=10$, we obtain the approximate solutions with the absolute error of a different values of α as in the Table 3.5. For $\alpha=1$ Fig. 3.4 shown the results.

Example 3.5 Consider the fractional Riccati differential equation as follow as

$$\mathrm{D}^{\alpha}y(t)=1+2y(t)-y^2(t), \qquad 0<\alpha\le 1, \tag{3.38}$$

subject to the initial condition

$$y(0)=0. \tag{3.39}$$

When $\alpha=1$ the exact solution for above problem is

$$y(t)=1+\sqrt{2}\tanh\left(\sqrt{2}\mathrm{t}+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{3.40}$$

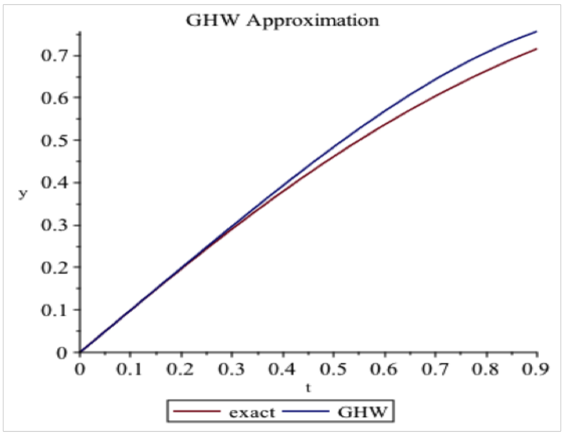


Figure 3.4 Exact and approximate solution when Example 3.4. $\alpha = 1$, $\lambda = 7$, $K = 4$ and $M = 10$.

By applying the same procedure of Example 3.4., we get the following system

$$C^T \Psi^{y,c}(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\, C^T P^{y,c,\alpha}_{m\times m}\, \Psi^{y,c}(t) - \begin{bmatrix} r_1^2 & r_2^2 & \cdots & r^2_{2^{k-1}M} \end{bmatrix}, \tag{3.41}$$

where

$$\begin{bmatrix} r_1^2 & r_2^2 & \cdots & r^2_{2^{k-1}M} \end{bmatrix} = C^T\, P^{y,c,\alpha}_{m\times m}\, \Psi^{y,c}_{2^{k-1}M\times 2^{k-1}M}(t). \tag{3.42}$$

We can find the unknown vector C, by solving the above system of a non-linear

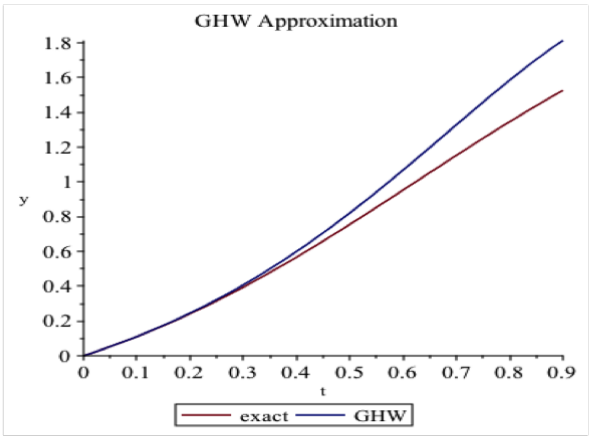


Figure 3.5 Exact and approximate solution when Example 3.5. $\alpha = 1$, $\lambda = 17$, $K = 4$ and $M = 5$.

equations. By applying the presented method for $\alpha = 1$, $\lambda = 17$, $y = 2$ and $c = 1$ with $k = 2$, $M = 5$ and $k = 4$, $M = 5$, we obtain the approximate solutions as in the Table

3.6. The vector of coefficients is:

$$
\begin{aligned}
C^T = [&0.1161583259, 0.02895681359, 0.5753193424E-3, 8.087912914E-6,\\
&-1.172128754E-8, 0.3946240035, 0.03693153239, 0.6967701480E-3,\\
&6.734888070E-6, -1.440913833E-7, 0.7466168465, 0.4619259575E-1,\\
&0.7720826378E-3, 1.975007409E-6, -3.709395724E-7, 1.179515313,\\
&0.05576436980, 0.7328828945E-3, -7.903104667E-6, -6.713842260E-7,\\
&1.688358841, 0.06367223048, 0.4911147820E-3, -0.2331505171E-4,\\
&-9.205477806E-7, 2.246846232, 0.06689887366, -0.2223928272E-4,\\
&-0.4101372289E-4, -8.633566121E-7, 2.800691413, 0.06205879703,\\
&-0.7830232256E-3, -0.5253842127E-4, -2.479331225E-7, 3.270835572,\\
&0.04703560937, -0.1606302758E-2, -0.4726610375E-4, 8.417351300E-7]
\end{aligned}
\tag{3.43}
$$

Fig.3.5 shown the results when $\alpha = 1$, $\lambda = 17$, $y = 2$, $c = 1$, $k = 4$ and $M = 5$. While in Table 3.7 and Table 3.8 we obtained the absolute error of a different values of y and c and these results obtained with $k = 4$, $M = 5$ and $\lambda = 17$, we can see the change of values of y and c for this example there is no a big different of error.

Table 3.5 Exact and Approximate solution for a different values of k, M, α in Example 3.4

t	**Exact solution**	**GHW Method k=2, M=3**	**Absolute Error** $\alpha = 1$	**GHW Method k=4, M=10,**	**Absolute Error** $\alpha = 0.5$	**Absolute Error** $\alpha = 0.7$	**Absolute Error** $\alpha = 1$
0	0	-0.158918E-2	0.15898E-2	-0.9100E-9	0.376174122E-1	0.71337195E-2	0.91000E-9
0.1	0.9968E-1	0.100090	0.422740E-3	0.999748E-1	0.2474934483	0.1188956408	0.30681E-3
0.2	0.197375	0.199739	0.236404E-2	0.199599	0.2710640905	0.1523423472	0.22245E-2
0.3	0.291312	0.297356	0.604409E-2	0.297984	0.2513949822	0.1613220518	0.66715E-2
0.4	0.379948	0.392942	0.129938E-1	0.393678	0.2104880241	0.1540577586	0.13729E-1
0.5	0.462117	0.480994	0.188776E-1	0.484752	0.1604248546	0.1347100156	0.22635E-1
0.6	0.537049	0.568240	0.311910E-1	0.568943	0.996134238E-1	0.1055419192	0.31893E-1
0.7	0.604367	0.643129	0.387615E-1	0.643852	0.500931258E-1	0.709009684E-1	0.39484E-1
0.8	0.664036	0.705660	0.416241E-1	0.707212	0.101737943E-1	0.138421720E-1	0.43177E-1
0.9	0.716297	0.755835	0.395376E-1	0.757152	0.1347159371	0.443972316E-1	0.40846E-1

Table 3.6 Exact and Approximate solution for the different values of k and M in Example 3.5

t	Exact ESolution	GHW Method k=2, M=5	Absolute Error	GHW Method k=4, M=5	Absolute Error
0	0.	0.222006E-2	0.22200E-2	0.156191E-3	0.156192E-3
0.1	0.110295	0.114168	0.38735E-2	0.110886	0.591400E-3
0.2	0.241976	0.249936	0.79599E-2	0.245629	0.365301E-2
0.3	0.395104	0.413538	0.18433E-1	0.407985	0.128802E-1
0.4	0.567812	0.606961	0.39149E-1	0.600243	0.324313E-1
0.5	0.756014	0.830446	0.74431E-1	0.822016	0.660017E-1
0.6	0.953566	1.076700	0.123134	1.068512	0.114946
0.7	1.152948	1.336847	0.183898	1.328861	0.175912
0.8	1.346363	1.591813	0.245450	1.585271	0.238907
0.9	1.526911	1.818491	0.291579	1.814090	0.287179

Table 3.7 The absolute error of the approximate solution in Example 3.5. for a different values of c.

t	**Absolute Error y=2,c=1**	**Absolute Error y=2,c=2**	**Absolute Error y=2,c=3**
0	0.1561919E-3	0.1561919E-3	0.1561919E-3,
0.1	0.5914011E-3	0.5914011E-3	0.5914011E-3
0.2	0.3653015E-2	0.3653015E-2	0.3653015E-2
0.3	0.1288026E-1	0.1288026E-1	0.1288026E-1
0.4	0.3243133E-1	0.3243133E-1	0.3243133E-1
0.5	0.6600171E-1	0.6600171E-1	0.6600171E-1
0.6	0.1149466	0.1149466	0.1149466
0.7	0.1759123	0.1759123	0.1759123
0.8	0.2389076	0.2389076	0.2389076
0.9	0.2871790	0.2871790	0.2871790

Table 3.8 The absolute error of the approximate solution in Example 3.5. for a different values of y.

t	**Absolute Error y=1,c=2**	**Absolute Error y=2,c=2**	**Absolute Error y=3,c=2**
0	0.1561919E-3	0.1561919E-3	0.1561919E-3,
0.1	0.5914011E-3	0.5914011E-3	0.5914011E-3
0.2	0.3653015E-2	0.3653015E-2	0.3653015E-2
0.3	0.1288026E-1	0.1288026E-1	0.1288026E-1
0.4	0.3243133E-1	0.3243133E-1	0.3243133E-1
0.5	0.6600171E-1	0.6600171E-1	0.6600171E-1
0.6	0.1149466	0.1149466	0.1149466
0.7	0.1759123	0.1759123	0.1759123
0.8	0.2389076	0.2389076	0.2389076
0.9	0.2871790	0.2871790	0.2871790

4
A NEW OPERATIONAL MATRIX OF FRACTIONAL DERIVATIVE BASED ON THE GENERALIZED GEGENBAUER- HUMBERT POLYNOMIALS TO SOLVE FRACTIONAL DIFFERENTIAL EQUATIONS

The intention of this chapter, develops a new operational matrix of fractional derivative based on the generalized Gegenbauer– Humbert polynomials and employ for solving linear and non-linear FDEs. The proposed method allows to examine some types of wavelets by one formula and choose the best approach to the exact solutions accurately.

The most common types of wavelets used to solve fractional differential equations based on their polynomials are Legendre, Chebyshev, Leguare and Bernoulli. For instance, the operational matrix of the fractional derivative of Chebyshev wavelets was used to solve Bagley– Trovik equations in [44]. Secer and Altun [45] introduced a new operational matrix for the fractional derivatives of Legendre wavelet to solve systems of FDEs. Chang and Isah applied the Legendre wavelet operational matrix of the fractional derivative to solve the Brusselator system of fractional order [34]. For FDEs with variable order, Heydari employed Chebyshev wavelets to find the solution [46]. Kumar et al. used the operational matrix of the Haar wavelet to solve the Lotka–Volterra model having a fractional order [47].

Moreover, this study aims to derive and investigate operational matrix of fractional derivative to be source gives chance for researcher utilize to solve different problems in the future.

4.1 Operational Matrix of The Derivative

In this part, we derived and developed a new operational matrices for the derivatives (integer or fractional) order.

Theorem 4.1. *Assume that the generalized Gegenbauer– Humbert polynomial $P_m^{\lambda,y,c}(t)$ is defined on [-1,1], then these polynomials satisfied the relation below:*

$$\mathrm{D}_t[P_m^{\lambda,y,c}(t)] = \sum_{\substack{k=0 \\ m+k\, odd}}^{m-1} \frac{2}{c}(k+\lambda)\gamma_{n,k} P_k^{\lambda,y,c}(t), \tag{4.1}$$

where

$$\gamma_{n,k} = \begin{cases} \frac{y}{c} & m \geq 3 \\ 1 & o.w. \end{cases}, \quad n = 0,1,...,m-3. \tag{4.2}$$

Proof. Let consider a function $h(t)$, that is approximated by generalized Gegenbauer–Humbert polynomial as follows:

$$h(t) = \sum_{k=0}^{\infty} \tilde{h}_k P_k^{\lambda,y,c}(t). \tag{4.3}$$

Derived both sides of Eq.4.3 with respect to t, given as the following form

$$\mathrm{D}_t h(t) = \sum_{k=0}^{\infty} \tilde{h}_k^{(1)} P_k^{\lambda,y,c}(t), \tag{4.4}$$

where $\tilde{h}_k^{(1)}$ is defined as:

$$\tilde{h}_k^{(1)} = \frac{2}{c}(k+\lambda) \sum_{\substack{q=m+1 \\ q+k\, odd}}^{m-1} \gamma_{n,k} \tilde{h}_q. \tag{4.5}$$

Next, taking into account that $h(t) = P_m^{\lambda,y,c}(t)$ into Eq.4.3, we obtained $\tilde{h}_l = 0$ for $l \neq m$ and $\tilde{h}_m = 1$, then

$$\tilde{h}_k^{(1)} = \begin{cases} \frac{2}{c}(k+\lambda)\gamma_{n,k} & \text{for } m+k \ odd,\ k \leq m-1, \\ 0 & o.w. \end{cases} \tag{4.6}$$

By the means of the above calculation of $\tilde{h}_k^{(1)}$ in Eq.4.4, we get the result in Eq.4.1. ∎

Theorem 4.2. *Suppose the vector of generalized Gegenbauer– Humbert wavelets that be defined as Eq.2.40. The derivative of $\Psi^{y,c}(t)$ satisfy the relation as follows:*

$$\mathrm{D}_t \Psi^{y,c}(t) = \mathfrak{D}\, \Psi^{y,c}(t), \tag{4.7}$$

where $\mathfrak{D}$ represent an operational matrix of derivative with $2^{k-1}M$ as the following form:

$$\mathfrak{D} = \begin{bmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda \end{bmatrix}, \tag{4.8}$$

here Λ is a matrix of order $M \times M$ and the (r,s)-th elements defined in Eq.4.9:

$$\Lambda_{r,s} = \begin{cases} \dfrac{2^{k+1}(s+\lambda-1)c^{-1}\gamma_{m,s}}{\sqrt{(\frac{y}{c})^{r-s}\frac{\Gamma(2\lambda+r-1)(\lambda+s-1)\Gamma(s)}{\Gamma(r)\Gamma(2\lambda+s-1)(\lambda+r-1)}}}, & r = 2,3,...,M; s = 1,2,...,r-1 \, and \, (r+s) \, odd \\ 0 & o.w. \end{cases} \tag{4.9}$$

where $\gamma_{m,s}$ defined as in Eq.4.2.

Proof. Assume that the r-th element of the GHW vector $\Psi^{y,c}(t)$ is given as follows:

$$\psi_r^{y,c}(t) = \psi_{n,m}^{y,c}(t) = \frac{1}{\sqrt{h_m}}\, 2^{k/2} P_m^{\lambda,y,c}(2^k t - \hat{n})\chi_{[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]}, \quad \text{for } r = 1,2,...,2^{k-1}M, \tag{4.10}$$

here $\hat{n} = 2n-1$, $r = M(n-1)+m+1$ and $\chi_{[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]}$ is the characteristic function which is defined by:

$$\chi_{[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]} = \begin{cases} 1 & t \in [\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}], \\ 0 & o.w. \end{cases} \tag{4.11}$$

The form in Eq.4.12 is the result after differentiate Eq.4.10 with respect to t.

$$\mathrm{D}_t \psi_r^{y,c}(t) = \frac{2^{k/2}}{\sqrt{h_m}}\, 2^k \, [P_m^{\lambda,y,c}(2^k t - \hat{n})]' \chi_{[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]}. \tag{4.12}$$

Outside the interval $[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}]$ the characteristic function equal zero, therefore, the generalized Gegenbauer– Humbert wavelets extension includes the elements of

$\Psi^{y,c}(t)$, that are non zero in the interval $[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}]$ which are

$$\psi_i^{y,c}(t);\ i = M(n-1)+1, M(n-1)+2, ..., M(n-1)+M. \tag{4.13}$$

Subsequently,the GHW expansions takes the form as follows:

$$\mathtt{D}_t[\psi_r^{y,c}(t)] = \sum_{i=M(n-1)+1}^{Mn} b_i\ \psi_i^{y,c}(t). \tag{4.14}$$

The matrix D in Eq.4.8 proceeds by the above expression.

Furthermore, $[P_0^{\lambda,y,c}(t)]' = 0$ then $[\psi_r^{y,c}(t)]' = 0$ when $r = 1, M+1, 2M+1, 3M+1, ..., (2^{k-1}-1)M+1$. Thus, the first row of matrix Λ is zero. By means of the relation Eq.4.1 in Eq.4.10 we get the relation below:

$$\mathtt{D}_t\psi_r^{y,c}(t) = \frac{2^{k/2}}{\sqrt{h_m}}\, 2^{k+1} \sum_{\substack{q=0 \\ q+l\, odd}}^{l-1} \frac{1}{c}(q+\lambda)\gamma_{n,q}\, P_q^{\lambda,y,c}(t), \chi_{[\frac{\hat{n}-1}{2^k}, \frac{\hat{n}+1}{2^k}]}. \tag{4.15}$$

After extending the expression Eq.4.15 using the GHWs basis $\Psi^{y,c}(t)$, we get the required result:

$$\mathtt{D}_t\psi_r^{y,c}(t) = 2^{k+1} \sum_{\substack{s=1 \\ r+s\, odd}}^{r-1} (s+\lambda-1)\, \frac{c^{-1}\gamma_{n,s}}{\sqrt{(\frac{y}{c})^{r-s}\frac{\Gamma(2\lambda+r-1)(\lambda+s-1)\Gamma(s)}{\Gamma(r)\Gamma(2\lambda+s-1)(\lambda+r-1)}}}\, \psi_{M(n-1)+s}^{y,c}(t). \tag{4.16}$$

Then consider $\Lambda_{r,s}$, such that

$$\Lambda_{r,s} = \begin{cases} \frac{2^{k+1}(s+\lambda-1)c^{-1}\gamma_{m,s}}{\sqrt{(\frac{y}{c})^{r-s}\frac{\Gamma(2\lambda+r-1)(\lambda+s-1)\Gamma(s)}{\Gamma(r)\Gamma(2\lambda+s-1)(\lambda+r-1)}}} & r = 2,3,...,M; s = 1,2,...,r-1\, and\ (r+s)\ odd \\ 0 & o.w. \end{cases} \tag{4.17}$$

■

If $k=2, M=3$, then the matrix below is the result of $\mathfrak{D}$:

$$\mathfrak{D}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4\sqrt{2(1+\lambda)}}{\sqrt{c\,y}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8\sqrt{2(1+\lambda)(2+\lambda)}}{\sqrt{c\,y\,(2\lambda+1)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4\sqrt{2(1+\lambda)}}{\sqrt{c\,y}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8\sqrt{2(1+\lambda)(2+\lambda)}}{\sqrt{c\,y\,(2\lambda+1)}} & 0 \end{bmatrix}. \tag{4.18}$$

Corollary 4.1. *By using Eq.4.7, the operational matrix of the GHW vector* $\Psi^{y,c}(t)$ *for the n-th order can be obtained as follows:*

$$\mathrm{D}_t^n \Psi^{y,c}(t) = \mathrm{D}^n\, \Psi^{y,c}(t). \tag{4.19}$$

To investigate the operational matrix for the derivative of fractional order, we defining the piecewise functions in [0,1] as in below:

$$\omega_{n,m}=\begin{cases} t^m & t\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}] \\ 0 & o.w. \end{cases}, \tag{4.20}$$

where $n=1,2,...,2^{k-1}$ and $m=0,1,...,M-1$. The set of piecewise functions in the above are not normalized and can be expressed $2^{k-1}M$-th set of these functions as Eq.4.21

$$\Xi(t)=\{\omega_1,\omega_2,\omega_3\cdots,\omega_{2^{k-1}M}\}. \tag{4.21}$$

Here, $\omega_r=\omega_{n,m}$ and the relation $r=M(n-1)+m+1$ help us to get r-th index.

Theorem 4.3. *Suppose that the* $\Xi(t)$ *be a vector defined in Eq.4.21 and*

$$\Xi(t)=\Theta\,\Psi^{y,c}(t), \tag{4.22}$$

where Θ *represent a matrix with* $2^{k-1}M \times 2^{k-1}M$ *order takes the following form:*

$$\Theta = \begin{bmatrix} \rho_1 & 0 & 0 & \cdots & 0 \\ 0 & \rho_2 & 0 & \cdots & 0 \\ 0 & 0 & \rho_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \rho_{2^{k-1}} \end{bmatrix}, \tag{4.23}$$

with

$$\rho_n = \begin{bmatrix} \phi(0,0) & \phi(0,1) & \phi(0,2) & \cdots & \phi(0,M-1) \\ \phi(1,0) & \phi(1,1) & \phi(1,2) & \cdots & \phi(1,M-1) \\ \phi(2,0) & \phi(2,1) & \phi(2,2) & \cdots & \phi(2,M-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi(M-1,0) & \phi(M-1,1) & \phi(M-1,2) & \cdots & \phi(M-1,M-1) \end{bmatrix}, \tag{4.24}$$

then prove the following relation:

$$\begin{aligned} \phi(l,p) = {} & \frac{2^{k/2}}{2^{(l+1)k}\sqrt{h_p}} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-y)^k \, c^{-\lambda+p+k} \, 2^{p-2k} \, \Gamma(\lambda+p-k)}{k! \, (p-2k)! \, \Gamma(\lambda)} \\ & \times \sum_{q=0}^{l} \binom{l}{q} \frac{(\hat{n})^{l-1}}{2} \frac{(1-(-1)^{p-2k+q}) \, \Gamma(\frac{p}{2}-k+\frac{q}{2}+\frac{1}{2}) \, \Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{p}{2}-k+\frac{q}{2}+1)}. \end{aligned} \tag{4.25}$$

Proof. Let

$$\Xi(t) = \Theta \, \Psi^{y,c}(t). \tag{4.26}$$

Using the means of Theorem 2.1, we have the following relation:

$$\phi(l,p) = \int_{-1}^{1} \omega_{n,l}(t)\,\psi_{l,p}^{y,c}(t)\,\vartheta_n^{\lambda}(t)\,dt = \frac{2^{k/2}}{\sqrt{h_p}} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-y)^k\, c^{-\lambda+p+k}\, 2^{p-2k}\,\Gamma(\lambda+p-k)}{k!\,(p-2k)!\,\Gamma(\lambda)} \times \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (2(2^k\,t-\hat{n}))^{p-2k}\, t^l\, (1-(2^k\,t-\hat{n})^2)^{\lambda-1/2}\, dt. \tag{4.27}$$

Next, let us substitute $\tau = 2^k\,t - \hat{n}$ implies that $dt = 2^{-k} d\tau$, then we have

$$\begin{aligned} &\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} (2(2^k\,t-\hat{n}))^{p-2k}\, t^l\, (1-(2^k\,t-\hat{n})^2)^{\lambda-1/2}\, dt \\ &= \frac{2^{p-3k}}{2^{kl}} \int_{-1}^{1} \tau^{p-2k}\,(\tau+\hat{n})^l (1-\tau^2)^{\lambda-1/2}\, d\tau \\ &= \frac{1}{2^{(l+1)k}} \sum_{q=0}^{l} \binom{l}{q} \frac{(\hat{n})^{l-1}}{2} \frac{(1-(-1)^{p-2k+q})\,\Gamma(\frac{p}{2}-k+\frac{q}{2}+\frac{1}{2})\,\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{p}{2}-k+\frac{q}{2}+1)}. \end{aligned} \tag{4.28}$$

Using the relation Eq.4.28 in Eq.4.27, we achieve the required result Eq.4.25. ■

As example of matrix Θ when $y, c = 1, k = 2$ and $M = 3$, we consider the following matrix:

$$\Theta = \frac{\pi\, 2^{1-2\lambda}\,\lambda^2\,\Gamma(2\lambda)}{\Gamma(\lambda)^2\,\sqrt{2\lambda}\,(\lambda+1)} \begin{bmatrix} \frac{(\lambda+1)}{\lambda^2} & 0 & 0 & 0 & 0 & 0 \\ \frac{(\lambda+1)}{4\lambda^2} & \frac{\sqrt{2}}{4\sqrt{\lambda+1}} & 0 & 0 & 0 & 0 \\ \frac{2\lambda+3}{32\lambda^2} & \frac{\sqrt{2}}{8\sqrt{\lambda+1}} & -\frac{(-2\lambda-1)\sqrt{2\lambda+1}}{32(\lambda+2)\sqrt{\lambda+1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(\lambda+1)}{\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{3(\lambda+1)}{4\lambda^2} & \frac{\sqrt{2}}{4\sqrt{\lambda+1}} & 0 \\ 0 & 0 & 0 & \frac{18\lambda+19}{32\lambda^2} & \frac{3\sqrt{2}}{8\sqrt{\lambda+1}} & -\frac{(-2\lambda-1)\sqrt{2\lambda+1}}{32(\lambda+2)\sqrt{\lambda+1}} \end{bmatrix}. \tag{4.29}$$

Lemma 4.1. *The differentiation of fractional order α of relation Eq.4.20 is defined as below:*

$$_0D_t^{\alpha}\,\omega_{n,m}(t) = \begin{cases} \frac{m!}{\Gamma(m-\alpha+1)}\,t^{m-\alpha} & m=\delta,\delta+1,\cdots,M-1,\ t\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}] \\ 0 & o.w. \end{cases}, \tag{4.30}$$

where $(\delta-1) < \alpha < \delta$ is a positive function.

Proof. The proof is simple by using the expression Eq.2.20. ■

Lemma 4.2. *Assume that* $[\Xi(t)]$ *is the vector which is defined by Eq.4.21, The fractional differentiation of order* α *is:*

$$ {}_0D_t^{\alpha}\,[\Xi(t)] = P^{\alpha}\,[\Xi(t)], \tag{4.31}$$

where $(\delta-1)<\alpha<\delta$ *is a positive function in [0,1]. P^{α} is a matrix order* $2^{k-1}M$ *as the following definition :*

$$P^{\alpha} = t^{-\alpha}\begin{bmatrix} \Omega^{\alpha} & 0 & 0 & \cdots & 0 \\ 0 & \Omega^{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \Omega^{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Omega^{\alpha} \end{bmatrix}, \tag{4.32}$$

where Ω^{α} *is the matrix of* $M\times M$ *order defined as following:*

$$\Omega^{\alpha} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\delta!}{\Gamma(\delta-\alpha+1)} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \frac{(\delta+1)!}{\Gamma(\delta-\alpha+2)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \frac{(M-2)!}{\Gamma(M-\alpha-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{(M-1)!}{\Gamma(M-\alpha)} \end{bmatrix}. \tag{4.33}$$

Proof. Using the means of Lemma 4.1 to prove this lemma. ■

Theorem 4.4. *Let* $\Psi^{y,c}(t)$ *be the GHW vector defined in Eq.2.40 and* $(\delta-1)<\alpha<\delta$ *is* *a positive function defined in [0,1]. Then the fractional differentiation of order α in the Caputo sense of GHW can be as below:*

$$ {}_0D_t^{\alpha}\,\Psi^{y,c}(t) = \Phi^{\alpha}\,\Psi^{y,c}(t) = (\Theta^{-1}\,P^{\alpha}\,\Theta)\,\Psi^{y,c}(t), \tag{4.34}$$

where Θ is defined in Eq.4.23 as the coefficients matrix, the operational matrix P^{α} of order α is defined in Eq.4.32 for piecewise functions and Φ^{α} is the operational matrix of fractional order α for the GHW.

Proof. By consider the equation Eq.4.22 and Lemma 4.2, we get

$$\Psi^{y,c}(t) = \Theta^{-1}\,\Xi(t), \tag{4.35}$$

and then

$$ {}_0\mathrm{D}_t^{\alpha}\,\Psi^{y,c}(t) = \Theta^{-1}\,{}_0\mathrm{D}_t^{\alpha}\,\Xi(t) = \Theta^{-1}\,P^{\alpha}\,\Xi(t) = (\Theta^{-1}\,P^{\alpha}\,\Theta)\,\Psi^{y,c}(t), \tag{4.36}$$

which is the required result. ■

For $\lambda, y, c = 1, k = 2$ and $M = 3$ the matrix Φ^{α} is given as :

$$\Phi^{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{4\Gamma(2-\lambda)} & \frac{1}{\Gamma(2-\lambda)} & 0 & 0 & 0 & 0 \\ \frac{-1}{8\Gamma(2-\lambda)} + \frac{3}{32\Gamma(3-\lambda)} & \frac{1}{2\Gamma(2-\lambda)} - \frac{1}{\Gamma(3-\lambda)} & \frac{2}{\Gamma(3-\lambda)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-3}{4\Gamma(2-\lambda)} & \frac{1}{\Gamma(2-\lambda)} & 0 \\ 0 & 0 & 0 & \frac{-9}{8\Gamma(2-\lambda)} + \frac{35}{32\Gamma(3-\lambda)} & \frac{3}{2\Gamma(2-\lambda)} - \frac{3}{\Gamma(3-\lambda)} & \frac{2}{\Gamma(3-\lambda)} \end{bmatrix}. \tag{4.37}$$

4.2 Error Estimates

It is interesting to know that the error bound for the presented algorithm.

Theorem 4.5. *Let us consider $f(t) \in C^M[0,1]$ where $t \in [0,1]$. Consider*

$$\sigma_n = Span\left\{\psi_{n0}^{y,c}, \psi_{n1}^{y,c}, ..., \psi_{nM-1}^{y,c}\right\}, \tag{4.38}$$

where $n = 1, \cdots, 2^{k-1}$ and $f(t) = \sum_{n=1}^{2^{k-1}} f_n(t)$. If $C^T\,\Psi^{y,c}$ represent the better approxi-

mation of $f_n(t)$ out of σ_n, thus

$$\left\|f(t)-C^T\Psi^{y,c}\right\|_{L^2[0,1]} \leq \frac{\delta\sqrt{2}}{M!\,2^{M(k-1)}\sqrt{2M+1}}, \tag{4.39}$$

where $C,\Psi^{y,c}$ as the matrices defined in Eq.2.40 and

$$\delta = max_{t\in[0,1]}\left|f^{(M)}(t)\right| \tag{4.40}$$

Proof. Let be consider Taylor series formula and applied for $f_n(t)$

$$\hat{f}_n(t) = f_n\left(\frac{\hat{n}-1}{2^k}\right)+f_n'\left(\frac{\hat{n}-1}{2^k}\right)\left(t-\frac{\hat{n}-1}{2^k}\right)+\cdots+f_n^{(M-1)}\left(\frac{\hat{n}-1}{2^k}\right)\frac{\left(t-\frac{\hat{n}-1}{2^k}\right)^{M-1}}{(M-1)!}. \tag{4.41}$$

And, we know that

$$\left|f_n(t)-\hat{f}_n(t)\right| \leq \left|f^{(M)}(t)\right|\frac{(t-\frac{\hat{n}-1}{2^k})^M}{(M+1)!},\ \exists t\in\left[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}\right). \tag{4.42}$$

As a result of hypothesis that $C^T\Psi^{y,c}$ is the best approximation of $f_n(t)\in\sigma_n$ and $\hat{f}_n(t)\in\sigma_n$. Therefore by the means of Eq.4.42 ,we obtain

$$\begin{aligned}\left\|f_n(t)-C^T\Psi^{y,c}(t)\right\| \leq \left\|f_n(t)-\hat{f}_n(t)\right\|_2^2 &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}(f_n(t)-\hat{f}_n(t))^2dt\\ &\leq \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}\left(\left|f^{(M)}(t)\right|\frac{(t-\frac{\hat{n}-1}{2^k})^M}{M!}\right)^2dt\\ &\leq \left(\frac{\delta}{M!}\right)^2\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}\left(t-\frac{\hat{n}-1}{2^k}\right)^{2M}dt\\ &= \left(\frac{\delta}{M!}\right)^2\frac{2}{2^{2M(k-1)}(2M+1)}.\end{aligned} \tag{4.43}$$

Next, taking the square root for the last relation we get the required result when k,M approaching ∞. ■

4.3 Proposed Methodology

In this section, devoted to explain the steps of the algorithm by utilizing the operational matrix for fractional derivative to find the solutions of differential equations with fractional order.

4.3.1 Linear Fractional Differential Equation

Consider the linear differential equation with fractional order of the form:

$$\mathrm{D}^{\eta}u(t)=\sum_{i=1}^{k}a_i\mathrm{D}^{\alpha_i}u(t)+a_0\,u(t)+g(t),\quad t\in(0,L), \tag{4.44}$$

with initial conditions

$$u^{j}(0)=b_j,\; j=0,\cdots,\nu-1, \tag{4.45}$$

where $a_i, i=0,\cdots,k$ are real constant coefficients,

$$\nu-1<\eta\leq\nu, 0<\alpha_1<\alpha_2<\cdots<\alpha_k<\eta, \tag{4.46}$$

b_j is the initial values of $u(t)$ and $g(t)$ is a given function. Now, to solve the fractional differential problem with initial values, Eq.4.44 and Eq.4.45 first step is approximate the unknown function $u(t)$ and $g(t)$ by the GHWs as:

$$\begin{aligned} u(t) &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\,\psi^{y,c}_{n,m}(t)=C^T\,\Psi^{y,c}(t),\\ g(t) &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} G_{nm}\,\psi^{y,c}_{n,m}(t)=G^T\,\Psi^{y,c}(t), \end{aligned} \tag{4.47}$$

where C is an unknown vector and $G=[G_{10},\cdots,G_{2^{k-1}M-1}]$ is a known vector.

Using theorems in Section 4.1 can be approximated integer and fractional order derivatives as follows:

$$\begin{aligned} \mathrm{D}^{\eta}u(t) &= C^T\,\Phi^{\eta}\,\Psi^{y,c}(t),\\ \mathrm{D}^{\alpha_i}u(t) &= C^T\,\Phi^{\alpha_i}\,\Psi^{y,c}(t),\\ \mathrm{D}^{n}u(t) &= C^T\,\mathfrak{D}^{n}\,\Psi^{y,c}(t). \end{aligned} \tag{4.48}$$

From Eqs. 4.47 and 4.48, then the residual $R(t)$ for Eq.4.44 can be written as:

$$C^T\,\Phi^{\eta}\,\Psi^{y,c}(t)-\sum_{i=1}^{k}a_iC^T\,\Phi^{\alpha_i}\,\Psi^{y,c}(t)-a_0\,C^T\,\Psi^{y,c}(t)-G^T\,\Psi^{y,c}(t)\cong R(t)\simeq 0, \tag{4.49}$$

also, with the initial conditions

$$C^T\,\mathrm{D}^{j}\,\Psi^{y,c}(0)=bj,\quad j=0,\cdots,\nu-1. \tag{4.50}$$

Next, to find the approximate solution, we need to generate a system of $2^{k-1}M$ equations for the unknown vector C. Then, $u(t)$ given in Eq.4.47 can be evaluated that give us the solution of the given problem.

4.3.2 Non-Linear Fractional differential Equation

Consider the non- linear FDE as the following form:

$$\mathrm{D}^{\eta}u(t) = F(t, u(t), \mathrm{D}^{\alpha_1}u(t), \cdots, \mathrm{D}^{\alpha_k}u(t)), \tag{4.51}$$

with initial conditions$u^j(0) = b_j,\ j = 0, \cdots, \nu - 1$, where

$$\nu - 1 < \eta \leq \nu, 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \eta. \tag{4.52}$$

To solve this problem, first we approximate $u(t), \mathrm{D}^{\eta}u(T), \mathrm{D}^{\alpha_i}u(t)$ for $i = 1, \cdots, k$ as in the previous section.

Next, we substitute these equations in Eq.4.51, we obtain

$$C^T\, \Phi^{\eta}\, \Psi^{y,c}(t) \approx F(t, C^T\, \Psi^{y,c}(t), C^T\, \Phi^{\alpha_1}\, \Psi^{y,c}(t), \cdots, \Phi^{\alpha_1}\, \Psi^{y,c}(t)), \tag{4.53}$$

where C is the unknown vector and for initial condition we approximating as Eq.4.50.

To find the solution we calculate Eq.4.53 at $2^{k-1}M - \nu$ collocation points. The system Eq.4.53 and Eq.4.50 obtained together contains $2^{k-1}M$ non-linear equations that can be find the solution of it using Newton's iterative method. Thus, $u(t)$ can be calculated as in Eq.4.47.

4.4 Numerical Experimental

Here, we solve some problems to show the effectively and accuracy of our proposed method.

Example 4.1 Let be consider the following fractional differential equation:

$$\mathrm{D}^2\, u(t) + \mathrm{D}^{3/2}\, u(t) + u(t) = g(t), \tag{4.54}$$

subject to

$$u(0) = 0,\ u(5) = 25, \tag{4.55}$$

and $g(t) = t^2 + 4\sqrt{t/\pi} + 2$. The exact solution is given by t^2.

By using the GHW operational matrices of derivatives as in Section 4.3. to solve this boundary value problem. We suppose that $k=1,\ M=3$ to find the solution of the problem Eq.4.54

$$u(t)=C^T\,\Psi^{y,c}(t). \tag{4.56}$$

After substitute the trial solution above in Eq.4.54 we obtain the following matrix equation

$$C^T\,\mathfrak{D}^2\,\Psi^{y,c}(t)+C^T\,\Phi^{3/2}\,\Psi^{y,c}(t)+C^T\,\Psi^{y,c}(t)-G^T\,\Psi^{y,c}(t)\simeq 0, \tag{4.57}$$

and the boundary conditions

$$C^T\,\Psi^{y,c}(0)=0,\quad C^T\,\Psi^{y,c}(5)=25, \tag{4.58}$$

where $\mathfrak{D}^2$, $\Phi^{3/2}$ as the following

$$\mathfrak{D}^2=\begin{bmatrix}0&0&0\\0&0&0\\32&0&0\end{bmatrix},\ \Phi^{3/2}=\frac{4\,t^{-3/2}}{\sqrt{\pi}}\begin{bmatrix}0&0&0\\0&0&0\\5&4&1\end{bmatrix},\ C=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix},\ \Psi^{y,c}(t)=\begin{bmatrix}1\\4\,t-2\\16\,t^2-16\,t+3\end{bmatrix}. \tag{4.59}$$

To find the unknown vector C we can solve the following system:

$$\begin{aligned}&c_1-2\,c_2+3\,c_3=0,\\&c_1+31\,c_3+32\,c_3\,\sqrt{2/\pi}-9/4-2\,\sqrt{2/\pi}=0,\\&c_1+2\,c_2+35\,c_3+64\,c_3/\sqrt{\pi}-3-4/\sqrt{\pi}=0.\end{aligned} \tag{4.60}$$

Next, solving the above system we obtain C value as

$$C=\begin{bmatrix}0.3125000016\\0.2500000007\\0.06249999996\end{bmatrix}, \tag{4.61}$$

Now, substituting C in Eq.4.56 to achieve the exact solution that is

$$u(t) = [0.3125000016, 0.2500000007, 0.06249999996] \begin{bmatrix} 1 \\ 4t-2 \\ 16t^2-16t+3 \end{bmatrix} = t^2. \tag{4.62}$$

We construct Table 4.1 to show the absolute error for a different values of k and M with $\lambda = 1, y = 1$ and $c = 1$.

Table 4.1 The absolute error of the present method for a different values of k, M in Example 4.1.

t	**Absolute Error k=1,M=3**	**Absolute Error k=1,M=6**	**Absolute Error k=2,M=3**	**Absolute Error k=2,M=6**	**Absolute Error k=1,M=12**
0	1.0E-10	5.99999870E-11	1.414213562E-11	4.242640686E-11	2.60006084E-10
0.1	4.4E-10	1.222810E-6	1.363E-9	2.491722E-6	0.781084139E-2
0.2	7.8E-10	2.43571E-6	2.69E-9	4.96368E-6	0.1555964888E-1
0.3	1.01E-9	3.63064E-6	4.01E-9	7.39906E-6	0.231935423E-1
0.4	1.3E-9	4.8002E-6	5.3E-9	9.7829E-6	0.306656303E-1
0.5	1.6E-9	5.9380E-6	7.20E-8	0.2944447E-3	0.379336652E-1
0.6	1.9E-9	7.0378E-6	7.54E-8	0.3334414E-3	0.449594269E-1
0.7	2.2E-9	8.0944E-6	7.82E-8	0.3704791E-3	0.517083829E-1
0.8	2.4E-9	9.1026E-6	8.07E-8	0.4054076E-3	0.581494804E-1
0.9	2.6E-9	0.100585E-4	8.27E-8	0.4380955E-3	0.642550043E-1

Example 4.2 Consider the following fractional differential equation:

$$\mathrm{D}^2 u(t) + \mathrm{D}^{1/2} u(t) + u(t) = g(t), \tag{4.63}$$

subject to

$$u(0) = 0, \ \ u'(0) = 0, \tag{4.64}$$

and

$$g(t) = 2 + t^2 + \frac{8\,t^{1.5}}{3\,\sqrt{\pi}}. \tag{4.65}$$

The exact solution is given by t^2.

Applying the proposed method to solve the above problem with y, c, λ and $k = 1, M = 4$ given us

$$C^T\,\mathfrak{D}^2\,\Psi^{y,c}(t) + C^T\,\Phi^{1/2}\,\Psi^{y,c}(t) + C^T\,\Psi^{y,c}(t) - G^T\,\Psi^{y,c}(t) \simeq 0, \tag{4.66}$$

with conditions

$$C^T\,\Psi^{y,c}(0) = 0,\;\; C^T\,\mathrm{D}\Psi^{y,c}(0) = 0. \tag{4.67}$$

Here $\mathfrak{D}^2$, $\Phi^{1/2}$ as the following

$$\mathfrak{D}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 32 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 \end{bmatrix},\; \Phi^{3/2} = \frac{4\,t^{-1/2}}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 \\ -2/3 & 2/3 & 2/3 & 0 \\ 6/5 & 1/5 & 4/5 & 4/5 \end{bmatrix},$$
$$C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},\; \Psi^{y,c}(t) = \begin{bmatrix} 1 \\ 4\,t - 2 \\ 16\,t^2 - 16\,t + 3 \\ 64\,t^3 - 96\,t^2 + 40\,t - 16\,t - 4 \end{bmatrix}. \tag{4.68}$$

By solving the following systems, we obtain the unknown vector of coefficients C

$$\frac{283\,c_3}{9}-\frac{1700\,c_4}{27}+\frac{8\sqrt{3}\,c_2}{3\sqrt{\pi}}-\frac{160\sqrt{3}\,c_3}{27\sqrt{\pi}}+\frac{784\sqrt{3}\,c_4}{135\sqrt{\pi}}+c_1-\frac{2\,c_2}{3}-2.400653340=0,$$
$$\frac{283\,c_3}{9}+\frac{1700\,c_4}{27}+\frac{8\sqrt{3}\,c_2}{3\sqrt{\pi}}-\frac{32\sqrt{2}\sqrt{3}\,c_3}{27\sqrt{\pi}}+\frac{16\sqrt{3}\,c_4}{135\sqrt{\pi}}+c_1+\frac{2\,c_2}{3}-3.263393539=0,$$
$$c_1-2\,c_2+3\,c_3-4\,c_4=0,$$
$$4\,c_2-16\,c_3+40\,c_4=0. \tag{4.69}$$

Next, substituting C to get the approximate solution as:

$$u(t)=\begin{bmatrix}0.3125000016\\0.2500000007\\0.06249999996\\0\end{bmatrix}^T\begin{bmatrix}1\\4\,t-2\\16\,t^2-16\,t+3\\64\,t^3-96\,t^2+40\,t-16\,t-4\end{bmatrix}=t^2. \tag{4.70}$$

Absolute errors obtained when using the propsed method with different values of

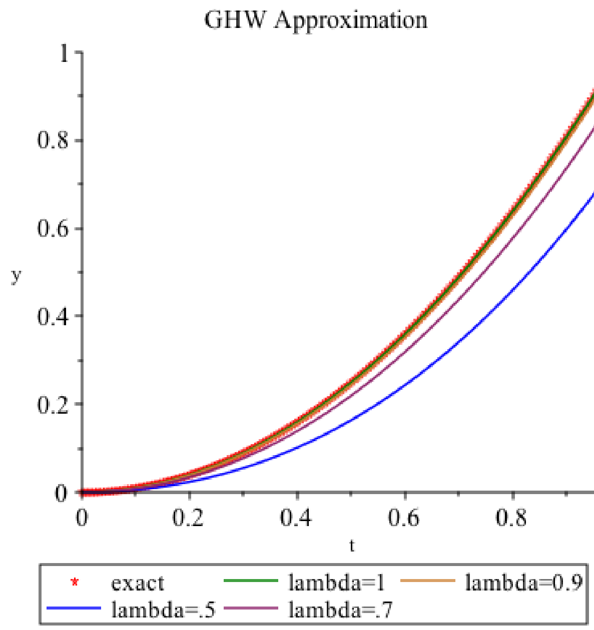


Figure 4.1 Exact and approximate solution of Example 4.2. with different values of λ and $k=1, M=4$

k, M with $y=1$, $c=1$ and $lambda=1$ are considered in Table 4.2, where the best

Table 4.2 The absolute error of the present method for a different values of $k,\ M$ in Example 4.2.

t	**Absolute Error** $k=1,\ M=3$	**Absolute Error** $k=1,\ M=4$	**Absolute Error** $k=1,\ M=5$	**Absolute Error** $k=1,\ M=7$	**Absolute Error** $k=1,\ M=11$
0.1	0.70E-10	**0**	0.296E-9	0.104E-9	0.191E-8
0.2	0.10E-10	**0**	0.36E-9	0.690E-9	0.454E-8
0.3	0.10E-10	**0**	0.58E-9	0.114E-8	0.694E-8
0.4	0.10E-9	**0**	0.70E-9	0.150E-8	0.930E-8
0.5	0.10E-9	**0**	0.90E-9	0.180E-8	0.114E-7
0.6	0.10E-9	**0**	0.11E-8	0.210E-8	0.131E-7
0.7	0.20E-9	**0**	0.12E-8	0.230E-8	0.146E-7
0.8	0.20E-9	**0**	0.14E-8	0.230E-8	0.157E-7
0.9	0.30E-9	**0**	0.15E-8	0.240E-8	0.166E-7
1	0.40E-9	**0**	0.15E-8	0.270E-8	0.179E-7

error with $k=1,\ M=4$ with notation the results obtained for 15th digits number. While Fig.4.1 shows the exact and approximate solution obtained by GHW method for various λ with $k=1,\ M=4$.

Example 4.3 Consider the following fractional differential problem

$$4(1+t)\mathrm{D}^{\frac{5}{2}}u(t)+4\mathrm{D}^{\frac{3}{2}}u(t)+\frac{1}{\sqrt{t+1}}u(t)=\sqrt{t}+\sqrt{\pi}, \tag{4.71}$$

subject to

$$u(0)=\sqrt{\pi},\ u'(0)=\sqrt{\pi}/2,\ u(1)=\sqrt{2\,\pi}. \tag{4.72}$$

The exact solution of this problem is $u(t)=\sqrt{\pi(t+1)}$. To obtain the solution of the above problem by the presented method procedure as in the previous examples. We examine the lest error obtained by a different values of $k,\ M$ as shown in Table 4.3.

If $k=1,\ M=3$

$$\Phi^{3/2}=\frac{4\,t^{-3/2}}{\sqrt{\pi}}\begin{bmatrix}0&0&0\\0&0&0\\5&4&1\end{bmatrix},\quad \Phi^{5/2}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\tag{4.73}$$

and the approximate solution

$$u(t)=\begin{bmatrix}2.168050908 & 0.1835436051 & -0.00950328156\end{bmatrix}\begin{bmatrix}1\\4\,t-2\\16\,t^2-16\,t+3\end{bmatrix}.\tag{4.74}$$

When $k=1,\ M=4$

$$\Phi^{3/2}=\frac{4\,t^{-3/2}}{\sqrt{\pi}}\begin{bmatrix}0&0&0&0\\0&0&0&0\\5&4&1&0\\-2&4&6&2\end{bmatrix},\quad \Phi^{5/2}=\frac{12\,t^{-5/2}}{\sqrt{\pi}}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\14&14&6&1\end{bmatrix},\tag{4.75}$$

and the approximate solution

$$u(t)=\begin{bmatrix}2.164530850\\0.1823702535\\-0.008329929\\0.0005866761\end{bmatrix}^T\begin{bmatrix}1\\4\,t-2\\16\,t^2-16\,t+3\\64\,t^3-96\,t^2+40\,t-16\,t-4\end{bmatrix}.\tag{4.76}$$

While $k=1,\ M=5$

$$\Phi^{3/2}=\frac{4\,t^{-3/2}}{\sqrt{\pi}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 \\ -2 & 4 & 6 & 2 & 0 \\ 77/5 & 68/5 & 57/5 & 48/5 & 16/5 \end{bmatrix},$$

$$\Phi^{5/2}=\frac{12\,t^{-5/2}}{\sqrt{\pi}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 \\ 0 & 16 & 24 & 13.33 & 2.67 \end{bmatrix}, \tag{4.77}$$

and the approximate solution

$$u(t)=\begin{bmatrix} 2.163305365 \\ 0.1821291746 \\ -0.007837726 \\ 0.0007072156 \\ -0.000050225 \end{bmatrix}^T \begin{bmatrix} 1 \\ 4\,t-2 \\ 16\,t^2-16\,t+3 \\ 64\,t^3-96\,t^2+40\,t-16\,t-4 \\ 256\,t^4-512\,t^3+336\,t^2-80\,t+5\,t-4 \end{bmatrix}. \tag{4.78}$$

If $k = 1,\ M = 7$

$$\Phi^{3/2} = \frac{4\,t^{-3/2}}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 6 & 2 & 0 & 0 & 0 \\ 77/5 & 68/5 & 57/5 & 48/5 & 16/5 & 0 & 0 \\ -88/7 & 16/7 & 104/7 & 120/7 & 96/7 & 32/7 & 0 \\ 250/7 & 568/21 & 130/7 & 464/21 & 496/21 & 128/7 & 128/21 \end{bmatrix}$$

$$\Phi^{5/2} = \frac{12\,t^{-5/2}}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 & 0 & 0 \\ 0 & 16 & 24 & 13.33 & 2.67 & 0 & 0 \\ 88 & 104 & 82.67 & 57.34 & 26.67 & 5.34 & 0 \\ -320/7 & 208/7 & 856/7 & 973.34/7 & 706.67/7 & 320/7 & 64/7 \end{bmatrix}, \tag{4.79}$$

then, the approximate solution

$$u(t)=\begin{bmatrix} 2.163109543 \\ 0.1821827501 \\ -0.007735020 \\ 0.0006685255 \\ -0.0000718225 \\ 7.9349292\ 10^{-6} \\ -6.15575858\ 10^{-7} \end{bmatrix}^T \begin{bmatrix} 1 \\ 4\,t-2 \\ 16\,t^2-16\,t+3 \\ 64\,t^3-96\,t^2+40\,t-16\,t-4 \\ 256\,t^4-512\,t^3+336\,t^2-80\,t+5\,t-4 \\ 1024\,t^5-2560\,t^4+2304\,t^3-896\,t^2+140\,t-6 \\ 4096\,t^6-12288\,t^5+14080\,t^4-7680\,t^3+2016\,t^2-224\,t+7 \end{bmatrix}. \quad (4.80)$$

If $k=1,\ M=9$

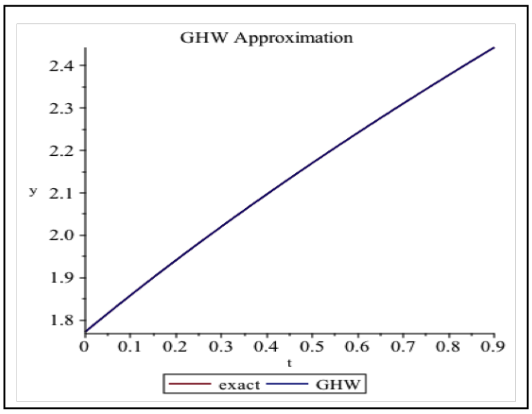
$$\Phi^{3/2}=\frac{4\,t^{-3/2}}{\sqrt{\pi}}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \\ \frac{77}{5} & \frac{68}{5} & \frac{57}{5} & \frac{48}{5} & \frac{16}{5} & 0 & 0 & 0 & 0 \\ -\frac{88}{7} & \frac{16}{7} & \frac{104}{7} & \frac{120}{7} & \frac{96}{7} & \frac{32}{7} & 0 & 0 & 0 \\ \frac{250}{7} & \frac{568}{21} & \frac{130}{7} & \frac{464}{21} & \frac{496}{21} & \frac{128}{7} & \frac{128}{21} & 0 & 0 \\ -\frac{740}{21} & -\frac{40}{7} & \frac{1676}{77} & \frac{6384}{231} & \frac{6967}{231} & \frac{2368}{77} & \frac{256}{11} & \frac{256}{33} & 0 \\ 70 & \frac{520}{11} & \frac{830}{33} & \frac{4528}{143} & \frac{5264}{143} & \frac{16768}{429} & \frac{5504}{143} & \frac{4096}{143} & \frac{4096}{429} \end{bmatrix}, \quad (4.81)$$

$$
\Phi^{5/2} = \frac{12\,t^{-5/2}}{\sqrt{\pi}} \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 14 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 16 & 24 & 13.33 & 2.67 & 0 & 0 & 0 & 0 \\
88 & 104 & 82.67 & 57.34 & 26.67 & 5.34 & 0 & 0 & 0 \\
-\frac{320}{7} & \frac{208}{7} & \frac{856}{7} & \frac{973.34}{7} & \frac{706.67}{7} & \frac{320}{7} & \frac{64}{7} & 0 & 0 \\
\frac{953.34}{3} & \frac{1070.67}{3} & 284 & \frac{776.67}{3} & \frac{693.34}{3} & 160 & \frac{213.34}{3} & \frac{42.67}{3} & 0 \\
-\frac{746.67}{3} & -\frac{746.67}{33} & \frac{9757.34}{33} & \frac{13949.34}{33} & \frac{13746.67}{33} & \frac{11626.67}{33} & \frac{7786.67}{33} & \frac{3413.34}{33} & \frac{682.67}{33}
\end{bmatrix}, \tag{4.82}
$$

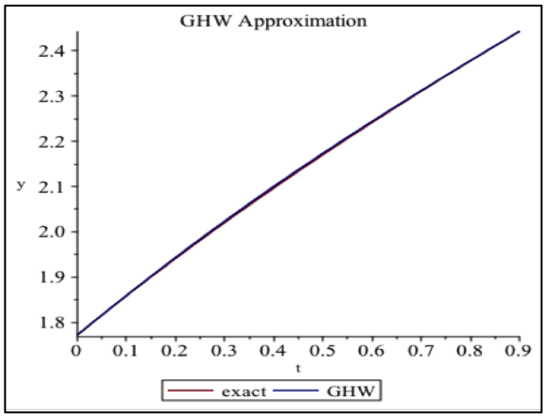
and the approximate solution

$$
u(t) = \begin{bmatrix}
2.163126846 \\
0.1821953400 \\
-0.00774146178 \\
0.0006611518811 \\
-0.000070764248 \\
8.484778327\ 10^{-6} \\
-1.069153906\ 10^{-6} \\
1.2699131\ 10^{-7} \\
-1.027850507\ 10^{-8}
\end{bmatrix}^{T}
\begin{bmatrix}
1 \\
4t-2 \\
16t^2-16t+3 \\
64t^3-96t^2+40t-16t-4 \\
256t^4-512t^3+336t^2-80t+5t-4 \\
1024t^5-2560t^4+2304t^3-896t^2+140t-6 \\
4096t^6-12288t^5+14080t^4-7680t^3+2016t^2-224t+7 \\
16384t^7-57344t^6+79872t^5-56320t^4+21120t^3-4032t^2+336t-8 \\
65536t^8-262144t^7+430080t^6-372736t^5+183040t^4-50688t^3+7392t^2-480t+9
\end{bmatrix}. \tag{4.83}
$$

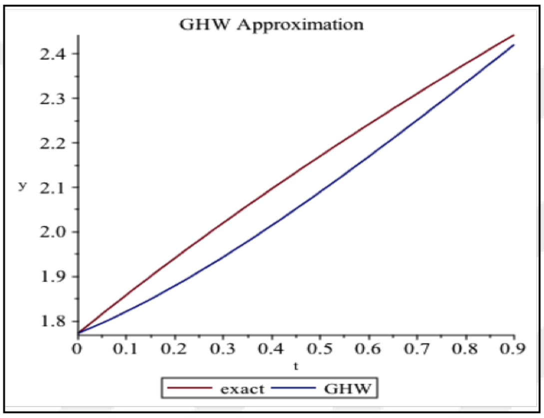
Using the GHW method we got a different wavelets (second kind of Chebyshev wavelet, Legendre wavelet, Morgan– Voyce wavelet, first kind of Fermat wavelet, Dickson wavelet with $a = 0.5$ and Gegenbauer wavelet with $\lambda = 5$ as shown in Fig.4.2. As a result we can see the best choice from these wavelet types to achieve best error is Chebyshev wavelet of the second kind.



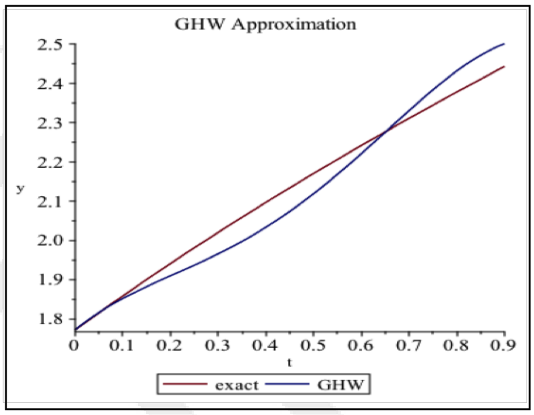
(a) Chebyshev wavelet with $k = 1, M = 9$



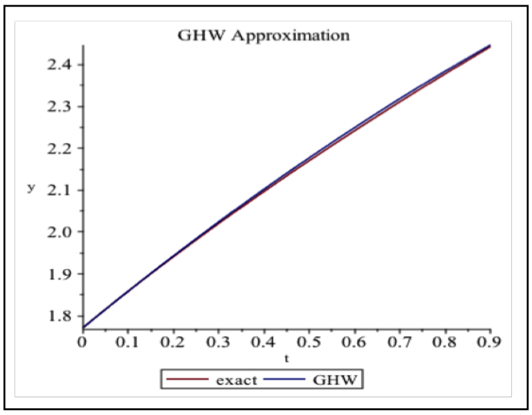
(b) Legendre wavelet with $k = 1, M = 7$



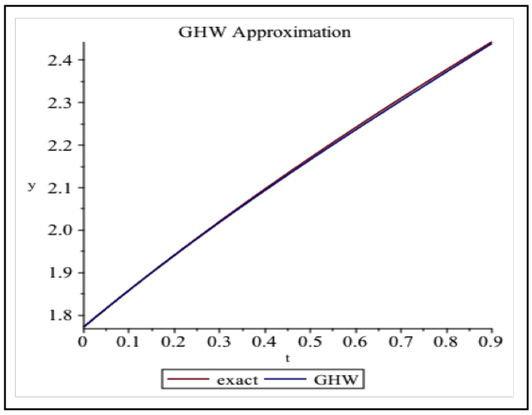
(c) Morgan- Voyce wavelet with $k = 1, M = 9$



(d) Fermat wavelet with $k = 1, M = 5$



(e) Dickson wavelet with $k = 1, M = 3$



(f) Gegenbauer wavelet with $k = 1, M = 4$

Figure 4.2 Different wavelets with different values of k, M that gave us a best error of Example 4.3.

Table 4.3 The error of the different wavelet methods of Example 4.3.

Errors $k=1$	**2nd kind of Chebyshev wavelet**	**Legendre wavelet**	**Morgan- Voyce wavelet**	**1st kind of Fermat wavelet**	**Dickson wavelet**	**Gegenbauer wavelet**
$L^2, M=3$	0.5036439084E-2	0.5036438E-2	0.76100216E-1	0.76100216E-1	**0.5036438E-2**	0.5036438E-2
$L^\infty, M=3$	0.7454618E-2	0.7454617E-2	0.104027942	0.104027942	**0.7454617E-2**	0.7454617E-2
$L^2, M=4$	0.1403526111E-2	0.26854837E-2	0.74395589E-1	0.47251705E-1	0.1411792276	**0.3816574E-2**
$L^\infty, M=4$	0.2057017E-2	0.31264770E-2	0.101671591	0.79170918E-1	0.216872048	**0.5987078E-2**
$L^2, M=5$	0.1978998265E-3	0.4629112E-2	0.74395087E-1	**0.42005410E-1**	0.8248089E-1	0.2679033E-1
$L^\infty, M=5$	0.334816E-3	0.4629104E-2	0.101670962	**0.63197821E-1**	0.109808466	0.3998057E-1
$L^2, M=7$	0.2581350456E-4	**0.2159070E-2**	0.71686671E-1	0.6226229948	1.510631880	0.1865785235
$L^\infty, M=7$	0.42796E-4	**0.3543494E-2**	0.97843785E-1	1.032964069	2.464872894	0.270017681
$L^2, M=9$	**1.544551585E-6**	0.5883541E-2	**0.59529267E-1**	2.314774036	2.797998212	0.7254666981
$L^\infty, M=9$	**2.707E-6**	0.9673486E-2	**0.82184932E-1**	3.376283928	4.884617647	1.024568600

Example 4.4 We consider the Riccati equation as the following

$$\mathrm{D}^{\alpha}u(t) = -u^2(t) + 1, \tag{4.84}$$

subject to $u(0) = 0$. The exact solution of this problem is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{4.85}$$

If $y,\ c,\ \lambda,\ k = 1,\ M = 5$ and $\alpha = 0.5$ used the same procedure in the Section 4.3.2 where

$$\Phi^{1/2} = \frac{4\,t^{\frac{-1}{2}}}{\sqrt{\pi}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{6}{5} & \frac{1}{5} & \frac{4}{5} & \frac{4}{5} & 0 \\ -\frac{6}{5} & \frac{2}{7} & \frac{2}{7} & \frac{32}{35} & \frac{32}{35} \end{bmatrix}, \tag{4.86}$$

and by solving the following system, we obtain the unknown coefficient vector C:

$$\begin{aligned}
&(c_1 - c_2 + c_4 - c_5)^2 + 2\,c_2\,\frac{\sqrt{4}}{\sqrt{\pi}} - 16\,c_3\,\frac{\sqrt{4}}{3\sqrt{\pi}} + 36\,c_4\,\frac{\sqrt{4}}{5\,\sqrt{\pi}} - 208\,c_5\frac{\sqrt{4}}{35\,\sqrt{\pi}} - 1 = 0,\\
&(c_1 - c_3 + c_5)^2 + 4\,c_2\frac{\sqrt{2}}{\sqrt{\pi}} - 16\,c_3\,\frac{\sqrt{2}}{3\,\sqrt{\pi}} + 8\,c_4\,\frac{\sqrt{2}}{5\,\sqrt{\pi}} - 16\,c_5\,\frac{\sqrt{2}}{7\,\sqrt{\pi}} - 1 = 0,\\
&(c_1 + c_2 - c_4 - c_5)^2 + 2\,c_2\,\frac{\sqrt{3}\,\sqrt{4}}{\sqrt{\pi}} + 4\,c_4\,\frac{\sqrt{3}\sqrt{4}}{5\sqrt{\pi}} - 128\,c_5\,\frac{\sqrt{3}\sqrt{4}}{35\sqrt{\pi}} - 1 = 0,\\
&(c_1 + 2\,c_2 + 3\,c_3 + 4\,c_4 + 5\,c_5)^2 + c_2\frac{8}{\sqrt{\pi}} + \frac{32\,c_3}{3\,\sqrt{\pi}} + \frac{144\,c_4}{5\,\sqrt{\pi}} + \frac{1184\,c_5}{35\,\sqrt{\pi}} - 1 = 0.\\
&c_1 - 2\,c_2 + 3\,c_3 - 4\,c_4 + 5\,c_5 = 0.
\end{aligned} \tag{4.87}$$

Then the approximate solution

$$u(t)=\begin{bmatrix} 0.5298731336 \\ 0.1298845171 \\ -0.0508413137 \\ 0.02179819046 \\ -0.0060774793 \end{bmatrix}^{T} \begin{bmatrix} 1 \\ 4\,t-2 \\ 16\,t^2-16\,t+3 \\ 64\,t^3-96\,t^2+40\,t-16\,t-4 \\ 256\,t^4-512\,t^3+336\,t^2-80\,t+5 \end{bmatrix}. \tag{4.88}$$

While at the value $\alpha=0.7$ the operational matrix of fractional derivative as

$$\Phi^{0.7}=t^{-0.7}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2.228485018 & 1.114242509 & 0 & 0 & 0 \\ -0.342843857 & 2.399906937 & 1.714219243 & 0 & 0 \\ 2.16140687 & 1.30429723 & 3.13031340 & 2.235938144 & 0 \\ -1.1721734 & 1.3754409 & 1.84973068 & 3.79431928 & 2.710228054 \end{bmatrix}, \tag{4.89}$$

and the following system of algebraic equations are obtained

$$\begin{aligned}
&(c_1-c_2+c_4-c_5)^2+2.94050361\,c_2-7.238162741\,c_3\ +8.162602040\,c_4\\
&-3.862260552\,c_5-1=0,\\
&(c_1-c_3+c_5)^2+3.620184593\,c_2-3.341708866\,c_3-1.573993302\,c_4\\
&-0.506319198\,c_5-1=0,\\
&(c_1+c_2-c_4-c_5)^2+4.088444376\,c_2+2.515965754\,c_3+1.504109942\,c_4\\
&-7.707009290\,c_5-1=0,\\
&(c_1+2\,c_2+3\,c_3+4\,c_4+5\,c_5)^2++4.456970036\,c_2+9.599627746\,c_3\\
&+23.10469411\,c_4+35.85631783\,c_5-1=0,\\
&c_1-2\,c_2+3\,c_3-4\,c_4+5\,c_5=0.
\end{aligned} \tag{4.90}$$

After solving the above system to find C then substitute to get the following solution

$$u(t) = \begin{bmatrix} 0.5005327487 \\ 0.1597777827 \\ -0.0430508723 \\ 0.01059647361 \\ -0.0018877344 \end{bmatrix}^T \begin{bmatrix} 1 \\ 4\,t-2 \\ 16\,t^2-16\,t+3 \\ 64\,t^3-96\,t^2+40\,t-16\,t-4 \\ 256\,t^4-512\,t^3+336\,t^2-80\,t+5 \end{bmatrix}. \tag{4.91}$$

Now, we test when the fractional order $\alpha = 0.9$, $\Phi^{0.9}$ as

$$\Phi^{0.9} = t^{-0.9} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2.102274011 & 1.051137006 & 0 & 0 & 0 \\ 1.146694919 & 3.440084753 & 1.911158193 & 0 & 0 \\ 1.91115817 & 2.86673731 & 4.91440678 & 2.730225991 & 0 \\ 0.8014533 & 2.8271049 & 4.20102510 & 6.34117003 & 3.522872244 \end{bmatrix}, \tag{4.92}$$

the following system are obtained

$$\begin{aligned}
&(c_1-c_2+c_4-c_5)^2+3.660271647\,c_2-7.986047250\,c_3+6.179679262\,c_4\\
&+2.76015433\,c_5-1=0,\\
&(c_1-c_3+c_5)^2+3.922982019\,c_2-1.426538911\,c_3-5.604260070\,c_4\\
&+0.230086764\,c_5-1=0,\\
&(c_1+c_2-c_4-c_5)^2+4.085314010\,c_2+5.942274938\,c_3+2.652801301\,c_4\\
&-8.078208108\,c_5-1=0,\\
&((c_1+2\,c_2+3\,c_3+4\,c_4+5\,c_5)^2+4.204548023\,c_2+13.76033900\,c_3+33.30875709\,c_4\\
&+62.03777974\,c_5-1=0,\\
&c_1-2\,c_2+3\,c_3-4\,c_4+5\,c_5=0.
\end{aligned} \tag{4.93}$$

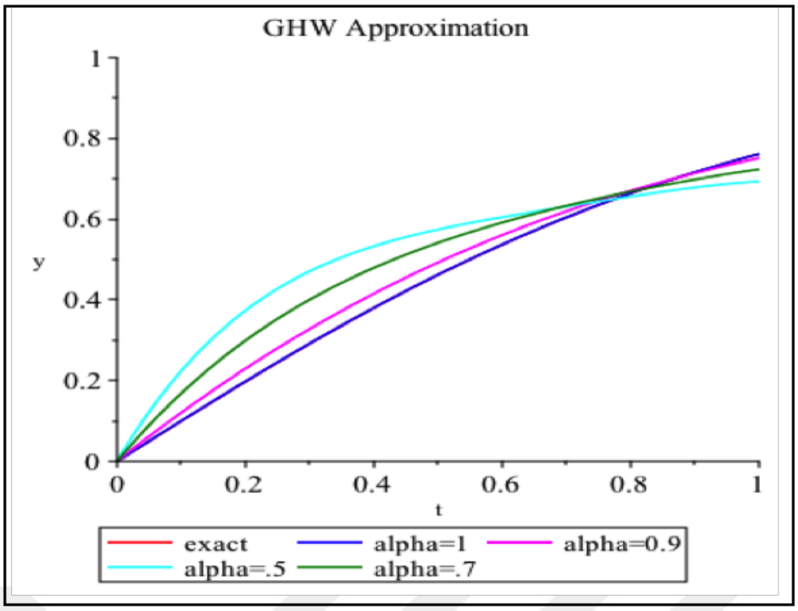


Figure 4.3 Exact and approximate solution of Example 4.4 for a different values of α when $K = 1$ and $M = 5$.

The approximate solution yield as

$$u(t) = \begin{bmatrix} 0.4634103472 \\ 0.1845998108 \\ -0.029441844 \\ 0.0016525389 \\ 0.00014499241 \end{bmatrix}^T \begin{bmatrix} 1 \\ 4t-2 \\ 16t^2-16t+3 \\ 64t^3-96t^2+40t-16t-4 \\ 256t^4-512t^3+336t^2-80t+5 \end{bmatrix}. \tag{4.94}$$

Fig.4.3 shows the nearest approximate solution to the exact solution is when $\alpha = 1$ by using the presented method where the figure present the result for a different values of $\alpha = 0.5, 0.7, 0.9, 1$ and $\lambda, y, c = 1$ with $k = 1, M = 5$. Moreover, Table 4.4 consider the comparison between the results obtained by the presented method with Ref. [48] when $k = 1, M = 12, \lambda, \alpha, y$ and $c = 1$.

Table 4.4 The comparison of the approximate solution using the presented method of Example 4.4

t	Ref. [48]	Our method	The exact solution	The error
0.1	0.0996679945	0.0996679271	0.0996679954	6.748E-8
0.2	0.1973753204	0.197375256	0.197375321	6.43E-8
0.3	0.2913126124	0.291312551	0.291312612	6.12E-8
0.4	0.3799489620	0.379948905	0.379948963	5.71E-8
0.5	0.4621171576	0.462117105	0.462117157	5.26E-8
0.6	0.5370495668	0.537049520	0.537049567	4.75E-8
0.7	0.6043677770	0.604367735	0.604367777	4.25E-8
0.8	0.6640367705	0.664036733	0.664036770	3.74E-8
0.9	0.7162978700	0.716297838	0.716297870	3.23E-8
1	0.7615941559	0.761594126	0.761594156	2.96E-8

Example 4.5 Consider the following problem

$$\mathrm{D}^{\alpha}u(t) = 2\,u(t) - u^2(t) + 1, \tag{4.95}$$

subject to $u(0) = 0$. The exact solution of this problem is

$$u(t) = 1 + \sqrt{2}\,tanh\left(\sqrt{2}\,t + \frac{1}{2}log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{4.96}$$

To solve the above nonlinear fractional differential problem, using the presented method in Section 4.3.2. The approximate solution $u(t)$ as following

$$u(t) \cong C^T\, \Psi^{y,c}(t),\;\; \mathrm{D}^{\alpha}u(t) \cong C^T\, \Phi^{\alpha}\, \Psi^{y,c}(t). \tag{4.97}$$

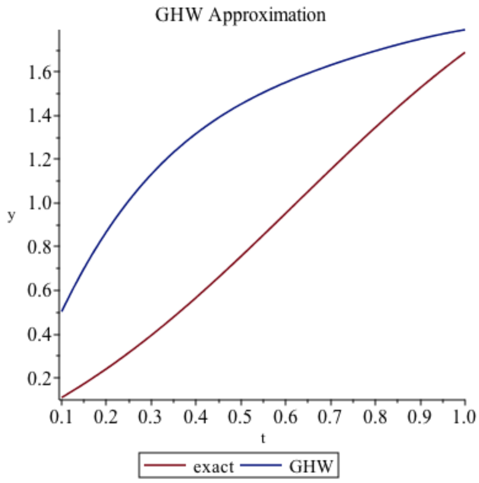
Then solving the algebraic system to find the unknown vector C. Table 4.5 consider the comparison of absolute error obtained by the presented method with different values of $k,\ M$ and $y,\ c,\ \lambda,\ \alpha = 1$. As a result, it is obvious the lest error gained when increased the value of $k,\ M$. Moreover, Fig.4.4 shows the approach of approximated solutions to the exact when α close to 1 with $y,\ c,\ \lambda,\ k = 1$ and $M = 5$, and the figure shows even though $\alpha = 1$ the approximate solution still far from the exact solution. At the same figure with $k = 1,\ M = 16$ the result is the same exact solution approximately when $\alpha = 1$.

Table 4.5 The absolute error of Example 4.5 for a different values of $k,\ M$

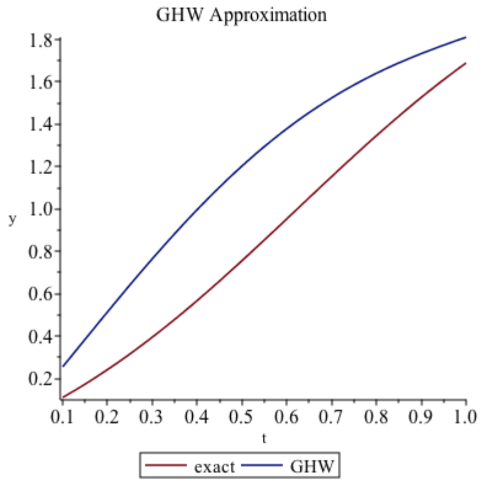
t	**Absolute Error** $k=1,\ M=3$	**Absolute Error** $k=1,\ M=7$	**Absolute Error** $k=1,\ M=12$	**Absolute Error** $k=1,\ M=16$
0.1	1.51586284	0.27851520E-2	0.143693E-4	**1.815E-7**
0.2	2.55068972	0.35034327E-2	0.166762E-4	**2.129E-7**
0.3	3.10442060	0.38990898E-2	0.191502E-4	**2.434E-7**
0.4	3.17892267	0.43358099E-2	0.212427E-4	**2.694E-7**
0.5	2.77828027	0.46675199E-2	0.227348E-4	**2.871E-7**
0.6	1.90863873	0.478006161E-2	0.2340729E-4	**2.9399E-7**
0.7	0.577516707	0.47185631E-2	0.231569E-4	**2.889E-7**
0.8	1.20728680	0.45264342E-2	0.220318E-4	**2.778E-7**
0.9	3.43887283	0.41141296E-2	0.201644E-4	**2.644E-7**
1	6.11214783	0.35199095E-2	0.179715E-4	**2.605E-7**

Example 4.6 Let be consider another fractional differential equation that be solved in before using Genocchi operational method in Ref. [49]

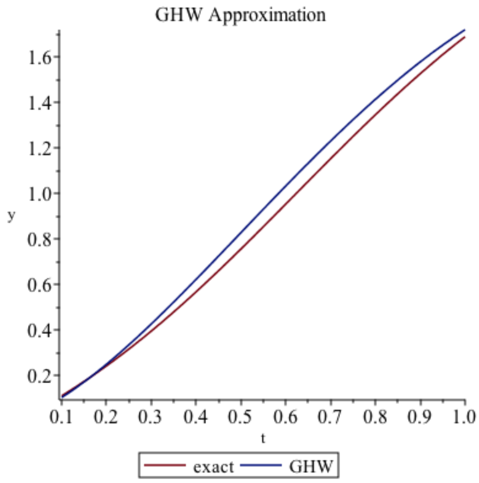
$$\mathrm{D}^2 u(t) + \Gamma\left(\frac{4}{5}\right) t^{\frac{6}{5}} \mathrm{D}^{\frac{6}{5}} u(t) + \frac{11}{9}\Gamma\left(\frac{5}{6}\right) t^{\frac{1}{6}} \mathrm{D}^{\frac{1}{6}} u(t) - (u'(t))^2 = 2 + \frac{1}{10} t^2, \tag{4.98}$$



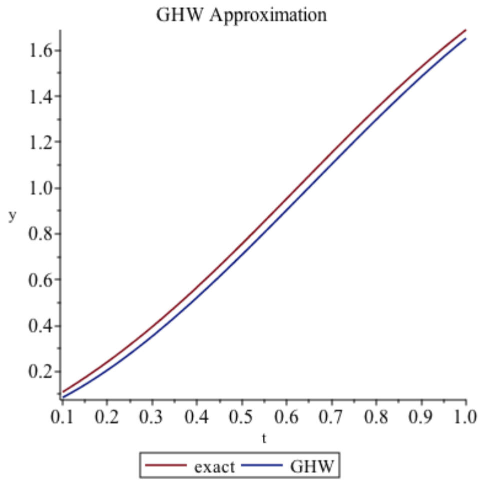
(a) Approximate solution with $k = 1, M = 5$ and $\alpha = 0.5$



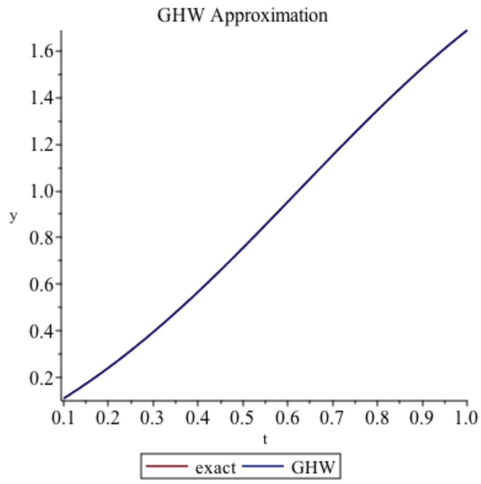
(b) Approximate solution with $k = 1, M = 5$ and $\alpha = 0.7$



(c) Approximate solution with $k = 1, M = 5$ and $\alpha = 0.9$



(d) Approximate solution with $k = 1, M = 5$ and $\alpha = 1$



(e) Approximate solution with $k = 1, M = 12$ and $\alpha = 1$

Figure 4.4 Approximate solution by GHW method with different values of α of Example 4.5.

with condition

$$u(0) = 1, \ \ u(1) = 2, \tag{4.99}$$

and the exact solution is given as

$$u(t) = 1 + t^2. \tag{4.100}$$

Applying the presented method to solve the above problem when $\lambda, y, c, k = 1$ and $M = 3, 4, 5, 7$ and 11. For $k = 1, \ M = 7$ the operational matrices of fractional order 6/5 and 1/6 as following

$$\Phi^{\frac{6}{5}} = t^{-\frac{6}{5}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10.73671 & 8.58937 & 2.147343 & 0 & 0 & 0 & 0 \\ -14.31562 & -1.431562 & 8.58937 & 3.578904 & 0 & 0 & 0 \\ 39.36795 & 24.95008 & 11.35024 & 12.27053 & 5.112720 & 0 & 0 \\ -56.83192 & -21.34426 & 12.35664 & 15.82252 & 16.145433 & 6.727264 & 0 \\ 102.16023 & 57.29476 & 13.65096 & 17.221795 & 20.51815 & 20.181791 & 8.4090795 \end{bmatrix}, \tag{4.101}$$

$$\Phi^{\frac{1}{6}} = t^{-\frac{1}{6}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.1261761 & 1.0630881 & 0 & 0 & 0 & 0 & 0 \\ -2.706042 & 0.38657748 & 1.159732 & 0 & 0 & 0 & 0 \\ 3.6611161 & -0.01136993 & 0.4093173 & 1.227952 & 0 & 0 & 0 \\ -4.46591 & 0.1295183 & 0.0088982 & 0.4271137 & 1.281341 & 0 & 0 \\ 5.38493 & -0.0162708 & 0.1382801 & 0.022092 & 0.4418418 & 1.3255254 & 0 \\ -6.2302 & 0.0767866 & -0.003419014 & 0.1451766 & 0.03156013 & 0.45447 & 1.3633975 \end{bmatrix}. \tag{4.102}$$

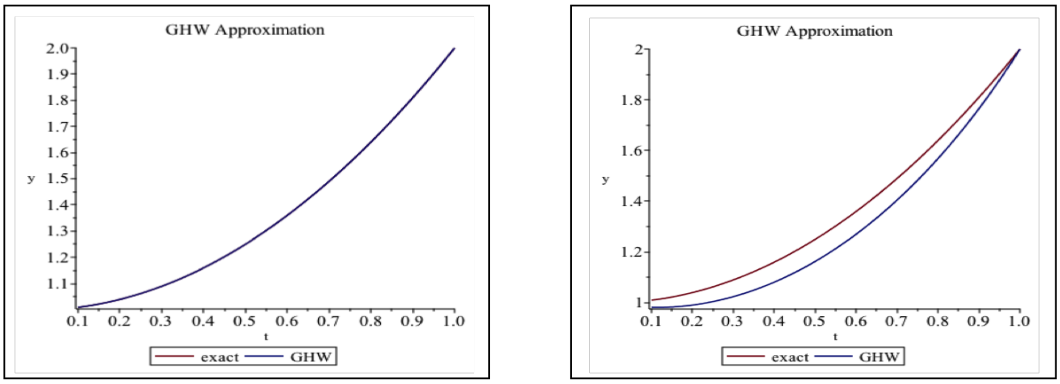
Then we compared the L^2, L^∞ errors of the results obtained with that obtained using Genocchi operational method [49] as shown in Table 4.6 with notation the results obtained for 15th digits number. As a result we can observe that the presented method got best error from the method in Ref. [49] when $M = 3, 7$ with zero error, when $M = 11$ we got a simple error that unaffected. While when $M = 4, 5$ the performance of our method less than in Ref. [49]. From Table 4.6 we can see our method got an accurate results better than in Ref. [49]. Fig.4.5 shows that the affect of changing λ on the result when $y, c, k = 1$ and $M = 7$, therefore the best value of λ for the given problem is equal to 1 .

Table 4.6 Comparison of the L^2, L^∞ error obtained by the our method and the operational method of Ref. [49] for Example 4.6

Errors $k = 1$	**Genocchi operational method Ref. [49]**	**Presented method**
$L^2, M = 3$	1.323E-4	0.
$L^\infty, M = 3$	1.8119E-4	0.
$L^2, M = 4$	3.377E-5	0.5639985335
$L^\infty, M = 4$	5.5528E-5	0.944754483
$L^2, M = 5$	1.698E-5	1.294231362
$L^\infty, M = 5$	1.8466E-5	1.875259588
$L^2, M = 7$	9.262E-6	0
$L^\infty, M = 7$	1.4556E-5	0
$L^2, M = 11$	not examined	1.140175425E-9
$L^\infty, M = 11$	not examined	2.0E-9

Example 4.7 Let consider another fractional differential equation as follows

$$\mathrm{D}^{0.25}u(t) + u^2(t) = \frac{2}{\Gamma(2.75)}\, t^{1.75} + t^4, \tag{4.103}$$



(a) GHW with $\lambda = 1$ **(b)** GHW with $\lambda = 1.5$

Figure 4.5 Different wavelets with different values of λ for Example 4.6

with condition

$$u(0) = 0. \tag{4.104}$$

The exact solution of the above equation is

$$u(t) = t^2. \tag{4.105}$$

Applied the proposed method to solve Eq.4.103, we get the approximate solution when $k = 2, M = 3, \lambda = 1, y = 1$ and $c = 1$ as in the Fig.4.6. Where the operational matrix of order 0.25 as the following matrix

$$\Phi^{0.25} = t^{-0.25} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2.176130505 & 1.088065252 & 0 & 0 & 0 & 0 \\ -2.487006292 & 0.6217515729 & 1.243503146 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.528391515 & 1.088065252 & 0 \\ 0 & 0 & 0 & -32.33108179 & 1.865254719 & 1.243503146 \end{bmatrix}. \tag{4.106}$$

By solving the following system of equations we obtain the unknown coefficients vector C

$$
\begin{array}{l}
(c_1\sqrt{2}-2\,c_2\dfrac{\sqrt{2}}{5}-21\,c_3\sqrt{2}\left(\dfrac{1}{25}\right)^2+1.740904404\,c_2\,5^{1/4}\sqrt{2}\\
-3.780249564\,c_3\,5^{1/4}\sqrt{2}-0.07597883649=0,\\
c_1\sqrt{2}+6\,c_2\dfrac{\sqrt{2}}{5}+11\,c_3\sqrt{2}\left(\dfrac{1}{25}\right)^2+3.481808807\,c_2\,2^{1/4}\,5^{1/4}\\
-1.193763020\,c_3\,2^{1/4}\,5^{1/4}-.2757795878=0,\\
(c_4\sqrt{2}-6\,c_5\,\dfrac{\sqrt{2}}{5}+11\,c_6\sqrt{2}\left(\dfrac{1}{25}\right)^2+1.740904404\,c_5\,3^{3/4}\,5^{1/4}\,\sqrt{2}\\
-11.34074869\,c_6\,3^{3/4}\,5^{1/4}\,\sqrt{2}-0.6382412480=0,\\
(c_4\,\sqrt{2}+2\,c_5\dfrac{\sqrt{2}}{5}-21\,c_6\sqrt{2}\left(\dfrac{1}{25}\right)^2+1.740904404\,c_5\,4^{3/4}\,5^{1/4}\,\sqrt{2}\\
-8.157380637\,c_6\,4^{3/4}\,5^{1/4}\,\sqrt{2}-1.251100475=0,\\
(c_4\sqrt{2}+2\,c_5\sqrt{2}+3\,c_6\,\sqrt{2})^2+8.704522019\,c_5\,\sqrt{2}-24.87006291\\
c_6\,\sqrt{2}-2.243503145=0,\\
c_1\sqrt{2}-2\,c_2\sqrt{2}+3\,c_3\sqrt{2}=0.
\end{array}
\quad . \tag{4.107}
$$

Next, the approximate solution is

$$
u(t)=\begin{bmatrix} 0.05524271727 \\ 0.04419417380 \\ 0.01104854344 \\ 0.4087961132 \\ 0.1325825206 \\ 0.01104854345 \end{bmatrix}^T
\begin{bmatrix} \sqrt{2} \\ (8t-2)\sqrt{2} \\ (64\,t^2-32\,t+3)\sqrt{2} \\ \sqrt{2} \\ (8t-6)\sqrt{2} \\ (64\,t^2-96\,t+35)\sqrt{2} \end{bmatrix}. \tag{4.108}
$$

Table 4.7 shows the proposed method approach to the exact solution with $k=2, M=3, \lambda=1, y=1$ and $c=1$ and its far from it with other values of k, M for example $k=1, M=7$.

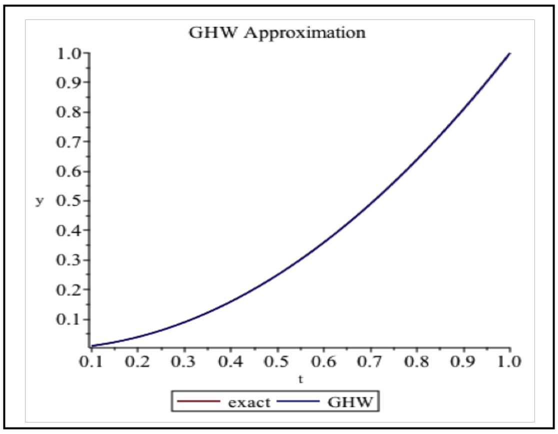


Figure 4.6 Approximate solution using the GHW method for Example 4.7

Table 4.7 Absolute error for Example 4.7

t	**Exact solution**	**The proposed method** $k = 1, M = 7$	**The proposed method** $k = 2, M = 3$
0.1	0.01	0.169306097	1×10^{-11}
0.2	0.04	0.0611645753	1×10^{-11}
0.3	0.09	0.0866363132	3×10^{-11}
0.4	0.16	0.191276130	2×10^{-10}
0.5	0.25	0.267164826	0.250000010
0.6	0.36	0.338061520	8.9×10^{-9}
0.7	0.49	0.517676305	8×10^{-9}
0.8	0.64	0.741063194	6.8×10^{-9}
0.9	0.81	0.269133388	5.8×10^{-9}

5
SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY USING GENERALIZED GEGENBAUER-HUMBERT WAVELETS

Fractional partial differential equations (FPDEs) attracted a lot of scientists, owing to effectively represent of the real world problems. A considerable techniques were evolved to find approximate solutions of FPDEs. However, these techniques can not be useful in general because of the features of each method. Wavelets beat the lacks of the methods (numerical or analytical)by utilizing a family of orthogonal functions to reduce the given problem to some of algebraic equations (linear or non- linear) as a system.

Recently wavelet techniques have a wide applications in disciplines of physics and engineering; especially signal analysis, optimal control, numerical analysis, time–frequency analysis and fast algorithms [50]. Some researchers investigated and employed a new algorithm of wavelets called scale-3 Haar wavelets to solve initial and boundary value problems and other PDEs problems [51], [52] and [53]. Massive interest have been dedicated to solve the FPDEs using operational matrices, orthogonal polynomials such as: Chebyshev polynomials, Legendre polynomials and Gegenbauer polynomials, Fourier approximation and wavelet methods. The authors used Haar wavelets and their operational matrix to solve FPDEs in [54]. R. Jiwari used the quasilinearization and uniform Haar wavelets to solve Burgers' equation [55] and [56]. In 2015 Rahimkhani and others to find the solution of pantograph fractional differential equation employed generalized form of Bernoulli wavelet with fractional order [57], while in 2018 he and Ordokhani solved the FPDEs with Dirichlet boundary conditions by using Bernoulli wavelets collocation and the fractional integral operator together [58]. In the other hand Heydari and others used Legendre wavelets with their operational matrices to solve the same type of FPDEs [59]. [60] Chohan and Shah solved FPDEs by using the operational matrices based on Jacobi polynomials. The researchers dedicated the collection method of Chebyshev wavelets (3rd kind) to solve FPDE with variable coefficients [61]. Firoozjaee and Yousefi used polynomial basis functions after transformed FPDEs into optimization problem, employed Ritz approximation to find the solution of FPDEs as a result in [62].

This chapter introduce an approximate method based on GHW method to reach the solution of the FPDEs subject to the two types of conditions (initial- boundary and boundary).

5.1 Convergence of The GHW Method

In this part, we investigate the convergence of the presented method.

Theorem 5.1. *If a continuous function $u(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$, and bounded on $[0,1) \times [0,1)$, namely $|u(x,t)| \le \delta$, then the GHW expansion of the function converges uniformly to $u(x,t)$.*

Proof. Suppose that $u(x,t)$ be a function defined over $[0,1)\times[0,1)$ and bounded such that:

$$|u(x,t)| \le \delta, \tag{5.1}$$

where δ is a positive constant. Coefficients of the GHW for the continuous function $u(x,t)$ can be defined as:

$$\begin{aligned} u_{ij} &= \int_0^1 \int_0^1 u(x,t)\, \psi_i^{y,c}(x)\, \psi_j^{y,c}(t)\, dx\, dt \\ &= 2^{k_1/2} \frac{1}{\sqrt{h_{m_1}}} \int_0^1 \int_{I_1} u(x,t)\, P_{m_1}^{\lambda,y,c}(2^{k_1}x - 2n_1 + 1)\, \psi_j^{y,c}(t)\, dx\, dt, \end{aligned} \tag{5.2}$$

where $I_1 = \left[\frac{2n_1-1}{2^{k_1}}, \frac{2n_1}{2^{k_1}}\right)$.

Now, by change of variable $2^{k_1}x - 2n_1 + 1 = v$, we get

$$u_{ij} = \frac{2^{k_1/2}}{2^{k_1}} \frac{1}{\sqrt{h_{m_1}}} \int_0^1 \psi_j^{y,c}(t) \left(\int_{-1}^1 u\left(\frac{v + 2n_1 - 1}{2^{k_1}}, t\right) P_{m_1}^{\lambda,y,c}(v)\, dv \right) dt. \tag{5.3}$$

By the mean value theorem of integral calculus, we will obtain

$$u_{ij} = \frac{1}{2^{k_1/2}} \frac{1}{\sqrt{h_{m_1}}} \int_0^1 \psi_j^{y,c}(t)\, u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, t\right) \left(\int_{-1}^1 P_{m_1}^{\lambda,y,c}(v)\, dv \right) dt, \tag{5.4}$$

where $\eta \in (-1,1)$.

$$u_{ij} = \frac{1}{2^{k_1/2}} \frac{1}{\sqrt{h_{m_1}}} \int_0^1 \psi_j^{y,c}(t)\, u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, t\right) \left(\int_{-1}^{1} \frac{c\, P_{m_1+1}^{\lambda,y,c\,\prime}(v)}{2\lambda(m_1+1)}\, dv \right) dt, \tag{5.5}$$

since $P_{m_1}^{\lambda,y,c\,\prime}(v) = \frac{2\lambda}{c}(m_1+1) P_{m_1+1}^{\lambda,y,c}(v)$.

$$\begin{aligned} u_{ij} &= \frac{1}{2^{k_1/2}} \frac{1}{\sqrt{h_{m_1}}} \int_0^1 \psi_j^{y,c}(t)\, u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, t\right) \left(\frac{c\, P_{m_1+1}^{\lambda,y,c}(v)}{2\lambda(m_1+1)} \right)_{-1}^{1} dt \\ &= \frac{c}{\lambda\, 2^{k_1/2}} \frac{1}{\sqrt{h_{m_1}}} \left(\frac{P_{m_1+1}^{\lambda,y,c}(1) - P_{m_1+1}^{\lambda,y,c}(-1)}{2(m_1+1)} \right) \\ &\times \int_0^1 \psi_j^{y,c}(t)\, u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, t\right) dt \\ &= \frac{c}{\lambda\, 2^{k_1/2}} \frac{1}{\sqrt{h_{m_1}}} \left(\frac{P_{m_1+1}^{\lambda,y,c}(1) - P_{m_1+1}^{\lambda,y,c}(-1)}{2(m_1+1)} \right) \\ &\times \int_{I_2} u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, t\right) 2^{k_2/2} \frac{1}{\sqrt{h_{m_2}}} P_{m_2}^{\lambda,y,c}(2^{k_2} t - 2n_2 + 1)\, dt \end{aligned} \tag{5.6}$$

where $I_2 = \left[\frac{2n_2-1}{2^{k_2}}, \frac{2n_2}{2^{k_2}}\right)$.

By changing the variable $2^{k_2} t - 2n_2 + 1 = \omega$, we get

$$\begin{aligned} u_{ij} = \frac{c\, 2^{k_2/2}}{\lambda\, 2^{k_1/2}\, 2^{k_2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P_{m_1+1}^{\lambda,y,c}(1) - P_{m_1+1}^{\lambda,y,c}(-1)}{2(m_1+1)} \right) \\ \times \int_{-1}^{1} u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\omega + 2n_2 - 1}{2^{k_2}}\right) P_{m_2}^{\lambda,y,c}(\omega)\, d\omega. \end{aligned} \tag{5.7}$$

Again using the mean value theorem of integral calculus, the following equation that we have

$$\begin{aligned}
u_{ij} &= \frac{c}{\lambda\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\times u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\xi + 2n_2 - 1}{2^{k_2}} \right) \int_{-1}^{1} P^{\lambda,y,c}_{m_2}(\omega)\, d\omega, where \xi \in (-1,1) \\
&= \frac{c}{\lambda\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\times u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\xi + 2n_2 - 1}{2^{k_2}} \right) \int_{-1}^{1} \frac{c\, P^{\lambda,y,c\,\prime}_{m_2}(\omega)}{2\lambda(m_2+1)}\, d\omega, \\
&= \frac{c^2}{\lambda^2\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\times u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\xi + 2n_2 - 1}{2^{k_2}} \right) \left(\frac{P^{\lambda,y,c}_{m_2}(\omega)}{2(m_2+1)} \right)_{-1}^{1} \\
&= \frac{c^2}{\lambda^2\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\times u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\xi + 2n_2 - 1}{2^{k_2}} \right) \left(\frac{P^{\lambda,y,c}_{m_2+1}(1) - P^{\lambda,y,c}_{m_2+1}(-1)}{2(m_2+1)} \right).
\end{aligned} \tag{5.8}$$

Therefore

$$\begin{aligned}
\left|u_{ij}\right| &= \frac{c^2}{\lambda^2\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\times \left| u\left(\frac{\eta + 2n_1 - 1}{2^{k_1}}, \frac{\xi + 2n_2 - 1}{2^{k_2}} \right) \right| \left(\frac{P^{\lambda,y,c}_{m_2+1}(1) - P^{\lambda,y,c}_{m_2+1}(-1)}{2(m_2+1)} \right) \\
&\leq \frac{c^2}{\lambda^2\, 2^{(k_1+k_2)/2}} \frac{1}{\sqrt{h_{m_1}\, h_{m_2}}} \left(\frac{P^{\lambda,y,c}_{m_1+1}(1) - P^{\lambda,y,c}_{m_1+1}(-1)}{2(m_1+1)} \right) \\
&\left(\frac{P^{\lambda,y,c}_{m_2+1}(1) - P^{\lambda,y,c}_{m_2+1}(-1)}{2(m_2+1)} \right) \delta,
\end{aligned} \tag{5.9}$$

since $u(x,t)$ is bounded. Hence $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij}$ is absolutely convergent.

■

5.2 Implementation of the proposed method

In this section, we will apply the GHW method to find the solution for the following problems:

5.2.1 Type 1

The partial differential equations with fractional order:

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} = F\left(x, t, u(x,t), \frac{\partial^{\gamma}u}{\partial x^{\gamma}}, \frac{\partial^{\vartheta}u}{\partial t^{\vartheta}}, \frac{\partial^{\beta}u}{\partial t^{\beta}}\right), \ \ \gamma > 0, \vartheta \leq 1, 1 < \alpha, \beta \leq 2 \tag{5.10}$$

with the Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= f_0(x), \quad u(0,t) = g_0(t), \\ u(x,1) &= f_1(x), \quad u(1,t) = g_1(t), \end{aligned} \tag{5.11}$$

where the functions $f_i(x)$ and $g_i(t)$ are twice continuously differentiable functions on $L^2[0,1]$. To solve the above problem we approximate

$$\frac{\partial^{\alpha+\beta}u}{\partial x^{\alpha}\partial t^{\beta}} \simeq \Psi^{y,c^T}(x)U\Psi^{y,c}(t), \tag{5.12}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ represent an unknown matrix which should be identified and $\Psi^{y,c}(.)$ is defined as in Eq.2.40. When applied the integration of fractional order β of Eq.5.12 with respect to t, we have

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} \simeq \Psi^{y,c^T}(x)U\,P^{y,c,\beta}\,\Psi^{y,c}(t) + \left.\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right|_{t=0} + t\,\frac{\partial}{\partial t}\left.\left(\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\right|_{t=0} \tag{5.13}$$

Putting $t = 1$ in Eq.5.13 and considering Eq.5.11, we have

$$\frac{\partial}{\partial t}\left.\left(\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\right|_{t=0} \simeq \frac{\partial^{\alpha}f_1}{\partial x^{\alpha}} - \frac{\partial^{\alpha}f_0}{\partial x^{\alpha}} - \Psi^{y,c^T}(x)U\,P^{y,c,\beta}\,\Psi^{y,c}(1). \tag{5.14}$$

By substituting Eq.5.14 into Eq.5.13, we yield

$$\begin{aligned} \frac{\partial^{\alpha}u}{\partial x^{\alpha}} \simeq\ & \Psi^{y,c^T}(x)U\,P^{y,c,\beta}\,\Psi^{y,c}(t) - t\,\Psi^{y,c^T}(x)U\,P^{y,c,\beta}\,\Psi^{y,c}(1) \\ & + (1-t)\,\frac{\partial^{\alpha}f_0}{\partial x^{\alpha}} + t\,\frac{\partial^{\alpha}f_1}{\partial x^{\alpha}}. \end{aligned} \tag{5.15}$$

On the other hand, by performing the integration of fractional order α of Eq.5.12 with respect to x, we obtain

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} \simeq (P^{y,c,\alpha}\, \Psi^{y,c}(x))^T\, U\, \Psi^{y,c}(t) + \left.\frac{\partial^{\beta} u}{\partial t^{\beta}}\right|_{x=0} + x\, \frac{\partial}{\partial x}\left(\frac{\partial^{\beta} u}{\partial t^{\beta}}\right)\Bigg|_{x=0}. \tag{5.16}$$

We putting $x = 1$ in Eq.5.16 and considering Eq.5.11, we have

$$\begin{aligned}\frac{\partial^{\beta} u}{\partial t^{\beta}} \simeq (P^{y,c,\alpha}\, \Psi^{y,c}(x))^T\, U\, \Psi^{y,c}(t) - x\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, U\, \Psi^{y,c}(t) \\ + (1-x)\, \frac{\partial^{\beta} g_0}{\partial t^{\beta}} + x\, \frac{\partial^{\beta} g_1}{\partial t^{\beta}}.\end{aligned} \tag{5.17}$$

Next, by fractional integrating of order α of Eq.5.15 with respect to x, and considering Eq.5.11, we get

$$\begin{aligned}u(x,t) \simeq &(P^{y,c,\alpha}\, \Psi^{y,c}(x))^T\, U\, P^{y,c,\beta}\, \Psi^{y,c}(t) - t\, (P^{y,c,\alpha}\, \Psi^{y,c}(x))^T \\ &U\, P^{y,c,\beta}\, \Psi^{y,c}(1) - x\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, U\, P^{y,c,\beta}\, \Psi^{y,c}(t) \\ &+ x\, t\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, U\, P^{y,c,\beta}\, \Psi^{y,c}(1) + R(x,t),\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}R(x,t) = &g_0(t) + (1-t)(f_0(x) - f_0(0) - x\, f_0'(0)) + t\, (f_1(x) \\ &- f_1(0) - x\, f_1'(0)) + x\, (g_1(t) - g_0(t)) - x\, (1-t) \\ &(f_0(1) - f_0(0) - f_0'(0)) - x\, t(f_1(1) - f_1(0) - f_1'(0)).\end{aligned} \tag{5.19}$$

Now, by fractional differentiation of order γ of Eq.5.18 with respect to x, we yield

$$\begin{aligned}\frac{\partial^{\gamma} u}{\partial x^{\gamma}} \simeq &\left(P^{y,c,\alpha-\gamma}\, \Psi^{y,c}(x)\right)^T\, UP^{y,c,\beta}\, \Psi^{y,c}(t) - t\left(P^{y,c,\alpha-\gamma}\, \Psi^{y,c}(x)\right)^T \\ &UP^{y,c,\beta}\, \Psi^{y,c}(1) - \frac{x^{1-\gamma}}{\Gamma(2-\gamma)}\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, UP^{y,c,\beta}\, \Psi^{y,c}(t) \\ &+ \frac{x^{1-\gamma}\, t}{\Gamma(2-\gamma)}\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, UP^{y,c,\beta}\, \Psi^{y,c}(1) + \frac{\partial^{\gamma} R(x,t)}{\partial x^{\gamma}}.\end{aligned} \tag{5.20}$$

By using fractional derivative of order θ of Eq.5.18 with respect to t, we get

$$\begin{aligned}\frac{\partial^{\theta} u}{\partial t^{\theta}} \simeq &(P^{y,c,\alpha}\, \Psi^{y,c}(x))^T\, UP^{y,c,\beta-\theta}\, \Psi^{y,c}(t) - \frac{t^{1-\theta}}{\Gamma(2-\theta)} \\ &(P^{y,c,\alpha}\, \Psi^{y,c}(x))^T\, UP^{y,c,\beta}\, \Psi^{y,c}(1) - x\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T \\ &UP^{y,c,\beta-\theta}\, \Psi^{y,c}(t) + \frac{x\, t^{1-\theta}}{\Gamma(2-\theta)}\, (P^{y,c,\alpha}\, \Psi^{y,c}(1))^T\, UP^{y,c,\beta} \\ &\Psi^{y,c}(1) + \frac{\partial^{\theta} R(x,t)}{\partial t^{\theta}}.\end{aligned} \tag{5.21}$$

By substituting Eq.5.15, 5.17-5.21 into Eq.5.10, and replacing $\simeq$ by $=$, and taking collocation points $x_i, t_i = (2i-1)/2\hat{m}; i = 1, 2, \cdots, \hat{m}$, into the generated equation, we get the non-linear system of algebraic equation as follows:

$$\frac{\partial^\alpha u}{\partial x^\alpha} - F\left(x, t, u(x,t), \frac{\partial^\gamma u}{\partial x^\gamma}, \frac{\partial^\vartheta u}{\partial t^\vartheta}, \frac{\partial^\beta u}{\partial t^\beta}\right)\Bigg|_{(x_i,t_i)} = 0, \;\; i, j = 1, \cdots, \hat{m}. \qquad (5.22)$$

To solve the above system and finding U, any iterative method such as Newton's iterative method can be used. We get the approximate solution by substituting U into Eq.5.18.

5.2.2 Type 2

Consider the partial fractional differential equations with the following form

$$\frac{\partial^\alpha u}{\partial x^\alpha} = F\left(x, t, u(x,t), \frac{\partial^\gamma u}{\partial x^\gamma}, \frac{\partial^\beta u}{\partial t^\beta}\right), \;\; \gamma > 0, \beta \le 1, 1 < \alpha \le 2 \qquad (5.23)$$

with initial condition

$$u(x,0) = f_0(x), \qquad (5.24)$$

and boundary conditions:

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \qquad (5.25)$$

where the functions $f_0(x)$ and $g_i(t)$ are given functions in $L^2[0,1]$. For solving this problem we approximate

$$\frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta} \simeq \Psi^{y,c^T}(x) U \Psi^{y,c}(t), \qquad (5.26)$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be found. By the integral of fractional order β of Eq. 5.26 with respect to t, we yield

$$\begin{aligned} \frac{\partial^\alpha u}{\partial x^\alpha} &\simeq \Psi^{y,c^T}(x) U \, P^{y,c,\beta} \, \Psi^{y,c}(t) + \frac{\partial^\alpha u}{\partial x^\alpha}\Bigg|_{t=0} \\ &\simeq \Psi^{y,c^T}(x) U \, P^{y,c,\beta} \, \Psi^{y,c}(t) + \frac{\partial^\alpha f_0}{\partial x^\alpha}. \end{aligned} \qquad (5.27)$$

Furthermore, by applying the fractional integration of order α of Eq.5.26 with respect to x, we obtain Eq.5.16. Putting $x = 1$ in Eq.5.16 and considering Eq.5.25, we get

Eq.5.17. Now, by the integration with fractional order α of Eq.5.27, we have

$$\begin{aligned} u(x,t) \simeq & (P^{y,c,\alpha}\ \Psi^{y,c}(x))^T\ U\ P^{y,c,\beta}\ \Psi^{y,c}(t) \\ & - f_0(x) - f_0(0) - x\ f_0'(0) + g_0(t) + x\ \frac{\partial u}{\partial x}\bigg|_{x=0}. \end{aligned} \tag{5.28}$$

By putting $x = 1$ in Eq.5.28 and concluding Eq's.5.24, 5.25, we can rewrite Eq.5.28 as

$$\begin{aligned} u(x,t) \simeq & (P^{y,c,\alpha}\ \Psi^{y,c}(x))^T\ U\ P^{y,c,\beta}\ \Psi^{y,c}(t) - x\,(P^{y,c,\alpha}\ \Psi^{y,c}(1))^T \\ & U\ P^{y,c,\beta}\ \Psi^{y,c}(t) + H(x,t) \end{aligned} \tag{5.29}$$

where

$$\begin{aligned} H(x,t) = & g_0(t) + f_0(x) - f_0(0) - x\ f_0'(0) + x\ (g_1(t) - g_0(t)) \\ & - x\ (f_0(1) - f_0(0) - f_0'(0)). \end{aligned} \tag{5.30}$$

By drive Eq.5.29 with fractional order γ with respect to x, we obtain

$$\begin{aligned} \frac{\partial^\gamma u}{\partial x^\gamma} \simeq & \left(P^{y,c,\alpha-\gamma}\ \Psi^{y,c}(x)\right)^T UP^{y,c,\beta}\ \Psi^{y,c}(t) - \frac{x^{1-\gamma}}{\Gamma(2-\gamma)} \\ & (P^{y,c,\alpha}\ \Psi^{y,c}(1))^T\ UP^{y,c,\beta}\ \Psi^{y,c}(t) + \frac{\partial^\gamma H(x,t)}{\partial x^\gamma}. \end{aligned} \tag{5.31}$$

Now, by substituting Eq.5.27,5.17,5.29 and 5.31 in Eq.5.23 with replacing $\simeq$ by $=$ and taking the collocation points as in the Type 1, we obtained the following nonlinear system of equations

$$\frac{\partial^\alpha u}{\partial x^\alpha} - F\left(x, t, u(x,t), \frac{\partial^\gamma u}{\partial x^\gamma}, \frac{\partial^\beta u}{\partial t^\beta}\right)\bigg|_{(x_i,t_i)} = 0, \quad i,j = 1,\cdots,\hat{m}, \tag{5.32}$$

which can be solved for the unknown matrix U.

5.3 Numerical Illustration

To illustrate the accuracy of the presented method, the results are examined by using L_∞, L_2 maximum absolute error and root mean square error respectively as:

$$L_\infty = \underset{1\le i\le \hat{m}}{Max} |u(x_i,t_i) - \tilde{u}(x_i,t_i)|, \tag{5.33}$$

$$L_2 = \sqrt{\frac{1}{\hat{m}} \sum_{i=1}^{\hat{m}} |u(x_i,t_i) - \tilde{u}(x_i,t_i)|^2}. \tag{5.34}$$

Example 5.1 Let consider the following fractional partial differential equation:

$$\frac{\partial^{3/2}u(x,t)}{\partial x^{3/2}}+\frac{\partial^{3/4}u(x,t)}{\partial x^{3/4}}+\frac{\partial^{4/3}u(x,t)}{\partial t^{4/3}}+u(x,t)=f(x,t), \tag{5.35}$$

with

$$f(x,t)=x^2+t+\frac{4\sqrt{x}}{\sqrt{\pi}}+\frac{16\sqrt{2}\,\Gamma(3/4)}{5\pi}, \tag{5.36}$$

and the boundary conditions:

$$\begin{aligned} &u(x,o)=x^2, \qquad u(o,t)=t,\\ &u(x,1)=x^2+1, \quad u(1,t)=1+t. \end{aligned} \tag{5.37}$$

Where the exact solution of the above problem is $u(x,t)=x^2+t$.

The approximate solution obtained by the method presented in Section 5.2.1 as follow as: Suppose

$$\frac{\partial^{\frac{3}{2}+\frac{4}{3}}u}{\partial x^{\frac{3}{2}}\partial t^{\frac{4}{3}}}\simeq \Psi^{y,c^T}(x)U\Psi^{y,c}(t). \tag{5.38}$$

By the fractional integration of order 4/3 of the Eq.5.38 with respect to t, then putting $t=1$ we obtain

$$\frac{\partial^{3/2}u}{\partial x^{3/2}}\simeq \Psi^{y,c^T}(x)U\,P^{y,c,4/3}\,\Psi^{y,c}(t)-t\,\Psi^{y,c^T}(x)U\,P^{y,c,4/3}\,\Psi^{y,c}(1)+\frac{4\sqrt{x}}{\sqrt{\pi}}. \tag{5.39}$$

While integrate Eq.5.38 of order 3/2 with respect to x, then putting $x=1$ we have

$$\frac{\partial^{4/3}u}{\partial t^{4/3}}\simeq \left(P^{y,c,3/2}\,\Psi^{y,c}(x)\right)^T U\,\Psi^{y,c}(t)-x\,\left(P^{y,c,3/2}\,\Psi^{y,c}(1)\right)^T U\,\Psi^{y,c}(t). \tag{5.40}$$

Now, integrate Eq.5.39 of order 3/2 with respect to x, yields Eq.5.18 where

$$R(x,t)=t+w^2. \tag{5.41}$$

Next, to get $\frac{\partial^{3/4}u}{\partial x^{3/4}}$ as Eq.5.20 by fractional integration of order 3/4 of Eq.5.18 with respect to x and

$$\frac{\partial^{3/4}R(x,t)}{\partial x^{3/4}}=\frac{16\,x^{5/4}\,\sqrt{2}\,\Gamma(3/4)}{5\,\pi}. \tag{5.42}$$

By substituting Eq's.5.39,5.40, 5.18 and 5.20 in Eq.5.35. Then substituting the collocation points into the obtained equation, solving the system of algebraic equations to find the unknown matrix U. The errors in some nodes $(x,t)\in[0,1]$ for $\lambda,y,c=$

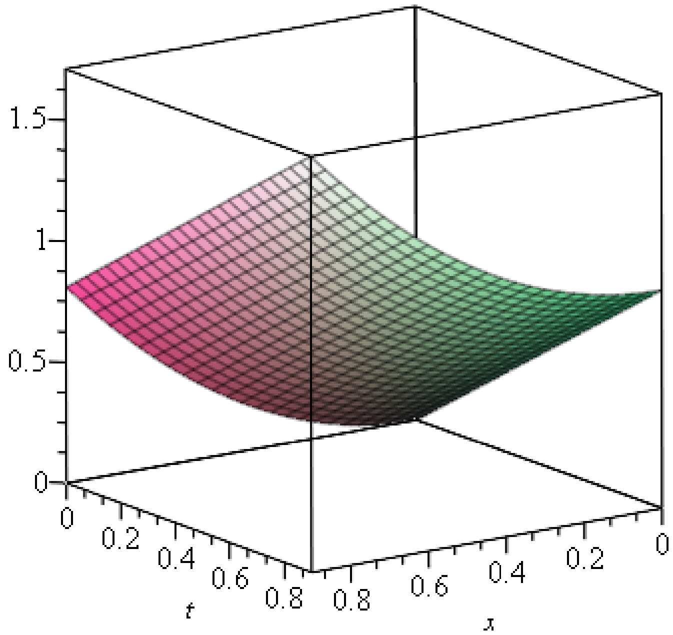


Figure 5.1 The approximate solution of Example 5.1.

$1, k = 3$ and $M = 3$ as in Table 5.1. Fig.5.1 represents the approximate solution by the GHW method for $\hat{m} = 12$. The results in Table 5.2 are compared the proposed method with the method that used by M.H. Hayderi et al [59].

Table 5.1 The errors for a different values of t of Example 5.1

t	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**
L_2	3.242×10^{-12}	2.309×10^{-11}	6.992×10^{-10}	8.834×10^{-10}	4.887×10^{-9}
L_∞	7.449×10^{-12}	4.146×10^{-11}	1.573×10^{-9}	2.245×10^{-9}	1.603×10^{-8}

Example 5.2 Consider the fractional partial differential equation:

$$\frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} + \frac{\partial^{1.2} u(x,t)}{\partial t^{1.2}} = f(x,t), \tag{5.43}$$

with

$$f(x,t) = \frac{4\sqrt{x}}{\sqrt{\pi}} + \frac{5\, t^{4/3}}{2\,\Gamma(4/5)}, \tag{5.44}$$

and the boundary conditions:

$$\begin{aligned} &u(x,o) = x^2, \qquad u(o,t) = t^2, \\ &u(x,1) = x^2+1, \quad u(1,t) = 1+t^2. \end{aligned} \tag{5.45}$$

Where the exact solution of the above problem is $u(x,t) = x^2 + t^2$.

Using the same procedure in Section 5.2.1 as: Let

$$\frac{\partial^{1.5+1.2} u}{\partial x^{1.5} \partial t^{1.2}} \simeq \Psi^{y,c^T}(x) U \Psi^{y,c}(t). \tag{5.46}$$

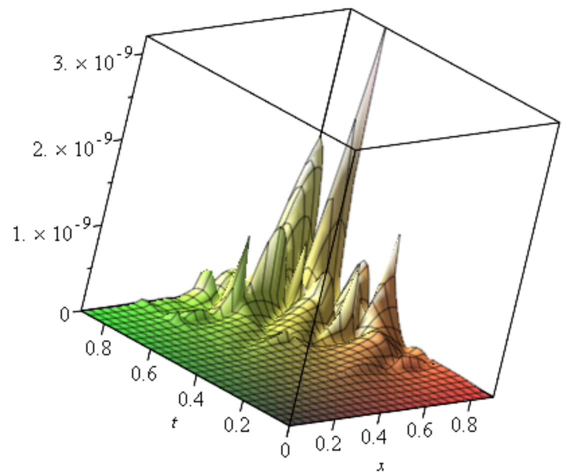
By the fractional integration of order 1.2 of the Eq.5.46 with respect to t, then putting $t = 1$ we obtain

$$\frac{\partial^{1.5} u}{\partial x^{1.5}} \simeq \Psi^{y,c^T}(x) U\, P^{y,c,1.2}\, \Psi^{y,c}(t) - t\, \Psi^{y,c^T}(x) U\, P^{y,c,1.2}\, \Psi^{y,c}(1) + 2.256758334\, \sqrt{x}. \tag{5.47}$$

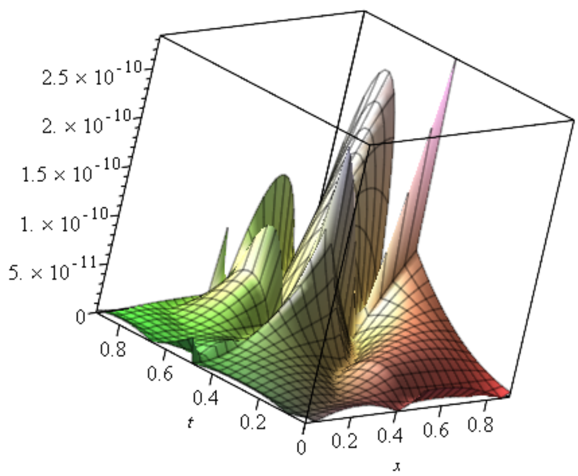
While integrate Eq.5.46 of order 1.5 with respect to x, then putting $x = 1$ we have

$$\frac{\partial^{1.2} u}{\partial t^{1.2}} \simeq \left(P^{y,c,1.5}\, \Psi^{y,c}(x)\right)^T U\, \Psi^{y,c}(t) - x\, \left(P^{y,c,1.5}\, \Psi^{y,c}(1)\right)^T U\, \Psi^{y,c}(t) + 2.147342548\, t^{4/5}. \tag{5.48}$$

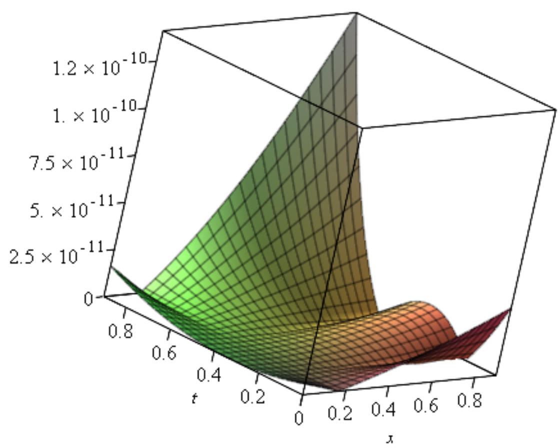
By substituting Eq's.5.47, 5.48 in Eq.5.43. Then solving the system of algebraic equations that be obtained after substitute the collocation points in the generated equation to find the unknown matrix U. Numerical results obtained by the proposed method for $y,\ c,\ \lambda = 1$ and for the different values of k, M consider Table 5.4 and Fig.5.2.



The absolute error of Example 5.2 for $k = 3,\ M = 3$.



The absolute error of Example 5.2 for $k = 2,\ M = 3$.



The absolute error of Example 5.2 for $k = 1,\ M = 3$.

Figure 5.2 Absolute errors with different values of k, M for Example 5.2.

Table 5.2 The comparison of errors between Ref. [59] and our method in different values of t of Example 5.1.

t	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**
L_∞ **Legendre wavelets method Ref. [59]**	4.90×10^{-5}	1.47×10^{-6}	2.28×10^{-6}	1.17×10^{-6}	6.45×10^{-6}
L_∞ **The proposed method**	7.449×10^{-12}	4.146×10^{-11}	1.573×10^{-9}	2.245×10^{-9}	1.603×10^{-8}
L_2 **Legendre wavelets method Ref. [59]**	8.64×10^{-7}	8.06×10^{-6}	1.38×10^{-6}	7.50×10^{-7}	8.60×10^{-8}
L_2 **The proposed method**	3.242×10^{-12}	2.309×10^{-11}	6.992×10^{-10}	8.834×10^{-10}	4.887×10^{-9}

Table 5.3 The comparison of errors between Ref. [63] and the proposed method of Example 5.3.

	Method Ref. [63] N=10	**The proposed Method** k=2, M=5	**Method Ref. [63]** N=20	**The proposed Method** k=3, M=5	**The proposed Method** k=2, M=4
L_2	1.2155×10^{-6}	7.230×10^{-10}	5.0195×10^{-7}	2.833×10^{-10}	4.493×10^{-11}
L_∞	6.3873×10^{-5}	2.100×10^{-9}	1.1114×10^{-4}	1.001×10^{-9}	7.301×10^{-11}

Table 5.4 The comparison of errors for different values of k, M in different values of t of Example 5.2.

t	0.1	0.3	0.5	0.7	0.9
L_∞ for $k=1,\ M=3$	8.065770E-12	1.798584E-11	4.476419E-12	4.859402E-11	1.412257E-10
L_2 for $k=1,\ M=3$	8.947909E-12	1.524394E-11	3.348171E-12	4.550491E-11	1.299698E-10
L_∞ for $k=2,\ M=3$	1.720468E-11	4.185996E-11	3.104470E-10	2.117184E-10	1.216400E-10
L_2 for $k=2,\ M=3$	1.468218E-11	3.080179E-11	1.777416E-10	1.268601E-10	8.001475E-11
L_∞ for $k=3,\ M=3$	2.058742E-11	1.019086E-10	1.370306E-9	5.407648E-10	1.711939E-9
L_2 for $k=3,\ M=3$	9.660581E-12	4.845856E-11	4.134269E-10	2.124966E-10	5.419098E-10

Example 5.3 Consider the following fractional partial differential equation:

$$\frac{\partial^{1/8}u(x,t)}{\partial x^{1/8}}+\frac{\partial^{1/3}u(x,t)}{\partial t^{1/3}}=\frac{8\,x^{7/8}}{7\,\Gamma\left(\frac{7}{8}\right)}+\frac{3\,t^{2/3}}{2\,\Gamma\left(\frac{2}{3}\right)},\ 0\le x,\ t\le 1, \tag{5.49}$$

with initial -boundary conditions as:

$$\begin{aligned} u(x,0)&=x, \qquad u(0,t)=2\,t,\\ u(x,1)&=x+2 \quad u(1,t)=1+2\,t. \end{aligned} \tag{5.50}$$

The exact solution of this problem $u(x,t)=x+2t$.

When we applied the GHW method to solve the above problem using the same procedure in the previous examples, Table 5.3 shows variation of the error values between the Ref. [63] method and the proposed method. In addition, GHW method given better results with less k, M and $y, c, \lambda = 1$. Fig.5.3 has been shown the absolute error when $k=2, M=4, y, c$ and $\lambda=1$. While 5.5 explained the absolute errors for a different values of k, M and $y, c, \lambda = 1$.

Example 5.4 Consider the following Burger's fractional differential equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}-\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}-u(x,t)\,\frac{\partial u(x,t)}{\partial x}=0,\ \ 0<\beta\le 1,\ 1<\alpha\le 2, \tag{5.51}$$

subject to the conditions:

$$\begin{aligned} &u(x,0)=2\,x,\\ &u(0,t)=0, \quad u(1,t)=\frac{2}{1+2\,t}. \end{aligned} \tag{5.52}$$

The exact solution of this problem $u(x,t)=\frac{2\,x}{1+2\,t}$.

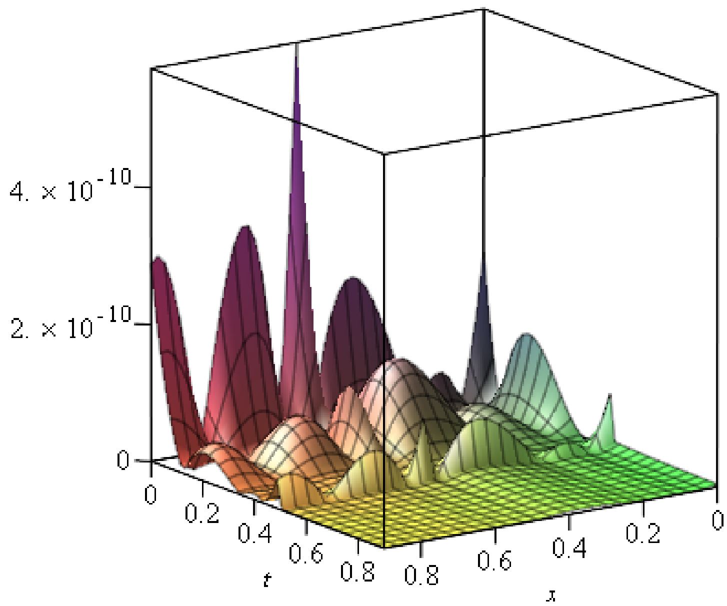


Figure 5.3 Absolute error of the approximate solution obtained by GHW method of Example 5.3

Table 5.5 The absolute errors of the proposed method for $y, c, \lambda = 1$ and different values of k, M for Example 5.3.

(x,t)	$k = 2, M = 4$	$k = 3, M = 3$	$k = 3, M = 4$
(0.1,.01)	1.6056×10^{-11}	5.2862×10^{-13}	1.2600×10^{-13}
(0.2,0.2)	9.8585×10^{-12}	3.5051×10^{-11}	1.2991×10^{-11}
(0.3,0.3)	4.0419×10^{-11}	5.3720×10^{-12}	1.4337×10^{-12}
(0.4,0.4)	2.0966×10^{-12}	1.3186×10^{-11}	1.2011×10^{-11}
(0.5,0.5)	4.9595×10^{-12}	2.2372×10^{-10}	1.1894×10^{-11}
(0.6,0.6)	1.4464×10^{-12}	9.9521×10^{-11}	6.7577×10^{-12}
(0.7,0.7)	4.9868×10^{-13}	3.7911×10^{-11}	5.6182×10^{-13}
(0.8,0.8)	5.6008×10^{-13}	2.3421×10^{-12}	3.0564×10^{-11}
(0.9,0.9)	3.5169×10^{-13}	2.3494×10^{-11}	1.1616×10^{-10}

To solve the above problem by the method in Section 5.2.2, we integrate Eg.5.26 of fractional order β with respect to t, we have

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} \simeq \Psi^{y,c^T}(x) U \, P^{y,c,\beta} \, \Psi^{y,c}(t). \tag{5.53}$$

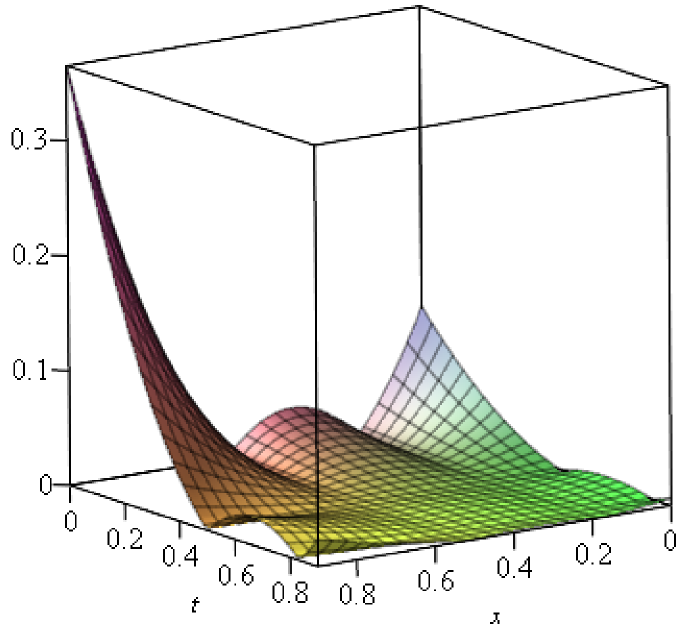


Figure 5.4 Absolute error of the approximate solution obtained by GHW method of Example 5.4.

Next by integrating Eg.5.26 of fractional order α with respect to x and putting $x = 1$, we yield

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} \simeq (P^{y,c,\alpha}\,\Psi^{y,c}(x))^{T}\,U\,\Psi^{y,c}(t) - x\,(P^{y,c,\alpha}\,\Psi^{y,c}(1))^{T}\,U\,\Psi^{y,c}(t) - \frac{4\,x}{(1+2\,t)^2}. \quad (5.54)$$

By the integration with fractional order α of Eq.5.53, and putting $x = 1$ we obtain Eq.5.29 where

$$H(x,t) = \frac{2\,x}{1+2\,t}. \quad (5.55)$$

By substituting $\gamma = 1$ in Eq.5.31 we obtain $\frac{\partial u}{\partial x}$ where

$$\frac{\partial H(x,t)}{\partial x} = \frac{2}{1+2\,t}. \quad (5.56)$$

Now, substitute Eq's 5.53,5.54, 5.29 and 5.31 in Eq.5.51 and solving the non-linear obtained system to find the unknown matrix U.

Fig.5.4 has been shown the absolute error when $k = 1, M = 3, y, c, \lambda = 1$ and $\alpha = 1.5$. The results in Table 5.6 consider the absolute errors for a different values of α with notation the results obtained for 15th digits number.

Table 5.6 The absolute errors of the proposed method for $y, c, \lambda = 1$, $k = 1$, $M = 3$ and different values of α for Example 5.4.

(x,t)	Absolute Error $\alpha = 1.2$	Absolute Error $\alpha = 1.5$	Absolute Error $\alpha = 1.75$
(0.1,0.1)	0.5361633e-1	0.2336636E-1	0
(0.2,0.2)	0.1254141	0.5675636E-2	0
(0.3,0.3)	0.1765109	0.8496859E-2	0
(0.4,0.4)	0.2235409	0.3020368E-2	0
(0.5,0.5)	0.2789657	4.889778E-13	0
(0.6,0.6)	0.3510744	0.3020368E-2	0
(0.7,0.7)	0.4439836	0.8496859E-2	0
(0.8,0.8)	0.5576372	0.5675636E-2	0
(0.9,0.9)	0.6878067	0.2336636E-1	0

Example 5.5 Consider the following time- fractional diffusion equation:

$$\frac{\partial^{\beta} u(x,t)}{\partial t^{\beta}} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\ 0 < x < 1,\ 0 < t \leq 1,\ 0 < \beta \leq 1, \tag{5.57}$$

where

$$f(x,t) = \frac{2\, t^{2-\beta}}{\Gamma(3-\beta)} - 2 \tag{5.58}$$

and the initial-boundary conditions:

$$u(x,0) = x^2,\ \ u(0,t) = t^2,\ \ u(1,t) = 1 + t^2. \tag{5.59}$$

The exact solution of this problem is $u(x,t) = x^2 + t^2$.

We reach the exact solution with error equal to zero when $k = 2, M = 2, 3,\ \beta = 0.5, 1$ and $y, c,\ \lambda = 1$. Absolute error of this problem when $k = 2, M = 3,\ \beta = 0.9$ shown in Fig.5.5 and Table 5.7 with different $k, M,\ \beta$, all the results obtained with $y, c,\ \lambda = 1$. Moreover, Table 5.8 shows the performance of GHW method when its error compared with method used in Ref [64] the results obtained when $y, c,\ \lambda = 1$ with notation the results obtained for 15th digits number.

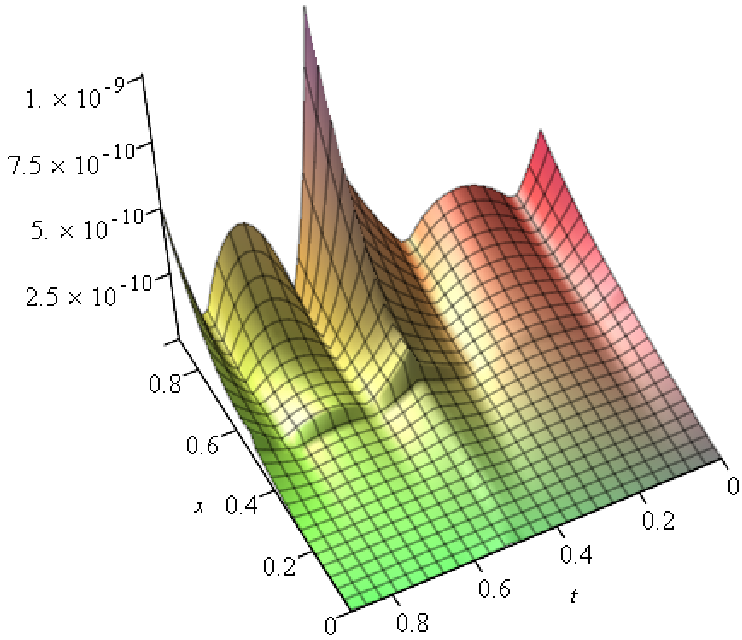


Figure 5.5 Absolute error of Example 5.5.

Table 5.7 The absolute errors for $y, c,\ \lambda = 1$ and different values of $k, M,\ \beta$ for Example 5.5.

(x,t)	$k=2, M=3,\ \beta=0.5$	$k=3, M=3,\ \beta=0.5$	$k=2, M=3,\ \beta=0.9$	$k=3, M=3,\ \beta=0.9$
(0.2,0.2)	0	1.2606×10^{-12}	6.8115×10^{-12}	1.2865×10^{-13}
(0.4,0.4)	0	1.6297×10^{-12}	2.4000×10^{-12}	8.9881×10^{-12}
(0.6,0.6)	5.8127×10^{-12}	3.4912×10^{-11}	1.6885×10^{-11}	5.0683×10^{-11}
(0.8,0.8)	1.2010×10^{-10}	2.2643×10^{-11}	9.2382×10^{-11}	4.0967×10^{-11}

Table 5.8 Comparison the absolute errors for Example 5.5.

	The method in Ref. [64]			**The proposed method**	
(x,t)	$J=1, m=2$	$J=1, m=3$	$J=2, m=2$	$k=2, M=3,\ \beta=0.5$	$k=3, M=3,\ \beta=0.5$
(0.2,0.25)	3.3×10^{-2}	4.4×10^{-3}	8.8×10^{-2}	0	7.6392×10^{-13}
(0.4,0.25)	1.9×10^{-2}	5.1×10^{-2}	9.8×10^{-2}	0	2.7958×10^{-13}
(0.6,0.25)	1.6×10^{-2}	7.1×10^{-2}	3.4×10^{-1}	0	4.4831×10^{-12}
(0.8,0.25)	1.2×10^{-1}	2.8×10^{-2}	4.3×10^{-1}	0	1.6001×10^{-11}

Example 5.6

$$\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^\beta u(x,t)}{\partial t^\beta} + x\frac{\partial u(x,t)}{\partial x} = f(x,t),\ \ 0 \le x,\ t < 1,\ 0 < \beta \le 1 \qquad (5.60)$$

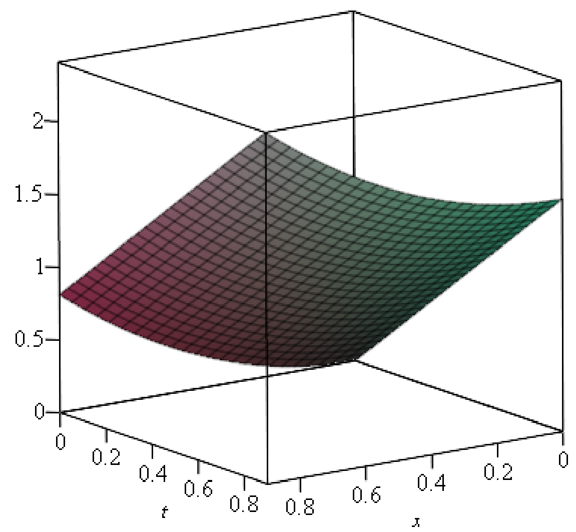


Figure 5.6 Approximate solution of Example 5.6 with $k = 2, M = 3,\ \beta = 0.5$

With

$$f(x,t) = 2\,t^{\beta} + 2\,x^2 + 2, \tag{5.61}$$

and the initial- boundary conditions as:

$$\begin{aligned} &u(x,0) = x^2, \\ &u(0,t) = \frac{2\,\Gamma(\beta+1)}{\Gamma(2\beta+1)}\,t^{2\beta},\ \ u(1,t) = 1 + \frac{2\,\Gamma(\beta+1)}{\Gamma(2\beta+1)}\,t^{2\beta}. \end{aligned} \tag{5.62}$$

The exact solution of the above problem is

$$u(x,t) = x^2 + \frac{2\,\Gamma(\beta+1)}{\Gamma(2\beta+1)}\,t^{2\beta}. \tag{5.63}$$

We applied the GHW method to solve this problem for $y, c,\ \lambda = 1$ and $k = 2, M = 3, \beta = 0.5$ as in Fig's 5.6 and 5.7. F. Zhou and X. Xu Ref. [61] established the preference of their method by a comparison between the 3rd kind of Chebyshev wavelets collection to solve this problem with some other methods. As a result, to prove the efficiency and accuracy of the GHW method, Table 5.9 and Table 5.10 comparing the results of the proposed method with Ref. [61] results with notation the results obtained for 17th digits number.

Table 5.9 The absolute errors of GHW and Ref. [61] method when $k = 2$, $M = 3$ with a different values of β of Example 5.6.

	The method in Ref. [61]					The proposed method				
(x,t)	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
(0.1,0.1)	0	0	2.775×10^{-17}	1.387×10^{-17}	3.122×10^{-17}	4.211×10^{-19}	2.107×10^{-18}	5.266×10^{-19}	5.263×10^{-19}	1.801×10^{-18}
(0.2,0.2)	0	1.110×10^{-16}	1.110×10^{-16}	5.551×10^{-17}	2.220×10^{-16}	6.770×10^{-18}	1.656×10^{-18}	4.274×10^{-19}	4.604×10^{-19}	9.973×10^{-18}
(0.3,0.3)	2.220×10^{-16}	0	2.220×10^{-16}	5.551×10^{-17}	2.220×10^{-16}	1.384×10^{-17}	1.677×10^{-18}	3.366×10^{-19}	1.718×10^{-19}	8.135×10^{-18}
(0.4,0.4)	0	0	1.110×10^{-16}	1.110×10^{-16}	1.776×10^{-15}	4.912×10^{-18}	1.248×10^{-18}	3.147×10^{-19}	4.469×10^{-19}	1.297×10^{-17}
(0.5,0.5)	0	2.220×10^{-16}	0	4.440×10^{-16}	2.564×10^{-12}	5.232×10^{-17}	1.751×10^{-19}	1.522×10^{-19}	6.463×10^{-19}	3.388×10^{-17}
(0.6,0.6)	0	4.440×10^{-16}	0	0	2.700×10^{-13}	9.554×10^{-18}	2.133×10^{-19}	1.399×10^{-19}	5.104×10^{-19}	1.472×10^{-17}
(0.7,0.7)	0	0	2.220×10^{-16}	2.220×10^{-16}	6.972×10^{-13}	5.414×10^{-17}	2.726×10^{-19}	8.199×10^{-20}	7.549×10^{-20}	4.138×10^{-18}
(0.8,0.8)	4.440×10^{-16}	0	0	2.220×10^{-16}	3.648×10^{-13}	6.441×10^{-17}	3.457×10^{-19}	5.195×10^{-20}	3.013×10^{-19}	1.146×10^{-17}
(0.9,0.9)	0	4.440×10^{-16}	4.440×10^{-16}	4.440×10^{-16}	1.209×10^{-12}	1.709×10^{-17}	4.581×10^{-19}	1.400×10^{-19}	3.495×10^{-20}	1.105×10^{-17}

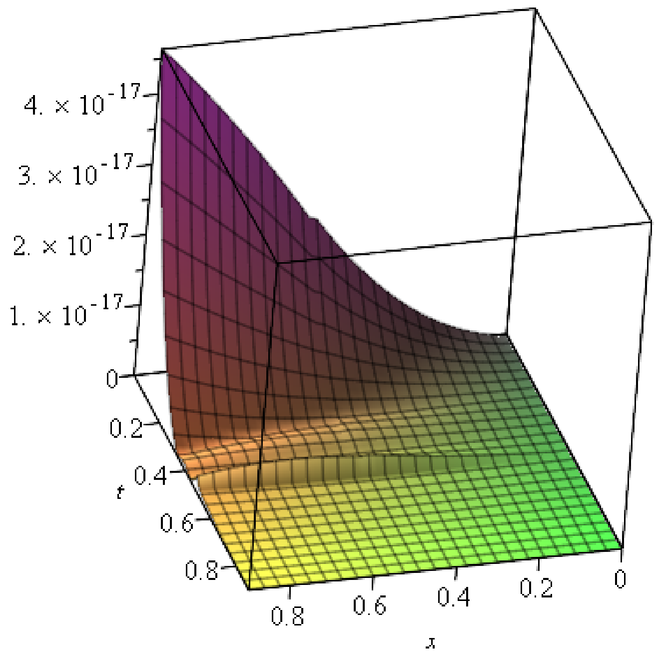


Figure 5.7 Absolute error of Example 5.6 with $k = 2, M = 3,\ \beta = 0.5$

Table 5.10 Comparison of the absolute error of Example 5.6 with $k = 2, M = 3, \beta = 0.5$ and $t = 0.5$.

x	Exact solution	Method of Ref. [61]	The present method
(0.1,0.1)	0.896226925452758	1.110×10^{-16}	0
(0.2,0.2)	0.926226925452758	1.110×10^{-16}	0
(0.3,0.3)	0.976226925452758	2.220×10^{-16}	0
(0.4,0.4)	1.04622692545276	2.220×10^{-16}	0
(0.5,0.5)	1.13622692545276	0	0
(0.6,0.6)	1.24622692545276	0	0
(0.7,0.7)	1.37622692545276	2.220×10^{-16}	0
(0.8,0.8)	1.52622692545276	0	0
(0.9,0.9)	1.69622692545276	0	0

6
SOLVING A COUPLED TIME- FRACTIONAL PARTIAL DIFFERENTIAL BY USING GENERALIZED GEGENBAUER- HUMBERT WAVELETS

Although, widespread of using partial differential equations to model different physical and mathematical problems, majority of the PDE applications arise when modeling these problems by using coupled systems of partial differential equations. For instance chemical and engineering [65], [66], modeling heart electrical activity (bio-mechanics) [67], [68] and modeling gravitational problems [69].

Fractional derivation and integration have attracted the attention of the researchers recently. Comparing with differential equations with integer order fractional-order differential equations have proven its efficiency and accuracy of describe the real problems. Therefore, many authors modeled most of physical and engineering problems by using systems of fractional differential equations for instance [70–75] and [76]. As a result, several methods investigated to solve fractional partial differential equations (FPDEs) analytically and numerically such as Kudryashov and Bernstein methods [77, 78], homotopy analysis method [79, 80], transform method like Sumudu and reduced differential [81, 82], and Adomain decomposition method [83]. Wavelet is one of the numerical techniques based on orthogonal polynomials used to find the approximate solution of FPDEs [84–87] and [50].

In fluid dynamics, the coupled systems of Whitham-Broer-Kaup (WBK) equations are described shallow water waves propagation [88] with the form:

$$\begin{aligned} &u_t + u\,u_x + \omega_x + \delta\,u_{xx} = 0, \\ &\omega_t + (u\,\omega)_x + \eta\,u_{xxx} - \delta\,\omega_{xx} = 0, \end{aligned} \tag{6.1}$$

where the horizontal velocity denotes by $u(x,t)$, $\omega(x,t)$ represent the height that deviates from the equilibrium postion of liquid and δ,η are constants to represent the different diffusion powers. In the last years, many techniques are developed to obtain the solutions of the (WBK) equations exactly and numerically like Xie et al. investigate some solutions of new solitary wave by using the hyperbolic function method [89], while Sayed and Kaya obtained the aapproximate solutions by applying Adomain decomposition method [90] and by homotopy perturbation method Mohyud-Din et al. [91] find the exact and approximate traveling wave solution of the (WBK) systems.

The time fractional form of (WBK) equations:

$$\begin{aligned} &D_t^{\mu} u + u\, u_x + \omega_x + \delta\, u_{xx} = 0, \\ &D_t^{\mu} \omega + (u\,\omega)_x + \eta\, u_{xxx} - \delta\, \omega_{xx} = 0, \end{aligned} \tag{6.2}$$

where $0 < \mu \leq 1$. When $\eta = 1$ and $\delta = 0$, system Eq.6.2 becomes modified Boussinesq equations (MB) with fractional order and if $\eta = 0,\ \delta = 1/2$ being approximate long wave equations(ALW). In [92] Wang and Chen used residual power series method to find the approximate travelling solutions of time fractional (WBK) equations. Ali et al. [93] employed Laplace transform with Adomian decomposition method to find the numerical solution of the fractional coupled nonlinear (WBK) systems. To construct approximate solutions for a nonlinear coupled WBK and Jaulent–Miodek system Al-Smadi et al.[94] implement the conformable residual power series.

In 19th century, Korteweg-de Vries equation (KdV) grew up the in the shallow water by Hirota and Satsuma and it takes a wide applications for instance wave of ion acoustic in plasma, in one dimensional long waves in shallow water waves and in the density-stratified of ocean. The form of time fractional KdV can be as:

$$\begin{aligned} &D_t^{\mu} u = \zeta\, u\, u_x + \gamma\, \omega\, \omega_x + \varsigma\, u_{xxx} + f(x,t), \\ &D_t^{\beta} \omega = \varsigma\, \omega_{xxx} - \gamma\, u\, \omega_x + g(x,t), \end{aligned} \tag{6.3}$$

where $0 < \mu,\ \beta \leq 1$ and $\zeta,\ \gamma,\ \varsigma$ are known constants. Bulut et al. [95] solved coupled systems of the KdV equations by Haar wavelets and in 2018 Albuohimad et al. solved these systems by using spectral collection method [96]. The authors in [97] and [98] studied the solutions of KdV equations of 5th order and generalized KdV equation. Ghany and Bab have investigated the Wick-type stochastic coupled KdV equation with fractional order and the exact solution of it are presented [99]. Based on Legendre polynomials Bhrawy and his friends solved the time fractional coupled Kdv equations [100].

In this chapter, we choose another orthogonal polynomials called generalized Gegnbauer -Humbert polynomials to construct generalized wavelets method for solving coupled systems of FPDEs. The presented method are new to solve two types of shallow waters as a coupled systems in addition to the known methods.

6.1 characterization of the proposed method

In this section, we applied the GHW with their operational matrices of integration to solve the following problems:

6.1.1 WBK equation

The partial differential equations with fractional order:

$$\begin{aligned} &D_t^{\mu} u + u\, u_x + \omega_x + \delta\, u_{xx} = 0, \\ &D_t^{\mu} \omega + (u\, \omega)_x + \eta\, u_{xxx} - \delta\, \omega_{xx} = 0, \end{aligned} \tag{6.4}$$

with the intial conditions:

$$u(x,0) = f(x), \;\; \omega(x,0) = g(x). \tag{6.5}$$

The procedure of the proposed method summarized as:

To solve this system, we suppose that

$$\frac{\partial^{\mu+3} u}{\partial t^{\mu} \partial x^3} = \Psi^{y,c\,T}(x) U \Psi^{y,c}(t), \tag{6.6}$$

and

$$\frac{\partial^{\mu+3} \omega}{\partial t^{\mu} \partial x^3} = \Psi^{y,c\,T}(x) W \Psi^{y,c}(t), \tag{6.7}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ and $W = [\omega_{ij}]_{\hat{m}\times\hat{m}}$ are unknown matrices which should be found and $\Psi^{y,c}(.)$ is the vector that is defined in Eq. 2.40. By fractional integration of order μ of Eq. 6.6 with respect to t and substituting the initial condition, we obtain

$$\frac{\partial^3 u}{\partial x^3} = \Psi^{y,c\,T}(x) U\, P^{y,c,\mu}\, \Psi^{y,c}(t) + f'''(x). \tag{6.8}$$

Now, integrating Eq.6.6 three times with respect to x we have

$$\begin{aligned} \frac{\partial^{\mu} u}{\partial t^{\mu}} = &\left(P^{y,c,3}\, \Psi^{y,c}(x)\right)^T\, U\, \Psi^{y,c}(t) + \left.\frac{\partial^{\mu} u}{\partial t^{\mu}}\right|_{x=0} + x\, \frac{\partial}{\partial x}\left(\frac{\partial^{\mu} u}{\partial t^{\mu}}\right)\Bigg|_{x=0} \\ &+ \frac{x^2}{2} \frac{\partial^2}{\partial x^2}\left(\frac{\partial^{\mu} u}{\partial t^{\mu}}\right)\Bigg|_{x=0}. \end{aligned} \tag{6.9}$$

Putting $x = 1$ in Eq.6.9 and let $u(0,t) = u_0(t)$, $u(1,t) = u_1(t)$ and $\frac{\partial u(0,t)}{\partial x} = u_3(t)$ and can be obtained from the exact solutions, we get

$$\frac{\partial^\mu u}{\partial t^\mu} = \left(P^{y,c,3}\, \Psi^{y,c}(x)\right)^T\, U\, \Psi^{y,c}(t) - x^2\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, U \\ \Psi^{y,c}(t) + x^2\, \frac{\partial^\mu u_1}{\partial t^\mu} + (1-x^2)\, \frac{\partial^\mu u_0}{\partial t^\mu} + (x-x^2)\, \frac{\partial^\mu u_3}{\partial t^\mu}. \tag{6.10}$$

By integrating of fractional order μ of Eq.6.10 with respect to t, we get

$$u(x,t) = \left(P^{y,c,3}\, \Psi^{y,c}(x)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) \\ -(x^2+x)\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) + H, \tag{6.11}$$

where H defined as the following equation

$$H = f(x) + x^2\,(u_1(t) - u_1(0)) + (1-x^2)\,(u_0(t) - u_0(0)) \\ +(x-x^2)\,(u_3(t) - u_3(0)). \tag{6.12}$$

Derive Eq.6.11 two times with respect to x we obtain the following equations

$$\frac{\partial u}{\partial x} = \left(P^{y,c,2}\, \Psi^{y,c}(x)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) \\ -(2\,x+1)\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) + \frac{\partial H}{\partial x}. \tag{6.13}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(P^{y,c,1}\, \Psi^{y,c}(x)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) \\ -2\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, U\, P^{y,c,\mu}\, \Psi^{y,c}(t) + \frac{\partial^2 H}{\partial x^2}. \tag{6.14}$$

The same procedure applied of the next systems, we get

$$\frac{\partial^3 \omega}{\partial x^3} = \Psi^{y,c\,T}(x) W\, P^{y,c,\mu}\, \Psi^{y,c}(t) + g'''(x). \tag{6.15}$$

Let $\omega(0,t) = \omega_0(t)$, $\omega(1,t) = \omega_1(t)$ and $\frac{\partial \omega(0,t)}{\partial x} = \omega_3(t)$ and can be obtained from the exact solutions

$$\frac{\partial^\mu \omega}{\partial t^\mu} = \left(P^{y,c,3}\, \Psi^{y,c}(x)\right)^T\, W\, \Psi^{y,c}(t) - x^2\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, W \\ \Psi^{y,c}(t) + x^2\, \frac{\partial^\mu \omega_1}{\partial t^\mu} + (1-x^2)\, \frac{\partial^\mu \omega_0}{\partial t^\mu} + (x-x^2)\, \frac{\partial^\mu \omega}{\partial t^\mu}. \tag{6.16}$$

$$\omega(x,t) = \left(P^{y,c,3}\, \Psi^{y,c}(x)\right)^T\, W\, P^{y,c,\mu}\, \Psi^{y,c}(t) \\ -(x^2+x)\, \left(P^{y,c,3}\, \Psi^{y,c}(1)\right)^T\, W\, P^{y,c,\mu}\, \Psi^{y,c}(t) + R, \tag{6.17}$$

and R defined as

$$R = g(x) + x^2\,(\omega_1(t) - \omega_1(0)) + (1 - x^2)\,(\omega_0(t) - \omega_0(0)) \\ + (x - x^2)\,(\omega_3(t) - \omega_3(0)). \tag{6.18}$$

$$\frac{\partial \omega}{\partial x} = \left(P^{y,c,2}\,\Psi^{y,c}(x)\right)^T\,W\,P^{y,c,\mu}\,\Psi^{y,c}(t) \\ -(2\,x+1)\,\left(P^{y,c,3}\,\Psi^{y,c}(1)\right)^T\,W\,P^{y,c,\mu}\,\Psi^{y,c}(t) + \frac{\partial R}{\partial x}. \tag{6.19}$$

$$\frac{\partial^2 \omega}{\partial x^2} = \left(P^{y,c,1}\,\Psi^{y,c}(x)\right)^T\,W\,P^{y,c,\mu}\,\Psi^{y,c}(t) \\ -2\,\left(P^{y,c,3}\,\Psi^{y,c}(1)\right)^T\,W\,P^{y,c,\mu}\,\Psi^{y,c}(t) + \frac{\partial^2 R}{\partial x^2}. \tag{6.20}$$

Finally, substituting Eq.6.8,(6.10-6.20) in Eq.6.4 then take the collocation points for t, x we obtain an algebraic nonlinear systems to find the matrices of coefficients U, W.

6.1.2 Kdv equation

Consider the time fractional coupled equation with the following form

$$D_t^{\mu} u = \zeta\,u\,u_x + \gamma\,\omega\,\omega_x + \varsigma\,u_{xxx} + f(x,t), \\ D_t^{\beta}\,\omega = \varsigma\,\omega_{xxx} - \gamma\,u\,\omega_x + g(x,t), \tag{6.21}$$

with initial condition

$$u(x,0) = f_0(x), \;\; \omega(x,0) = g_0(x) \tag{6.22}$$

and boundary conditions:

$$u(0,t) = v_1(t), \;\; u(1,t) = v_2(t), \;\; u_x(0,t) = v3(t) \\ \omega(0,t) = r_1(t), \;\; \omega(1,t) = r_2(t), \;\; \omega_x(0,t) = r3(t) \tag{6.23}$$

where ζ, γ, ς are known constants. Now, to solve the above coupled systems we approximate the highest order using GHW wavelets as:

$$\frac{\partial^3 u}{\partial x^3} \simeq \Psi^{y,c\,T}(x) U \Psi^{y,c}(t), \tag{6.24}$$

$$\frac{\partial^3 \omega}{\partial x^3} \simeq \Psi^{y,c\,T}(x) W \Psi^{y,c}(t), \tag{6.25}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ and $W = [\omega_{ij}]_{\hat{m}\times\hat{m}}$ are unknown coefficients matrices which should be found and $\Psi^{y,c}(.)$ is the vector defined in Eq.2.24. By integrating Eq.6.24

three times with respect to x then substitute the boundary conditions, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \left(P^{y,c,1}\ \Psi^{y,c}(x)\right)^T\ U\ \Psi^{y,c}(t) + \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0}. \tag{6.26}$$

$$\frac{\partial u}{\partial x} = \left(P^{y,c,2}\ \Psi^{y,c}(x)\right)^T\ U\ \Psi^{y,c}(t) + x\ \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} + \left.\frac{\partial u}{\partial x}\right|_{x=0}. \tag{6.27}$$

$$\begin{aligned} u(x,t) = &\left(P^{y,c,3}\ \Psi^{y,c}(x)\right)^T U\ P^{y,c,\beta}\ \Psi^{y,c}(t) + v_1(t) \\ &+ \frac{x^2}{2}\ \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} + x\ \left.\frac{\partial u}{\partial x}\right|_{x=0}. \end{aligned} \tag{6.28}$$

Putting $x = 1$ in last equation, we get

$$\begin{aligned} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} =& 2\ v_2(t) - 2\ v_1(t) - 2\ v_3(t) - 2\ \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T \\ & U\ P^{y,c,\beta}\ \Psi^{y,c}(t) \end{aligned} \tag{6.29}$$

Next, substituting Eq.6.29 in Eqs.6.26,6.27 and 6.28 we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} =& \left(P^{y,c,1}\ \Psi^{y,c}(x)\right)^T\ U\ \Psi^{y,c}(t) - 2\ \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T\ U \\ & \Psi^{y,c}(t) - 2\ (v_2(t) - v_1(t) - v_3(t)). \end{aligned} \tag{6.30}$$

$$\begin{aligned} \frac{\partial u}{\partial x} =& \left(P^{y,c,2}\ \Psi^{y,c}(x)\right)^T\ U\ \Psi^{y,c}(t) - 2\ x\ \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T \\ & U\ P^{y,c,\beta}\ \Psi^{y,c}(t) + 2\ x\ (v_2(t) - v_1(t) - v_3(t)) + v_3(t). \end{aligned} \tag{6.31}$$

$$\begin{aligned} u(x,t) =& \left(P^{y,c,3}\ \Psi^{y,c}(x)\right)^T U\ P^{y,c,\beta}\ \Psi^{y,c}(t) - x^2 \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T \\ & U\ P^{y,c,\beta}\ \Psi^{y,c}(t) + v_1(t) + (x - x^2)\ v_3(t) \\ & + x^2\ (v_2(t) - v_1(t)). \end{aligned} \tag{6.32}$$

We applied the same steps of Eq.6.25 as

$$\begin{aligned} \frac{\partial^2 \omega}{\partial x^2} =& \left(P^{y,c,1}\ \Psi^{y,c}(x)\right)^T\ W\ \Psi^{y,c}(t) - 2\ \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T \\ & W\ \Psi^{y,c}(t) - 2\ (r_2(t) - r_1(t) - r_3(t)). \end{aligned} \tag{6.33}$$

$$\begin{aligned} \frac{\partial \omega}{\partial x} =& \left(P^{y,c,2}\ \Psi^{y,c}(x)\right)^T\ W\ \Psi^{y,c}(t) - 2\ x\ \left(P^{y,c,3}\ \Psi^{y,c}(1)\right)^T \\ & U\ P^{y,c,\beta}\ \Psi^{y,c}(t) + 2\ x\ (r_2(t) - r_1(t) - r_3(t)) + r_3(t). \end{aligned} \tag{6.34}$$

$$\omega(x,t) = \left(P^{y,c,3}\,\Psi^{y,c}(x)\right)^T W\,P^{y,c,\beta}\,\Psi^{y,c}(t) - x^2$$
$$\left(P^{y,c,3}\,\Psi^{y,c}(1)\right)^T\,W\,P^{y,c,\beta}\,\Psi^{y,c}(t) + r_1(t) + (x - x^2) \tag{6.35}$$
$$r_3(t) + x^2\,(r_2(t) - r_1(t)).$$

We substitute Eqs.6.24,6.25 and 6.30-6.35 in Eqs.6.21 to get an algebraic nonlinear systems after taking the collocation points for t, x, The solution of the algebraic systems are the coefficient matrices U, W.

6.2 Numerical Exterminates

In order to evaluate the difference between analytic and numerical solutions, we concern the root mean square error L_2 and maximum absolute error L_∞ as:

$$L_\infty = \underset{1\le i\le \hat{m}}{Max}\,|u(x_i,t_i) - \tilde{u}(x_i,t_i)|\,, \tag{6.36}$$

$$L_2 = \sqrt{\frac{1}{\hat{m}}\sum_{i=1}^{\hat{m}}|u(x_i,t_i) - \tilde{u}(x_i,t_i)|^2}. \tag{6.37}$$

Example 6.1 Let be consider the following fractional WBK equations:

$$\begin{aligned} &D_t^{\mu} u + u\,u_x + \omega_x + \delta\,u_{xx} = 0,\\ &D_t^{\mu}\omega + (u\,\omega)_x + \eta\,u_{xxx} - \delta\,\omega_{xx} = 0, \end{aligned} \tag{6.38}$$

with the initial conditions:

$$\begin{aligned} &u(x,0) = \vartheta - 2\,B\,\xi\,coth(\xi\,(x+\tau)),\\ &\omega(x,0) = -2\,B\,(B+\delta)\,\xi^2\,csch^2(\xi\,(x+\tau)), \end{aligned} \tag{6.39}$$

where $B = \sqrt{\eta + \delta^2}$ and $\vartheta,\ \xi,\ \tau$ are arbitrary constants. The exact solutions of this problem are

$$\begin{aligned} &u(x,t) = \vartheta - 2\,B\,\xi\,coth(\xi\,(x+\tau-\vartheta\,t)),\\ &\omega(x,t) = -2\,B\,(B+\delta)\,\xi^2\,csch^2(\xi\,(x+\tau-\vartheta\,t)). \end{aligned} \tag{6.40}$$

Assume that, we take $\vartheta = 0.005,\ \xi = 0.1,\ \eta = \delta = 1.5$ and $\tau = 10$. Figures 6.1 and 6.2 shows the solutions of the above systems using the GHW method comparison with the exact solutions each of $u(x,t)$ and $\omega(x,t)$ respectively. Tables 5.1 and 5.2 present the absolute errors between the numerical solutions by The Adomian's decomposition method (ADM), the variational iteration method (VIM), the optimal homotopy asymptotic method (OHAM), the proposed method (GHW) for $\mu, y, c, k =$

1, $M = 3$ $\lambda = 0.75$ and the exact solution of Example 6.1. As the results shown in Table 6.1 and Table 6.2 the approximate solutions using GHW method are converge to the exact solutions more than the other methods used in Ref.[101].

Table 6.1 The absolute errors of $u(x,t)$ obtained by GHW method for μ, y, c and $k = 1$, $M = 3$ $\lambda = 0.75$ and other different methods for Example 6.1.

(x,t)	$\lvert u_{exact} - u_{ADM}\rvert$	$\lvert u_{exact} - u_{VIM}\rvert$	$\lvert u_{exact} - u_{OHAM}\rvert$	$\lvert u_{exact} - u_{GHW}\rvert$
(0.1, 0.1)	1.04892E-4	1.23033E-4	1.07078E-4	5.8025294E-9
(0.1, 0.3)	9.64474E-5	3.69597E-4	3.04565E-4	5.44560154E-8
(0.1, 0.5)	8.88312E-5	6.16873E-4	4.81303E-4	1.82970833E-8
(0.2, 0.1)	4.25408E-4	1.19869E-4	1.04388E-4	8.42524053E-8
(0.2, 0.3)	3.91098E-4	3.60098E-4	2.97260E-4	3.200613516E-7
(0.2, 0.5)	3.60161E-4	6.01006E-4	4.70138E-4	4.207822714E-7
(0.3, 0.1)	9.71922E-4	1.16789E-4	1.01776E-4	1.413342633E-7
(0.3, 0.3)	8.93309E-4	3.50866E-4	2.90150E-4	5.209119286E-7
(0.3, 0.5)	8.22452E-4	5.85610E-4	4.59590E-4	7.11230198E-7
(0.4, 0.1)	1.75596E-3	1.13829E-4	9.92418E-5	1.829480986E-7
(0.4, 0.3)	1.61430E-3	3.41948E-4	2.83229E-4	6.768077318E-7
(0.4, 0.5)	1.48578E-3	5.70710E-4	4.49118E-4	9.22440849E-7
(0.5, 0.1)	2.79519E-3	1.10936E-4	9.67808E-4	2.15394022E-7
(0.5, 0.3)	2.56714E-3	3.33274E-4	2.76492E-4	8.05449061E-7
(0.5, 0.5)	2.36184E-3	5.56235E-4	4.38895E-4	1.083714796E-6

Example 6.2 Consider the time modified Boussinesq equation that is represent a special case of WBK equation when $\eta = 1, \delta = 0$ as:

$$\begin{aligned} D_t^{\mu} u + u u_x + \omega_x &= 0, \\ D_t^{\mu} \omega + (u\,\omega)_x + u_{xxx} &= 0, \end{aligned} \tag{6.41}$$

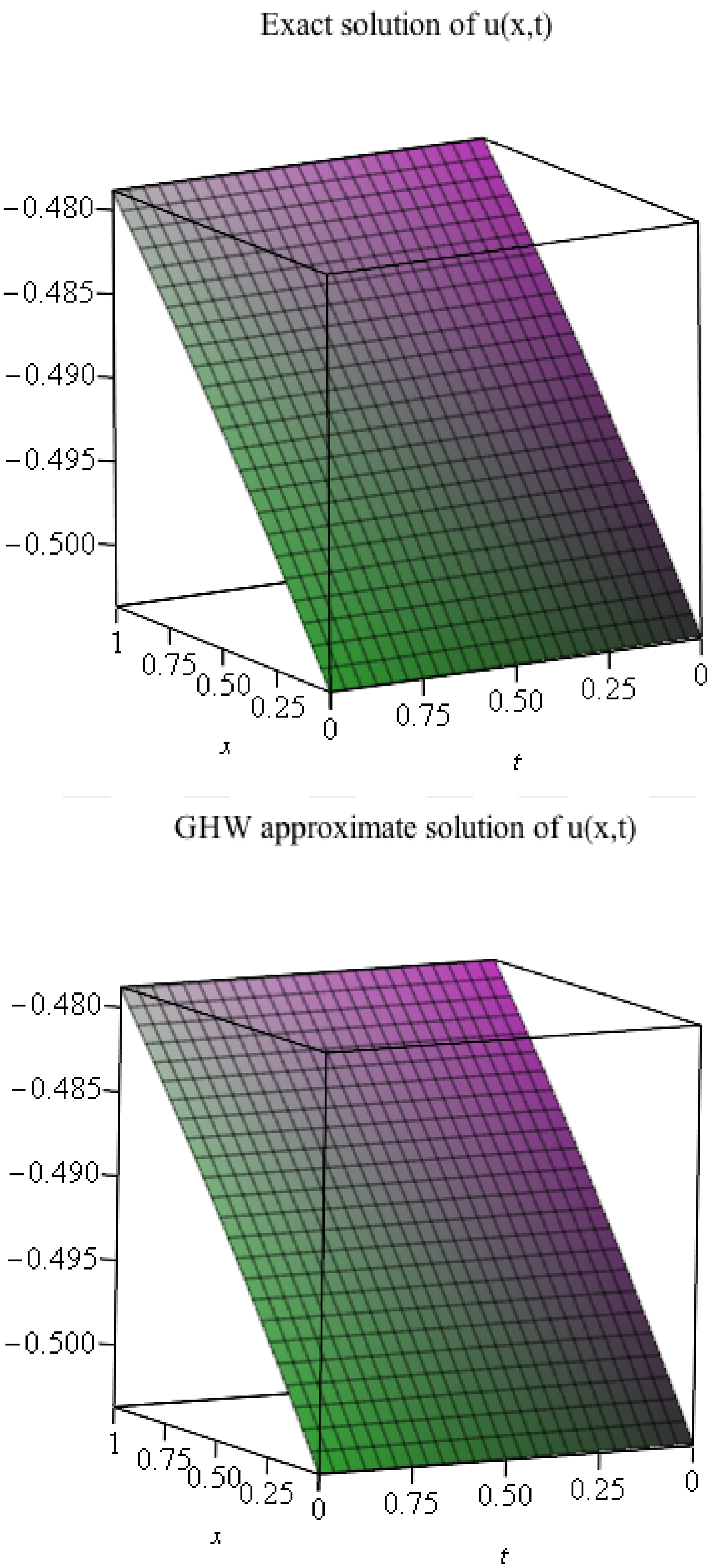
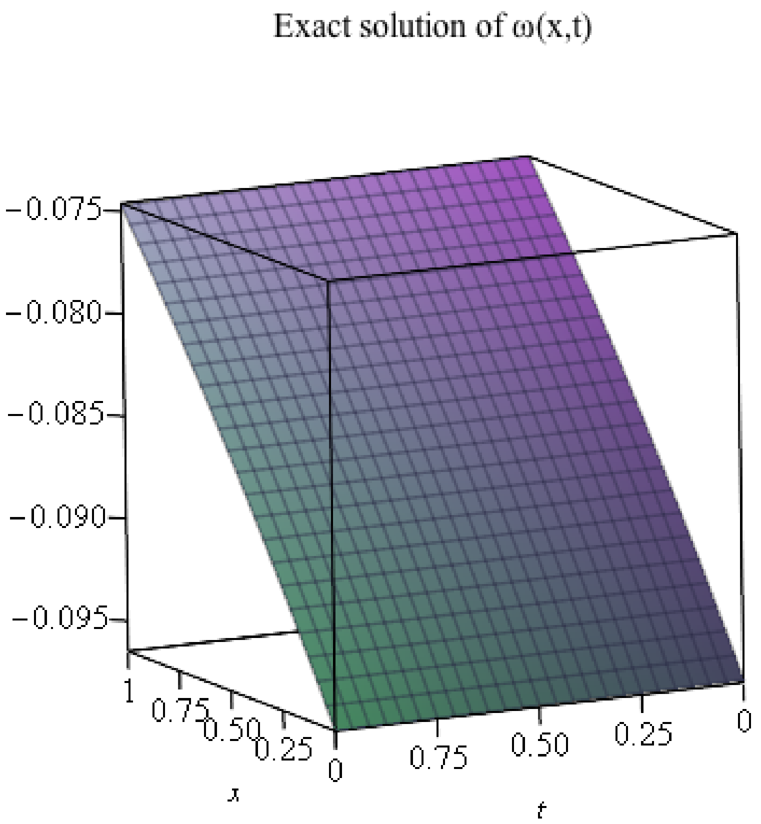


Figure 6.1 Exact and GHW approximate solution of $u(x,t)$ Example 6.1 for $\mu, k, y, c = 1,\ \lambda = 0.75,\ M = 3$.



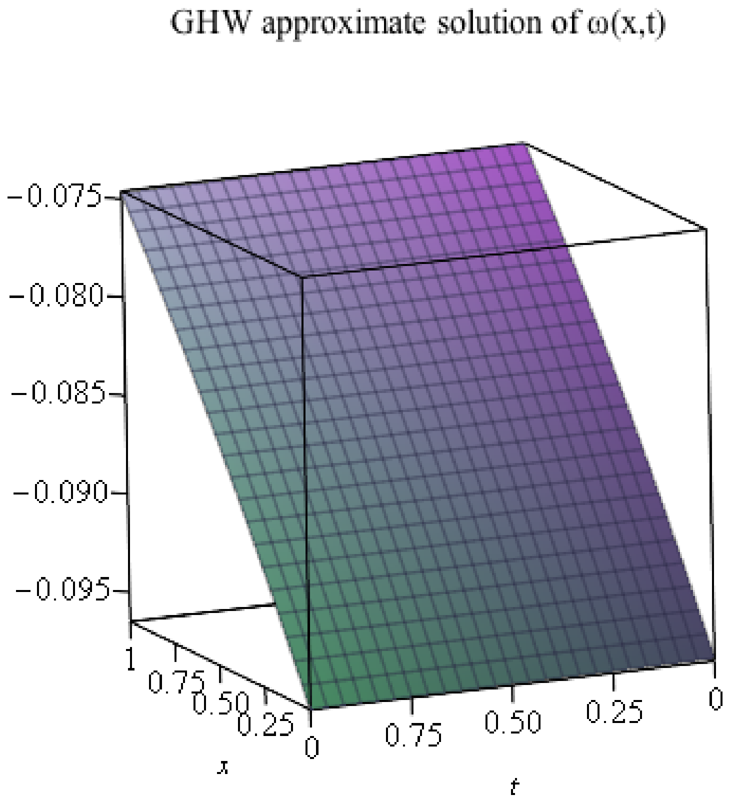


Figure 6.2 Exact and GHW approximate solution of $\omega(x,t)$ Example 6.1 for $\mu, k, y, c = 1,\ \lambda = 0.75,\ M = 3$.

Table 6.2 The absolute errors of $\omega(x,t)$ obtained by GHW method for μ, y, c and $k=1,\ M=3\ \lambda=0.75$ and other different methods for Example 6.1.

(x,t)	$\lvert\omega_{exact}-\omega_{ADM}\rvert$	$\lvert\omega_{exact}-\omega_{VIM}\rvert$	$\lvert\omega_{exact}-\omega_{OHAM}\rvert$	$\lvert\omega_{exact}-\omega_{GHW}\rvert$
(0.1, 0.1)	6.41419E-3	1.10430E-4	5.86860E-5	2.87481624E-7
(0.1, 0.3)	5.99783E-3	3.31865E-4	3.04565E-4	7.87636054E-7
(0.1, 0.5)	5.61507E-3	5.54071E-4	3.08812E-4	1.74257851E-6
(0.2, 0.1)	1.33181E-2	1.07016E-4	5.56884E-5	3.8768900E-10
(0.2, 0.3)	1.24441E-2	3.21601E-4	2.97260E-4	1.1092180E-8
(0.2, 0.5)	1.16416E-2	5.36927E-4	2.92626E-4	1.4388176E-7
(0.3, 0.1)	2.07641E-2	1.03737E-4	5.28609E-5	1.85719630E-7
(0.3, 0.3)	1.93852E-2	3.11737E-4	2.90150E-4	5.23858578E-7
(0.3, 0.5)	1.81209E-2	5.20447E-4	2.77382E-4	8.70705450E-7
(0.4, 0.1)	2.88100E-2	1.00579E-4	5.01929E-5	2.77264136E-7
(0.4, 0.3)	2.68724E-2	3.02245E-4	2.83229E-4	7.77062976E-7
(0.4, 0.5)	2.50985E-2	5.04593E-4	2.63019E-4	1.34517282E-7
(0.5, 0.1)	3.75193E-2	9.75385E-5	4.76741E-5	2.83022620E-7
(0.5, 0.3)	3.49617E-2	2.93107E-4	2.76492E-4	7.94669230E-7
(0.5, 0.5)	3.26239E-2	4.89335E-4	2.49480E-4	1.31951855E-6

with the intial conditions:

$$\begin{aligned} u(x,0) &= \vartheta - 2\ \xi\ coth(\xi\ (x+\tau)), \\ \omega(x,0) &= -2\ \xi^2\ csch^2(\xi\ (x+\tau)), \end{aligned} \tag{6.42}$$

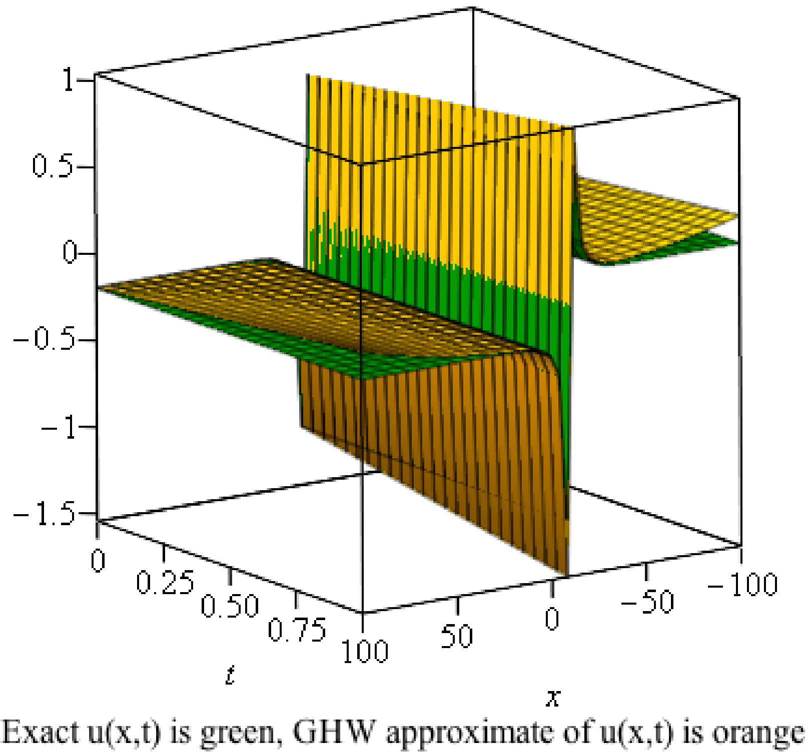
where $\vartheta,\ \xi,\ \tau$ are arbitrary constants. The exact solutions of this problem are

$$\begin{aligned} u(x,t) &= \vartheta - 2\ \xi\ coth(\xi\ (x+\tau-\vartheta\ t)), \\ \omega(x,t) &= -2\ \xi^2\ csch^2(\xi\ (x+\tau-\vartheta\ t)). \end{aligned} \tag{6.43}$$

Figure 6.3 shows the coupled surface of the exact and GHW approximate solution for $u(x,t)$ and $\omega(x,t)$ at μ, y, c and $k = 1,\ M = 3\ \lambda = 0.75$, respectively. Table 6.3 and Table 6.4 show the absolute errors of the exact solutions and the approximate solutions by methods in Ref.[101] and GHW method for $u(x,t)$ and $\omega(x,t)$ Ref.[101], respectively. The results show that the approximate solutions by GHW method are converge to the exact solutions more than the other methods.

Table 6.3 The absolute errors of $u(x,t)$ obtained by GHW method for μ, y, c and $k = 1,\ M = 3\ \lambda = 0.75$ and other different methods for Example 6.2.

(x,t)	$\lvert u_{exact} - u_{ADM}\rvert$	$\lvert u_{exact} - u_{VIM}\rvert$	$\lvert u_{exact} - u_{OHAM}\rvert$	$\lvert u_{exact} - u_{GHW}\rvert$
(0.1, 0.1)	8.16297E-7	6.35269E-5	6.35267E-5	1.19382200E-9
(0.1, 0.3)	7.64245E-7	1.90854E-4	1.90854E-4	2.30988830E-8
(0.1, 0.5)	7.16083E-7	3.18549E-4	3.18548E-4	4.40134333E-8
(0.2, 0.1)	3.26243E-6	6.18930E-5	6.18931E-5	1.00805065E-8
(0.2, 0.3)	3.05458E-6	1.85945E-4	1.85945E-4	7.25761641E-8
(0.2, 0.5)	2.86226E-6	3.10352E-4	3.10352E-4	1.424631520E-7
(0.3, 0.1)	7.33445E-6	6.03095E-5	6.03098E-5	1.276219628E-8
(0.3, 0.3)	6.86758E-6	1.81187E-4	1.81187E-4	1.323644038E-7
(0.3, 0.5)	6.43557E-6	3.02408E-4	3.02408e-4	2.568112803E-7
(0.4, 0.1)	1.30286E-5	5.87746E-5	5.87749E-5	1.243889115E-8
(0.4, 0.3)	1.22000E-5	1.76574E-4	1.76574E-4	1.926636025E-7
(0.4, 0.5)	1.14333E-5	2.94707E-4	2.94708E-4	3.700578182E-7
(0.5, 0.1)	2.03415E-5	5.72867E-5	5.72865E-5	1.221059764E-8
(0.5, 0.3)	1.90489E-5	1.72102E-4	1.72102E-4	2.440737588E-7
(0.5, 0.5)	1.78528E-5	2.87241E-4	2.87240E-4	4.667027683E-7



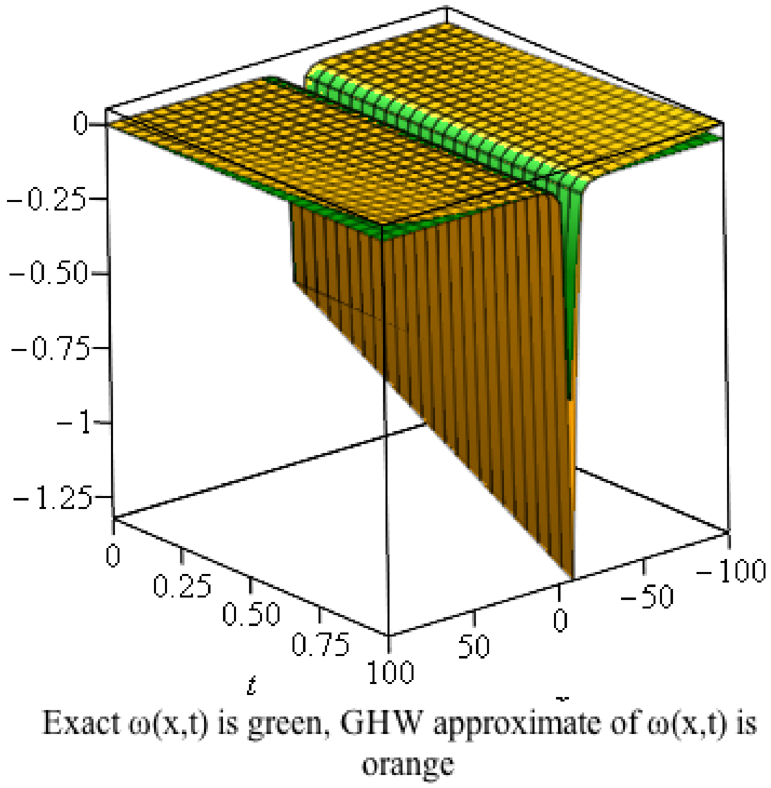


Figure 6.3 Exact and GHW approximate solution of $u(x,t)$, $\omega(x,t)$ of Example 6.2 for $k, y, c = 1$, $\lambda = 0.75$, $M = 3$ and $\mu, \beta = 0.9$.

Table 6.4 The absolute errors of $\omega(x,t)$ obtained by GHW method for μ, y, c and $k=1,\ M=3\ \lambda=0.75$ and other different methods for Example 6.2.

(x,t)	$\lvert\omega_{exact}-\omega_{ADM}\rvert$	$\lvert\omega_{exact}-\omega_{VIM}\rvert$	$\lvert\omega_{exact}-\omega_{OHAM}\rvert$	$\lvert\omega_{exact}-\omega_{GHW}\rvert$
(0.1, 0.1)	5.88676E-5	1.65942E-5	1.65942E-5	1.58982921E-8
(0.1, 0.3)	5.56914E-5	4.98691E-5	4.98691E-5	2.09296183E-7
(0.1, 0.5)	5.27169E-5	8.32598E-5	8.26491E-4	4.71065550E-7
(0.2, 0.1)	1.18213E-4	1.60813E-5	1.60812E-5	2.57052552E-8
(0.2, 0.3)	1.11833E-4	4.83269E-5	4.83269E-5	2.41109060E-8
(0.2, 0.5)	1.05858E-4	8.06837E-5	7.94290E-4	1.30548110E-8
(0.3, 0.1)	1.78041E-4	1.55880E-5	1.55880E-5	9.78459069E-8
(0.3, 0.3)	1.68429E-4	4.68440E-5	4.68439E-5	1.62540137E-7
(0.3, 0.5)	1.59428E-4	7.82068E-5	7.63646E-4	1.57921304E-7
(0.4, 0.1)	2.38356E-4	1.51135E-5	1.51135E-5	1.410219206E-7
(0.4, 0.3)	2.25483E-4	4.54174E-5	4.54174E-5	2.383027180E-7
(0.4, 0.5)	2.13430E-4	7.58243E-5	7.34471E-4	2.31763850E-7
(0.5, 0.1)	2.99162E-4	1.46569E-5	1.46569E-5	1.564239375E-7
(0.5, 0.3)	2.83001E-4	4.40448E-5	4.40448E-5	2.549799550E-7
(0.5, 0.5)	2.67868E-4	7.35317E-5	7.06678E-4	2.406138800E-7

Example 6.3 Consider the coupled KdV time- fractional equation as:

$$\begin{aligned} D_t^{\mu} u &= \zeta\, u\, u_x + \gamma\, \omega\, \omega_x + \varsigma\, u_{xxx} + f(x,t), \\ D_t^{\beta} \omega &= \varsigma\, \omega_{xxx} - \gamma\, u\, \omega_x + g(x,t), \end{aligned} \tag{6.44}$$

where $\zeta=-6,\ \varsigma=-1$ and $\gamma=3$ with the initial conditions

$$u(x,0)=0, \quad \omega(x,0)=0, \tag{6.45}$$

and boundary conditions

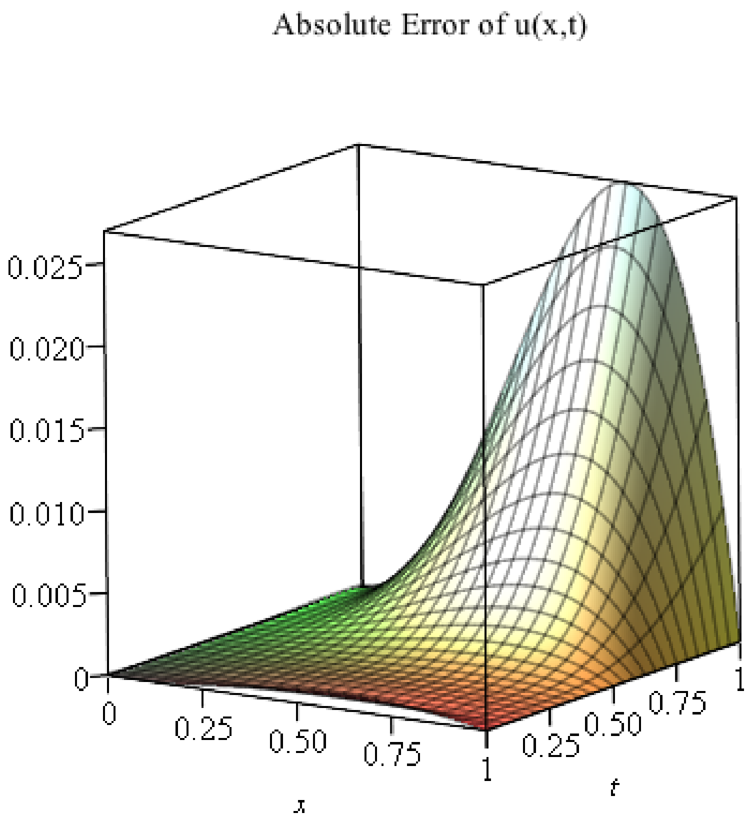
$$\begin{aligned} &u(0,t)=0, \quad u_x(0,t)=u(1,t)=t^2, \\ &\omega(0,t)=0, \quad \omega_x(0,t)=\omega(1,t)=t^2, \end{aligned} \tag{6.46}$$

where

$$f(x,t)=3xt^4+\frac{2x\ t^{2-\mu}}{\Gamma(3-\mu)}, \quad g(x,t)=3xt^4+\frac{2x\ t^{2-\beta}}{\Gamma(3-\beta)}.$$

The exact solution of this problem is $u(x,t)=\omega(x,t)=xt^2$.

In Fig. 6.4 shows that the absolute error of $u(x,t)$, $\omega(x,t)$ of this problem when applied the proposed method for $k, y, c = 1$, $\lambda = 0.5$, $M = 5$ and $\mu, \beta = 1$. In the other hand, Table 6.5 proved that when applied GHW method at $\mu = \beta = 1$ is closer to the exact solution more than $\mu = \beta = 0.5$



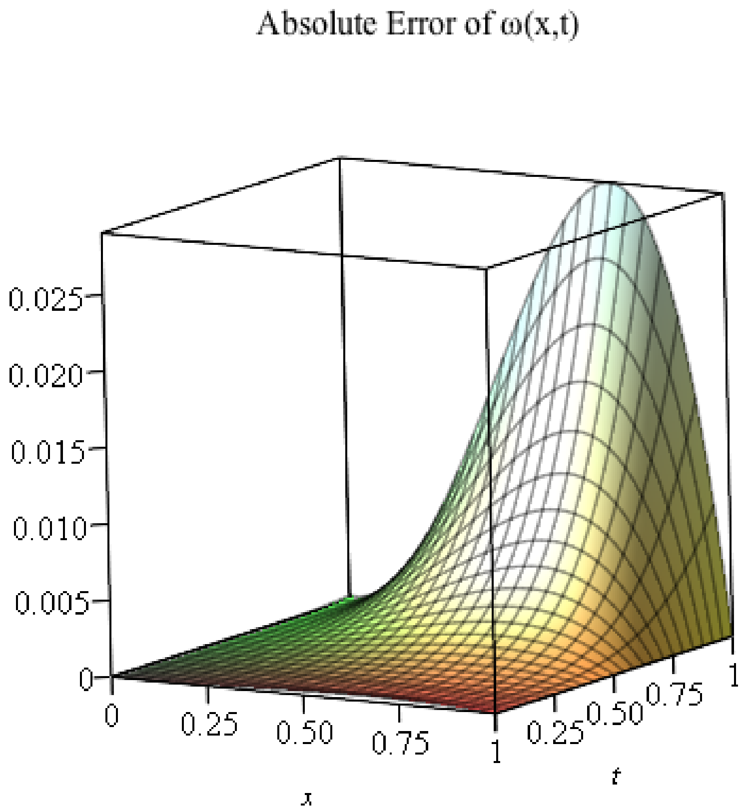


Figure 6.4 Absolute Errors of $u(x,t)$, $\omega(x,t)$ of Example 6.3 for $k, y, c = 1$, $\lambda = 0.5$, $M = 5$ and $\mu = \beta = 1$.

Table 6.5 The absolute errors of $u(x,t)$, $\omega(x,t)$ obtained by GHW method for $y, c, k = 1$, $M = 5, \lambda = 0.5$, $t = 0.1$ and different values of μ, β for Example 6.3.

x	**Error of** $u(x,t)$ **at** $\mu = \beta = 0.5$	**Error of** $\omega(x,t)$ **at** $\mu = \beta = 0.5$	**Error of** $u(x,t)$ **at** $\mu = \beta = 0.9$	**Error of** $\omega(x,t)$ **at** $\mu = \beta = 1$
0.1	1.062771516E-7	1.062771516E-7	6.04608780E-8	6.04608780E-8
0.2	4.154917686E-7	4.154917686E-7	1.41795648E-7	1.41795648E-7
0.3	8.870216424E-7	8.870216424E-7	5.42272150E-7	5.42272150E-7
0.4	1.454632318E-6	1.454632318E-6	9.65317400E-7	9.65317400E-7
0.5	2.026939693E-6	2.026939693E-6	1.21458981E-6	1.21458981E-6
0.6	2.487409523E-6	2.487409523E-6	1.19402992E-6	1.19402992E-6
0.7	2.694359008E-6	2.694359008E-6	9.07745690E-7	9.07745690E-7
0.8	2.480954098E-6	2.480954098E-6	4.60174320E-7	4.60174320E-7
0.9	1.655212619E-6	1.655212619E-6	5.59180100E-8	5.59180100E-8

Example 6.4 We consider the coupled KdV time- fractional equation of Example 6.3 with the initial conditions

$$u(x,0)=0, \quad \omega(x,0)=0, \tag{6.47}$$

and boundary conditions

$$\begin{aligned} &u(0,t)=0, \quad u_x(0,t)=u(1,t)=\sqrt{t^5}, \\ &\omega(0,t)=0, \quad \omega_x(0,t)=\omega(1,t)=\sqrt{t^5}, \end{aligned} \tag{6.48}$$

where

$$f(x,t)=3xt^5+\frac{x\,\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\mu)}t^{5/2-\mu}, \quad g(x,t)=3xt^5+\frac{x\,\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\beta)}t^{5/2-\beta}. \tag{6.49}$$

The exact solution of this problem is $u(x,t)=\omega(x,t)=x\sqrt{t^5}$.

Table 6.6 compared the results by using the absolute errors of each of method in Ref. [102] and the proposed method for a different values of μ, β. As a result, the numerical solution using the GHW method more accuracy and closed to the exact solution.

Table 6.6 Comparison the absolute errors of $u(x,t),\ \omega(x,t)$ for Example 6.4 by method in Ref.[102] and GHW method with $k=1,\ M=5, y=c=1,\ \lambda=0.5,\ t=0.1$

x	**Error of** $u(x,t)=\omega(x,t)$ **at** $\mu=\beta=0.3$ **in Ref.[102]**	**Error of** $u(x,t)=\omega(x,t)$ **at** $\mu=\beta=0.3$ **GHW**	**Error of** $u(x,t)=\omega(x,t)$ **at** $\mu=\beta=0.5$ **in Ref.[102]**	**Error of** $u(x,t)=\omega(x,t)$ **at** $\mu=\beta=0.5$ **GHW**
0.1	1.83153578E-5	2.80476450E-6	5.83378313E-5	1.095288366E-8
0.2	3.28200845E-5	1.397350520E-3	9.95389615E-5	4.294756724E-8
0.3	4.40034358E-5	2.373852170E-3	1.29963809E-4	9.127761900E-8
0.4	5.36473432E-5	3.062470200E-3	1.55760305E-4	1.488569703E-7
0.5	6.24020972E-5	3.372168550E-3	1.78938362E-4	2.062841622E-7
0.6	7.06254728E-5	3.268206000E-3	2.00569756E-4	2.518419797E-7
0.7	7.85563020E-5	2.7726479500E-3	2.213493522E-4	2.714983569E-7
0.8	8.63773316E-5	1.9635734000E-3	2.41796227E-4	2.489050063E-7
0.9	9.42444364E-5	9.7584375000E-6	2.62344601E-4	1.653988104E-7

7
RESULTS AND DISCUSSION

In order to widespread applications of partial differential equations and the feature of fractional order to represent most of the phenomena problem by the best way, deriving of new numerical methods for solving these types of problems are necessary. The main idea of this thesis is to build a new wavelet method and utilize it to solve fractional partial differential equations. Three published papers has been provided for this purpose in Chapters 3 to 6 as follows:

In Chapter 1, we look at the background, and literature related to the research topic that is studied in this thesis, and the purpose of the chosen topic. Some basic definitions of fractional calculus, wavelet, the generalized Gegenbauer- Humbert wavelets and their operational matrix of fractional order integration are provided and derived in Chapter 2.

In Chapter 3, a numerical technique of the generalized Gegenbauer- Humbert wavelet method is constructed by using their operational matrix of fractional integration, and employed to solve linear and non-linear fractional differential problems. The obtained results show the effect of various parameters and the fractional order α, of the accuracy of the approximate solutions.

On the other hand, the operational matrices of integer and fractional order of the GHW method are derived in Chapter 4. The proposed method demonstrated the efficiency and accuracy when applied to solve fractional differential problems (linear and non-linear) as compared with other methods, and comparison between different special cases of the proposed method it self in some examples.

In Chapter 5, developed the GHW technique to solve fractional partial differential equations with (initial-boundary and boundary) conditions. Convergence analysis of GHW method are established for two variables. The obtained results are good compared with different methods, for instance Legender, and 3rd kind of Chebyshev wavelet methods.

Extending of GHW technique to apply to coupled systems of two types of shallow waters (WBK and KdV) equstions in Chapter 6. The observed results are sufficient and accurate comparing with the Adomian's decomposition method, the variational iteration method, the optimal homotopy asymptotic method, and other.

All the numerical results and graphs are yielded by algorithms created in Maple, and the consequences are shown that the proposed method is successful in solving different problems and systems of equations with high accuracy.

The GHW method can be developed to solve other real -life and physical phenomena like modeling of diseases and engineering problems.

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PUBLICATIONS FROM THE THESIS

Papers

1. Alkhalissi, J. HS, Emiroglu, I., Secer, A. and Bayram, M.,(2020)."The generalized Gegenbauer-Humberts wavelet for solving fractional differential equations", Thermal Science, 24(1):107–118.

2. Alkhalissi, J. HS, Emiroglu, I.,Bayram, M., Secer, A. and Tasci, F.,(2021)."A new operational matrix of fractional derivative based on the generalized Gegenbauer–Humbert polynomials to solve fractional differential equations", Alexandria Engineering Journal, 60(4):3509–3519.

3. Alkhalissi, J. HS, Emiroglu, I., Bayram, M., Secdr, A. and Tasci, F.,(2021). "Generalized Gegenbauer–Humbert wavelets for solving fractional partial differential equations", Engineering with Computers, Springer, 1–12.

Conference Papers

1. Alkhalissi, J. HS, Emiroglu, I., Bayram, M., Secer, A. and Tasci, F.,(2021),"Solving a coupled time -fractional partial differential equations by using generalized Gegenbauer - Humbert wavelets", ICAAMM21, 11–13 June, Istanbul,Turkey.